DUALITY OF WEIGHTS, MIRROR SYMMETRY AND ARNOLD'S STRANGE DUALITY

Masanori KOBAYASHI

Department of Mathematics, Tokyo Institute of Technology

Introduction.

The hypersurfaces in weighted projective spaces often appear as important examples in the context of mirror symmetry. The Batyrev's idea using polar duality for reflexive polytopes in toric geometry[5] provides many examples which are not weighted hypersurfaces.

In this paper, first we describe the relation between polar duality in toric geometry and duality of weighted hypersurfaces.

As an application, we will show that Arnold's strange duality for the fourteen exceptional unimodal hypersurface singularities[1] reduces to polar duality.

K. Saito defined the weight systems and used effectively for isolated hypersurface singularities with a $\mathbb{C}^*$-action[32]. Those singularities correspond to weight systems as follows; take a germ of analytic function $f : \mathbb{C}^n \to \mathbb{C}$ with $f(0) = 0$, which determines a germ of hypersurface singularity $(\{f = 0\}, 0) \subset (\mathbb{C}^n, 0)$. The function f can be expressed as a weighted homogeneous polynomial of n variables $(x_1, \ldots, x_n)$ after some suitable analytic change of coordinates near the origin if and only if there is a holomorphic tangent vector field D such that $Df = f$[31]. In this case, we can assign to f a weight system $W = (wt(x_1), \ldots, wt(x_n); wt(f))$.

We associate some hypersurface singularity with $\mathbb{C}^*$-action a compact complex surface with trivial canonical sheaf in the following manner.

Let $X_0 \subset \mathbb{C}^n$ be a hypersurface singularity with $\mathbb{C}^*$-action whose weights are all positive. Such X_0 is known to be an algebraic variety defined by a weighted homogeneous polynomial[26].

Let X_t be its Milnor fiber (if it exists; e.g. when X_0 is an isolated singularity). We will not restrict ourselves to isolated singularities. Let $\bar{X}$ be the natural compactification of X_t in $\mathbb{P}(1, a_1, \ldots, a_n)$ by sending the points $(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ to $(1 : x_1 : \cdots : x_n)$.

We treat only the case that total degree h is greater than the sum of the weights $\sum_{i=1}^n a_i$ here. We denote the difference $h - \sum_{i=1}^n a_i$ by a_0.

Let $\hat{X}$ be the image of $\bar{X}$ by the $\mathbb{Z}/a_0\mathbb{Z}$-quotient: $\mathbb{P}(1, a_1, \ldots, a_n) \to \mathbb{P}(a_0, a_1, \ldots, a_n)$. By the adjunction formula, the canonical sheaf of $\hat{X}$ is trivial. We denote a crepant resolution of $\hat{X}$ by X if it exists, in which case $\hat{X}$ has only canonical Gorenstein singularities.

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In the case X_0 is an isolated singularity, it is known that the Milnor lattice or exceptional lattice of resolution and the exceptional lattice at infinity enjoy some dual property[21].

We will state how polar duality of polytopes related to X descends to the duality of weight systems up to Chapter 3.

The duality of weights partly suggests why [10] produced a mirror symmetric phenomena using only a resolution of weighted hypersurface in weighted $\mathbb{P}^4$. In fact, recently it is shown that those examples in weighted 4-spaces correspond to some reflexive polytopes [9].

When $n = 3$, X is a K3 surface. First we assume X is a K3 surface corresponding to one of the fourteen exceptional unimodal isolated hypersurface singularities[27][15].

Aspinwall and Morrison[3] explains why mirror symmetry for algebraic K3 surface is said to correspond to Arnold's strange duality. The transcendental lattice T_X decomposes as $L_G \oplus U$, where the dual graph of L_G is a tree of (-2)-elements with three branches and U is the even unimodular hyperbolic lattice of rank 2. We will write $L_D := S_X$ in this case. Then the whole cohomology ring decomposes as $H^*(X, \mathbb{C}) = H^{0,0} \oplus (L_G \oplus U \oplus L_D)_{\mathbb{C}} \oplus H^{2,2}$, where the subscript $_{\mathbb{C}}$ means the tensor product by $\mathbb{C}$. The mirror map should interchange $(L_G)_{\mathbb{C}}$ and $(L_D)_{\mathbb{C}}$, $H^{0,0} \oplus H^{2,2}$ and $U_{\mathbb{C}}$, respectively, since the vector space $(L_G)_{\mathbb{C}}$ corresponds to the tangent space of the deformation space of complex structures with a fixed Picard number and $(L_D)_{\mathbb{C}}$ corresponds to a deformation of complexified algebraic Kähler structures.

Roan[29] applied orbifold construction for four of the dual pairs and constructed non-linear coordinate changes for them. Borcea[8] used Batyrev's polar dual construction for the reflexive pyramids and calculated some of dual weights.

We shall give a proof that polar dual of polytopes corresponding Arnold's fourteen exceptional unimodal hypersurface singularities gives corresponding dual singularity.

More generally, take a K3 surface X which is the minimal resolution of toric hypersurface with only rational double points. Let L_D be the restriction of the toric divisors of the ambient space and L_0 be the orthogonal complement $L_D^{\perp}$ in S_X. then $(S_X)_{\mathbb{Q}}$ decomposes as a direct sum $(L_D)_{\mathbb{Q}} \oplus (L_0)_{\mathbb{Q}}$.

The reason why the full algebraic lattice and L_G does not interchange is that the induced complexified Kähler class is trivial on L_0-part while embedded deformation has the full dimension. This L_0-part is a generalization of $H_n^{1,1}$ in [29], where Roan showed a similar mirror property for a quotient construction. Batyrev pointed out in a non-published version of [5] that for general Δ, $\operatorname{rk} L_D(\Delta) + \operatorname{rk} L_D(\Delta^*)$ can be smaller than 20. For the 14 exceptional unimodal singularities, we found polytopes for which the sum above equals 20.

Since the mirror of generic marked K3 surface seems to be again a generic one, mirror phenomena for special marked K3 surfaces are of particular interest, where 'special' means e.g. the surface is algebraic and its (complexified) Kähler class is also ample.

For general K3 surfaces L_0 is defined as the orthogonal complement of the fixed lattice by the automorphism group which fixes $H^{2,0}$ and the (complexified) Kähler class. We show this group is finite using Nikulin's results on the automorphism of a K3 surface and Torelli's theorem. H. Tsuji kindly informed the author that the finiteness also follows from Yau's Theorem.

We refer to [3][18] and their reference for an introduction to some background and [23][37] for general information about mirror symmetry. A preprint by Dolgachev [14]

includes relevant general results and examples using lattice theory.

In Chapter 1, we shall formulate a dual correspondence of reflexive pairs in $\mathbb{Q}$-Fano toric projective varieties after reviewing some results of toric geometry. In Chapter 2, we prepare the language of weight systems according to K. Saito and introduce our notion of duality of weight systems. Our duality partly coincides with Saito's duality. In Chapter 3, we state the relation between duality of weight systems and Batyrev's toric mirror symmetry using polar polytopes. We found that our construction is partly a generalization of those of Batyrev[5], Borcea[8] and Berglund-Katz[7] for reflexive simplices and pyramids. In Chapter 4, we review some general results of K3 surfaces, and we argue some lattice theory and automorphism. We also construct a monomial divisor mirror map of Aspinwall-Greene-Morrison in the case of K3 surfaces as toric hypersurfaces and prove Arnold's strange duality. We add some examples which are not exceptional unimodal isolated singularities.

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Notation.

$\mathbb{N}$: the monoid of nonnegative integers.
$\mathbb{Z}_+$: the semigroup of positive integers.
$\mathbb{Z}$: the ring of integers.
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$: the field of rational, real, complex numbers.
$\mathbb{T}$: n-dimensional complex torus $(\mathbb{C}^*)^n$.

1. Polar duality.

1.1 Reflexive polytopes.

We fix here some notations about geometry of convex body and review some results. We refer [25] for generality of toric geometry and geometry of convex bodies.

Let M be a free $\mathbb{Z}$-module of rank n. We naturally imbed M in the $\mathbb{Q}$-vector space $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. Similarly for the dual module $N := M^\vee = \mathrm{Hom}(M, \mathbb{Z})$. We denote the natural pairing by $(\ ,\ ) : M \times N \to \mathbb{Z}$, which we also use for its $\mathbb{Q}$-extensions.

Definition 1.1.1. A subset Δ in $M_{\mathbb{Q}}$ is called a *convex polytope* if it is a convex hull of a finite subset of $M_{\mathbb{Q}}$. Δ is *integral* if the finite subset can be taken from M. A *face* of Δ is a nonempty intersection with a hyperplane whose closed half space contains whole Δ. We denote the *k-skeleton* of Δ by $\Delta^{[k]}$, which is the union of all faces of Δ whose dimension is no more than k. We call a codimension-one face a *facet*. We denote the nubers of points $|M \cap \Delta|$ and $|M \cap \mathrm{Int}\,\Delta|$ by $l(\Delta)$ and $l^*(\Delta)$, respectively.

Remark 1.1.2. We use those words above similarly for subsets in $N_{\mathbb{Q}}$. We shall sometimes omit the word "convex" in this paper since we always assume so for polytopes. A face of a polytope Δ is also a polytope.

Definition 1.1.3. Let K, K_1 and K_2 be subsets of $M_{\mathbb{Q}}$ and $c \in \mathbb{Q}$. $cK := \{cx \in M_{\mathbb{Q}} | x \in K\}$, $K_1 + K_2 := \{x_1 + x_2 \in M_{\mathbb{Q}} | x_i \in K_i \text{ for } i = 1, 2\}$.

Definition 1.1.4. K be a subset in $M_{\mathbb{Q}}$. The *polar dual* of K is the following subset of $N_{\mathbb{Q}}$: $K^*=\{y \in N_{\mathbb{Q}} | (x, y) \geq -1 \text{ for all } x \in K\}$.

Lemma 1.1.5.

(1) *K^* is a convex set containing 0. If $K_1 \subset K_2$ then $K_1^* \supset K_2^*$.*
(2) *If Δ is an n-dimensional convex polytope with $0 \in \operatorname{Int}\Delta$, then all the same for Δ^*. Moreover, $\Delta^{**} = \Delta$ by natural identification $M^{\vee\vee} = M$.*

Definition 1.1.6. [11] For an n-dimensional polytope Δ in $M_{\mathbb{Q}}$, define an n-dimensional projective variety

$$\mathbb{P}_{\Delta,M} = \mathbb{P}_{\Delta} := \operatorname{Proj} \bigoplus_{k=0}^{\infty} \mathbb{C}(k\Delta \cap M) T^k,$$

where $\mathbb{C}(k\Delta \cap M)$ denotes the $\mathbb{C}$-vector space generated the elements of M in $k\Delta$, and the multiplication comes from the addition of M. $(\mathbb{P}_{\Delta}, \mathcal{O}(1))$ is the polarized variety associated to the polytope Δ.

Definition 1.1.7. [5] Let Δ be an n-dimensional integral polytope with $0 \in \operatorname{Int}\Delta$. Δ is said to be *reflexive* if one of the following equivalent conditions is satisfied:

(1) Δ^* is also an integral polytope;
(2) there exists a finite subset $\{y_1, \ldots, y_k\}$ in N such that $\Delta = \{x \in M_{\mathbb{Q}} \mid (x, y_i) \geq -1 \text{ for } 1 \leq i \leq k\}$;
(3) for each facet δ of Δ, there exists an integral element $y \in N$ such that $\delta \subset \{x \in M_{\mathbb{Q}} \mid (x, y) = -1\}$;
(4) for each facet δ, there are no points of M between the origin and the hyperplane containing δ.

If Δ is reflexive, Δ^* is reflexive. We cite here some results in [5].

Theorem 1.1.8. [5 Theorem 4.1.9] *Let Δ be an n-dimensional integral polyhedron in $M_{\mathbb{Q}}$, $\mathbb{P}_{\Delta}$ the corresponding n-dimensional projective toric variety, and $\mathcal{F}(\Delta)$ the family of projective Δ-regular hypersurfaces $\overline{Z}_f$ in $\mathbb{P}_{\Delta}$. Then the following conditions are equivalent:*

(1) *the family $\mathcal{F}(\Delta)$ of Δ-regular hypersurfaces in $\mathbb{P}_{\Delta}$ consists of Calabi-Yau varieties with canonical singularities;*
(2) *the ample invertible sheaf $\mathcal{O}_{\Delta}(1)$ on the toric variety $\mathbb{P}_{\Delta}$ is anticanonical(i.e. $\mathbb{P}_{\Delta}$ is a toric Fano variety with Gorenstein singularities);*
(3) *Δ contains only one integral point m_0 in its interior, and $(\Delta - m_0, M)$ is a reflexive pair.*

Theorem 1.1.9. [5 Theorem 4.4.3] *For any reflexive polyhedron Δ of dimension $n \geq 4$, the Hodge number $h^{n-2,1}(\widehat{Z}_f)$ of a MPCP-desingularization of a Δ-regular Calabi-Yau hypersurface $\overline{Z}_f \subset \mathbb{P}_{\Delta}$ equals the Picard number $h^{1,1}(\widehat{Z}_g)$ of a MPCP-desingularization*

of a Δ^*-regular projective Calabi-Yau hypersurface $\overline{Z}_g \subset \mathbb{P}_{\Delta^*}$ corresponding to the dual reflexive polyhedron Δ^*.

In the above two theorems, "polyhedron" means "polytope" in our paper. We reserve the word "polyhedron" for a three dimensional polytope. MPCP-desingularization is a maximal projective crepant partial toric embedded resolution.

1.2. Dual correspondence for toric projective varieties.

In this section, we shall treat a family of anticanonical members in a $\mathbb{Q}$-Fano toric variety $\mathbb{P}_\Delta$ where Δ may not be integral. We will think not only the quotient family corresponding to the same Δ but also specialization corresponding to a sub-Newton polytope.

Definition 1.2.1. Let $\mathcal{P}_n$ be the set of n-dimensional pairs (Δ, M) where M is a free $\mathbb{Z}$-module of rank n and Δ is a n-dimensional polytope in $M_{\mathbb{Q}}$ which contains 0 in its interior, such that $\mathbb{P}_{\Delta,M}$ is $\mathbb{Q}$-Fano and Δ represents anticanonical divisor.

We usually assume that $n \geq 2$. Let $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ be two elements of $\mathcal{P}_n$. We denote the dual groups $(\overline{M})^\vee$ by $\underline{N}$ and $(\overline{N})^\vee$ by $\underline{M}$. We also denote the polar duals $\overline{\Delta}^*$ and $\overline{\nabla}^*$ by $\underline{\nabla}$ and $\underline{\Delta}$, respectively.

Definition 1.2.2. A $\mathbb{Q}$-linear isomorphism $\sigma : \underline{M}_{\mathbb{Q}} \to \overline{M}_{\mathbb{Q}}$ such that $\sigma(\underline{\Delta}) \subset \overline{\Delta}$, is called a *correspondence* between $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ if $\sigma(\underline{\Delta}) \cap \overline{M}$ generates $\overline{M}$ and $\underline{\Delta}$ has a nonempty intersection with each facet of $\overline{\Delta}$.

If $\sigma(\underline{M}) = \overline{M}$ and ${}^t\sigma$ is correspondence, we call it a *dual correspondence*.

We shall identify $\underline{M}_{\mathbb{Q}}$ and $\overline{M}_{\mathbb{Q}}$, $\overline{N}_{\mathbb{Q}}$ and $\underline{N}_{\mathbb{Q}}$ by σ and ${}^t\sigma$, respectively. We shall sometimes omit σ.

When $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ have a dual correspondence σ, we will use $M = \overline{M} = \sigma(\underline{M})$ and $N = \overline{N} = {}^t\sigma(\underline{N})$.

A basic observation is the following

Proposition 1.2.3. *Assume there exists a dual correspondence between two elements $(\overline{\Delta}, M)$ and $(\overline{\nabla}, N)$ of $\mathcal{P}_n$. Let (Δ, M) be a reflexive pair such that $\underline{\Delta} \subset \Delta \subset \overline{\Delta}$.*

Then there is a natural birational map $\Phi : \mathbb{P}_{\overline{\Delta}} \dashrightarrow \mathbb{P}_\Delta$ coming from the inclusion between the polytopes. A general anticanonical member of $\mathbb{P}_\Delta$ is birational to the corresponding member of $\mathbb{P}_{\overline{\Delta}}$.

Similarly, from $\underline{\nabla} \subset \Delta^* \subset \overline{\nabla}$, a general anticanonical member of $\mathbb{P}_{\Delta^*}$ is birational to that of $\mathbb{P}_{\overline{\nabla}}$.

Proof. We denote the sublinear system of the anticanonical linear system of $\mathbb{P}_{\overline{\Delta}}$ corresponding to the polytope Δ by $\Lambda(\Delta)$.

By definition, any nonempty nonzero subsystem Λ is free from base points on the n-dimensional orbit of $\mathbb{T}$.

Since the set $\Delta \cap M$ generates the whole M, $\Lambda(\Delta)$ defines a birational map $\overline{\varphi} : \mathbb{P}_{\overline{\Delta}} \dashrightarrow \mathbb{P}^{l(\Delta)-1}$. The image V is the projective variety $\operatorname{Proj} \bigoplus_{k=0}^{\infty} \left(\sum_{i=1}^{k} (\Delta \cap M) \right)$, which is associated to the graded algebra generated by $\Delta \cap M$.

Each facet of $\overline{\Delta}$ corresponds to an $(n-1)$-dimensional orbit of $\mathbb{T}$ on $\mathbb{P}_{\overline{\Delta}}$, which has nonempty intersection with Δ, since Δ contains $\underline{\Delta}$. Thus $\Lambda(\Delta)$ is free from base components and the birational transform of a hyperplane section of V in $\mathbb{P}_{\overline{\Delta}}$ is a member of $\Lambda(\Delta)$. In particular, a general member of $\Lambda(\Delta)$ is an irreducible divisor if $n > 1$.

On the other hand, the anticanonical complete linear system on $\mathbb{P}_\Delta$ is free from base points, since Δ is an integral polytope, and defines a morphism φ to the same V, which is birational by the same reason as above. Take $\Phi := \varphi^{-1} \circ \overline{\varphi}$. $\square$

Remark 1.2.4. Thus, in the situation above, if a Calabi-Yau variety $\bar{Z}_\Delta$ is a specialization of $\bar{Z}_{\Delta'}$, then $\bar{Z}_{\Delta^*}$ is a generalization of $\bar{Z}_{\Delta'^*}$. When you specialize $\bar{Z}$, the singularity goes bad. As a result, we have more Kähler moduli and less complex structure moduli for the desingularization $\tilde{Z}_\Delta$, and contrary for $\tilde{Z}_{\Delta^*}$. We hope this relation holds also outside toric varieties.

Remark 1.2.5. $\mathbb{P}_\Delta$ is isomorphic to V if $n \leq 3$. Note that we have unique minimal models in this case.

Proposition 1.2.6. *Let Δ be a reflexive polytope of dimension at most three, and $(\mathbb{P}_\Delta, \mathcal{O}(1))$ be the polarized variety associated to it. Then $\mathcal{O}(1)$ is simply generated and defines projective normal embedding in $\mathbb{P}^{l(\Delta)-1}$.*

Proof. The proof follows from the following

Proposition 1.2.7. *Let Δ be a reflexive polytopes in $M_\mathbb{Q}$ whose dimension is at most three, and k be an positive integer. Then for any integral point $P \in k\Delta$, there exist integral points $P_i \in \Delta$ $(i = 1, \ldots, k)$ with $P = P_1 + \cdots + P_k$.*

Proof. We assume $n = 3$, since other cases are similar and easier. For each 2-dimensional face δ of Δ, we have a subdivision by integral simplices (triangles) such that

(1) each simplex is elementary, that is, it does not contain points of M other than the vertices,
(2) each simplex has common points with other simplices only on its edges and
(3) the union of all simplices covers δ.

We note that an elementary simplex of dimension two is regular. That means, if we denote the vertices of the simplex by Q_0, Q_1 and Q_2 and the plane containing δ by α, $Q_1 - Q_0$ and $Q_2 - Q_0$ spans $\alpha \cap M$. Let C be the closed cone over the simplex with apex at 0. Then any integral point in C can be written as an integral combination of Q_i's, since Δ is reflexive.

Remark 1.2.8. In higher dimensions, there exist elementary non regular simplices.

In general for an ample divisor A on an algebraic K3 surface, it is known that $3A$ is very ample.

Corollary 1.2.9. *For a general toric hypersurface $\bar{Z}$ of dimension 1 or 2, the induced morphism $\bar{Z} \to \mathbb{P}^{l(\Delta)-2}$. is a projectively normal embedding.*

Proof. $H^1(\mathbb{P}_\Delta, \mathcal{O}(k))$ vanishes for $k \geq 0$. $\square$

In Chapter 3, we shall apply these notions to the case of weighted projective spaces with one particular choice of a homogeneous coordinate.

2. Duality for weight systems.

2.1. Weighted projective space.

We fix here some notations. For general properties about weighted projective spaces, see for example, [22][13][17].

Definition 2.1.1. Let $a_0, \ldots, a_n$ be fixed positive integers. We denote by $\mathbb{P}(a_0, \ldots, a_n)$ the projective variety $\operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$ where the degree of x_i is a_i. We call it a *weighted projective space* of weight $(a_0, \ldots, a_n)$.

For a positive integer k, there is a natural isomorphism $\mathbb{P}(a_0, \ldots, a_n) \cong \mathbb{P}(ka_0, \ldots, ka_n)$.

Definition 2.1.2. A weight $(a_0, \ldots, a_n)$ is *reduced* if $\gcd(a_0, \ldots, a_n) = 1$.

We assume the weight is reduced. Let k be $\gcd(a_1, \ldots, a_n)$. Then $\mathbb{P}(a_0, a_1, \ldots, a_n) \cong \mathbb{P}(a_0, a_1/k, \ldots, a_n/k)$, which leads to the following

Definition 2.1.3. A reduced weight $(a_0, \ldots, a_n)$ is *well-formed* if for all $i, \gcd(a_j)_{j \neq i} = 1$.

We shall usually treat only well-formed weights.

2.2. Weight systems.

We refer [32] for general statements of weight systems, especially for $n = 3$.

Definition 2.2.1. A $(n+1)$-uple of positive integers $W = (a_1, \ldots, a_n; h)$ is called a *system of weights* or simply a *weight system*. We always assume that $h \in \sum_{i=1}^n a_i$. We call the integers a_i as *weights* of W and the last weight h the *degree* of W.

Definition 2.2.2. W is said to be *reduced* if $\gcd(a_1, \ldots, a_n, h) = 1$ holds.

Remark 2.2.3. W is reduced then $\gcd(a_1, \ldots, a_n) = 1$, since $h \in \sum_{i=1}^n a_i$.

Definition 2.2.4. $W = (a_1, \ldots, a_n; h)$ and $W' = (a'_1, \ldots, a'_n; h')$ are *equivalent* if for some rational number k and a permutation $\sigma \in \mathfrak{S}_n$, $ka_{\sigma(i)} = a'_i$ $(1 \leq i \leq n)$ and $kh = h'$ hold.

For each equivalence class of weight systems there is exactly one reduced weight system satisfying $a_1 \leq a_2 \leq \cdots \leq a_n$, which we usually use in this paper.

2.3. Duality for weight systems.

Let n be a positive integer. Take a weight system $W_a = (a_1, \ldots, a_n; h)$. Assume that $h \in \sum_{i=1}^n \mathbb{N} a_i$ as always, and that W_a is reduced, for the sake of brevity. We denote the integral vector ${}^t(a_1, \ldots, a_n)$ by a.

Definition 2.3.1. $a_0 := h - \sum_{i=1}^n a_i$.

Assume also that $a_0 \neq 0$.

For a rational monomial $X_0^{m_0} \cdots X_n^{m_n}$ of degree h, we assign an n-tuple of integers $(\alpha_1, \ldots, \alpha_n) = (m_1 - 1, \ldots, m_n - 1)$. Note that $X_0 \cdots X_n$ corresponds to the origin. The whole such n-tuples constitute a set

Definition 2.3.2.

$$M(W_a) := \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^{\oplus n} \;\middle|\; \sum_{i=1}^n a_i(\alpha_i + 1) \equiv h \mod a_0 \right\}.$$

Lemma 2.3.3. *$M(W_a)$ is a subgroup of $\mathbb{Z}^{\oplus n}$ of index $|a_0|$.*

Proof. This follows from $M(W_a) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^{\oplus n} \mid \sum_{i=1}^n a_i\alpha_i \equiv 0 \mod a_0\}$ and $\gcd(a_1, \ldots, a_n) = 1$. □

Remark 2.3.4. If W_a and W_b are equivalent, $M(W_a)$ and $M(W_b)$ are the same subset of $\mathbb{Z}^{\oplus n}$ up to the permutation of the coordinates. We sometimes abbreviate this set as M in this chapter.

Let $C = (c_{ij})$ be an $n \times n$-matrix whose elements are nonnegative integers. Let B be the $n \times n$-matrix $(c_{ij} - 1)$.

Lemma 2.3.5. *Assume $Ca = {}^t(h, \ldots, h)$. Then*

(1) *$(\det C)/h = (\det B)/a_0$, and this is an integer.*
(2) *The following three conditions are equivalent: (a) $\{(c_{i1} - 1, \ldots, c_{in} - 1) | 1 \le i \le n\}$ is a basis of M, (b) $|\det B| = |a_0|$ and (c) $|\det C| = h$.*

Proof. Note that for $1 \le i \le n$, $(c_{i1} - 1, \ldots, c_{in} - 1) \in \pi(a_0) \cap M$ and $(c_{i1}, \ldots, c_{in}) \in \pi(h)$, where $\pi(t)$ is the hyperplane defined by $\sum_{i=1}^n a_i\alpha_i = t$. The rest is clear. □

Remark 2.3.6. We will say a few words for the case W_a is not reduced. Let d_0 be $\gcd(a_1, \ldots, a_n)$. Then M is a subgroup of index $|a_0|/d_0$, and the row vectors of B is a basis $\iff |\det B| = |a_0|/d_0 \iff |\det C| = h/d_0$.

Let $W_a = (a_1, \ldots, a_n; h)$ and $W_b = (b_1, \ldots, b_n; k)$ be two weighted systems.

Definition 2.3.7. An integer matrix $C \in M_n(\mathbb{N})$ is said to be a *weighted magic square* of weight $(W_a; W_b)$ if $C{}^t(a_1, \ldots, a_n) = {}^t(h, \ldots, h)$ and $(b_1, \ldots, b_n)C = (k, \ldots, k)$.

Remark 2.3.8. In the case $a_1 = \cdots = a_n = b_1 = \cdots = b_n = 1$ and $h = k$, C is called a *integer stochastic matrix* or *magic square*, which appears in classical combinatorics theory[35].

Definition 2.3.9. A weighted magic square C is *primitive* if $|\det C| = h/\gcd(a_1, \ldots, a_n) = k/\gcd(b_1, \ldots, b_n)$.

Remark 2.3.10. C is primitive if and only if the row vectors of B span $M(W_a)$ and the column vectors span $M(W_b)$.

Definition 2.3.11. When there exists a primitive weighted magic square C of weight $(W_a; W_b)$, we say W_a and W_b are *dual* with respect to C.

Dual weight systems are *strongly dual* if all rows and columns of C contain 0.

Remark 2.3.12. If W_a and W_b are reduced dual weights, it follows that $h = k$, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $a_0 = b_0$.

A permutation on weights $(a_1, \ldots, a_n)$ (resp. $(b_1, \ldots, b_n)$) interchanges the corresponding columns (resp. rows) of C.

From this definition, one can calculate the dual weights.

The following Proposition is handy for the calculation.

Proposition 2.3.13. *Let $W_a = (a_1, \ldots, a_n; h)$ and $W_b = (b_1, \ldots, b_n; h)$ be dual weight systems with a_0 be $h - \sum_{i=1}^n a_i > 0$ and $C = (c_{ij})$ be the corresponding weighted magic square. Let Θ be the $(n-1)$-simplex in $M(W_a)_{\mathbb{Q}}$ with vertices $\{(c_{i1} - 1, \ldots, c_{in} - 1) | 1 \leq i \leq n\}$. Then*

(1) $\sum_{j=1}^n c_{ij} a_j = h$ *for* $1 \leq i \leq n$,
(2) $\dfrac{a_0}{h - a_0}(1, \ldots, 1) \in \operatorname{Int} \Theta$ *and*
(3) Θ *is elementary, i.e.,* $\Theta \cap M(W_a)$ *are the set of all vertices of* Θ.

Conversely, for a given $(a_1, \ldots, a_n) \in (\mathbb{Z})^{\oplus n}$, if there exists a $(n-1)$-simplex Θ with vertices $\{(c_{i1} - 1, \ldots, c_{in} - 1) | 1 \leq i \leq n\}$ which satisfies (1),(2) and (3) above, then there exists a weight system $W_b = (b_1, \ldots, b_n; h)$ such that C is a weighted magic square of weight $(W_a; W_b)$. If W_b is reduced, then it is a dual weight system.

Proof. The assertions (1) and (3) are trivial. The remaining (2) is equivalent to that the origin sits inside of the n-simplex Δ which is the cone over Θ with apex $(-1, \ldots, -1)$. This is equivalent to that all the coefficients of the linear relation between the vertices of Δ has a same sign. Since we can take nothing but b_i's as coefficients: ${b_0}^t(-1, \ldots, -1) + \sum_{i=1}^n {b_i}^t(c_{i1} - 1, \ldots, c_{in} - 1) = 0$, (2) follows.

We will show the converse. From the argument above, b_i's are determined as positive integers up to ratio. We will take the reduced b_i's at first. We define $k := \sum_{i=0}^n b_i$ and $W_b := (b_1, \ldots, b_n; k)$. Then C is a weighted magic square of weight $(W_a; W_b)$. We note that $|\det C|$ is a multiple of k, and by (3), $|\det C| = h$. Thus multiplying some integer to b_i's, we can achieve that $h = k$. In particular, if this new W_b is reduced then C is primitive since $k = |\det C|$. $\square$

Definition 2.3.14. A weight system W is *self-dual* if W and W are dual.

Remark 2.3.15. If one can take a symmetric matrix as C, then W is self-dual. In general, there may be several dual weights as shown in the Proposition below. Similarly, self-duality does not mean that it is the only dual weight. They are determined completely by a finite check.

2.4. Examples.

First we show examples of case $a_0 = -1$, which is hopefully to be generalized as a mirror of log Calabi-Yau manifolds.

Proposition 2.4.1. *$A_{l-1} = \{(1, k, l-k; l),\ 1 \leq k \leq l/2)\}$ $(l \geq 2)$ are closed under duality for each l. $D_{l+1} = (2, l-1, l; 2l)$ $(l \geq 3)$, $E_6 = (3, 4, 6; 12)$, $E_7 = (4, 6, 9; 18)$ and $E_8 = (6, 10, 15; 30)$ are self-dual.*

Proof. Case A_{l-1}.

For an arbitrary positive integer j such that $jk < l$, we can take C as $\begin{pmatrix} l - jk & j & 0 \\ k & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then the dual weight is $(1, j, l-j; l)$. These are the only choice for C up to permutation.

For $D_{l+1}(l$:odd), $D_{l+1}(l$:even), E_6, E_7 and E_8, take the following matrices as C:

$$\begin{pmatrix} (l+1)/2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} l/2 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

They are all symmetric and unique up to the permutation of the rows. They are strongly dual. □

Next, we will treat the case of the exceptional unimodal singularities[1]. They are all weighted homogeneous with $a_0 = 1$.

Theorem 2.4.2. *Let $W = (a_1, a_2, a_3; h)$ be a weight system which corresponds to one of the 14 exceptional unimodal singularities. Then W has a unique dual weight system W^* up to equivalence. And W^* is the weight system which corresponds to Arnold's strange duality.*

Proof. We list below the choice of C. These are, in fact, unique choice for primitive C up to permutation. They are automatically strongly dual.

Let us take coordinate as x, y and z. In the column of C, each monomial $x^\alpha y^\beta z^\gamma$ means a row vector (α, β, γ) of C. For example, in the case E_{14}, $C = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, which is nothing but the transposed matrix ${}^tC$ for Q_{10}.

We list also C_0, which shows the vertices of the maximum Newton polytope of polynomials of degree h of x, y and z, and the Gabrielov numbers and the Dolgachev numbers[1].

Table of exceptional unimodal isolated hypersurface singularities.

class	a_1 a_2 a_3	h	C_0	C	Gab	Dol
E_{12}	6 14 21	42	x^7 y^3 z^2	x^7 y^3 z^2	2 3 7	2 3 7
E_{13}	4 10 15	30	x^5y y^3 z^2	x^5y y^3 z^2	2 3 8	2 4 5
Z_{11}	6 8 15	30	x^5 xy^3 z^2	x^5 xy^3 z^2	2 4 5	2 3 8
E_{14}	3 8 12	24	x^8 y^3 z^2	x^4z y^3 z^2	2 3 9	3 3 4
Q_{10}	6 8 9	24	x^4 y^3 xz^2	x^4 y^3 xz^2	3 3 4	2 3 9
Z_{12}	4 6 11	22	x^4y xy^3 z^2	x^4y xy^3 z^2	2 4 6	2 4 6
W_{12}	4 5 10	20	x^5 y^4 z^2	x^5 z^2 y^2z	2 5 5	2 5 5
Z_{13}	3 5 9	18	x^6 xy^3 z^2	x^3z xy^3 z^2	2 4 7	3 3 5
Q_{11}	4 6 7	18	x^3y y^3 xz^2	x^3y y^3 xz^2	3 3 5	2 4 7
W_{13}	3 4 8	16	x^4y y^4 z^2	x^4y z^2 y^2z	2 5 6	3 4 4
S_{11}	4 5 6	16	x^4 xz^2 y^2z	x^4 xz^2 y^2z	3 4 4	2 5 6
Q_{12}	3 5 6	15	x^5 y^3 xz^2	x^3z y^3 xz^2	3 3 6	3 3 6
S_{12}	3 4 5	13	x^3y xz^2 y^2z	x^3y xz^2 y^2z	3 4 5	3 4 5
U_{12}	3 4 4	12	x^4 y^3 z^3	x^4 y^2z yz^2	4 4 4	4 4 4

□

Remark 2.4.3. K. Saito defines a duality of weights using a duality of poset diagrams coming from the eigenvalues of the monodromy of minimally elliptic singularities[34]. His duality also reproduces Arnold's strange duality and he also computed the dual weights for the 49 weight systems corresponding to the minimally elliptic singularities which are not simple elliptic. His results include the uniqueness of duality for isolated singularities.

As shown below, our duality for $n = 3$ coincides with Saito's in the case (0) the fourteen unimodal singularities and (3) $a_0 \geq 2$. For cases (2) $a_0 = 1$ and modality m is more than one, our duality gives new dual weights, where nonisolated singularities appear as dual partners of isolated singularities. These of course also correspond to the polar duality, as shown in Chapter 5.

It is natural to treat the simple elliptic cases (1) for $n = 2$, since these correspond to weighted elliptic curves. Their list is the same as that of the weighted 2-spaces which correspond to reflexive polytopes. The weights are self-dual for $n = 2$.

Notice that the singularities of dual weights are not dual itself as Arnold's strange duality but its (explicit) specializations enjoy the duality as shown in the later Chapters.

Proposition 2.4.4.

(1) $(1,1;3)$, $(1,2;4)$, $(2,3;6)$ *are self-dual.*
(2) $(2,2,3;8)$, $(2,2,5;10)$, $(2,3,4;10)$, $(2,4,7,14)$ *and* $(2,6,9,18)$ *are self-strongly dual;* $(2,3,6;12)$ *and* $(2,4,5;12)$ *are strongly dual.* $(2,3,3;9)$ *has no strongly dual weight.*
(3) *Duality of weights for minimally elliptic weight systems with* $a_0 > 1$ *coincides with Saito's duality*[34].

Proof. Here we give the full list of primitive C up to permutation. The notation of classes is after [1] and [19].

(1) Simple elliptic singularities.

cl.	C		a_1 a_2	h	dual
P_8	x^2y	xy^2	1 1	3	1 1
X_9	y^2	x^2y	1 2	4	1 2
J_{10}	x^3	y^2	2 3	6	2 3

(2) Minimal elliptic singularities with $a_0 = 1$ and $m > 1$.

cl.			C			a_1 a_2 a_3	h	dual
$V_{1,0}$	=	V_{15}	xz^2	yz^2	x^2y^2	2 2 3	8	2 2 3
			x^2y^2	xy^3	xz^2		*	1 2 4
		N_{16}	x^3y^2	x^2y^3	z^2	2 2 5	10	2 2 5
$U_{1,0}$	=	U_{14}		-		2 3 3	9	-
			y^2z	x^3y	yz^2		*	1 3 4
$S_{1,0}$	=	S_{14}	y^2z	x^2y^2	xz^2	2 3 4	10	2 3 4
			xz^2	x^3z	y^2z		*	1 3 5
$W_{1,0}$	=	W_{15}	y^2z	x^3y^2	z^2	2 3 6	12	2 4 5
			z^2	x^3z	y^2z		*	1 4 6
$Q_{2,0}$	=	Q_{14}	y^3	x^2y^2	xz^2	2 4 5	12	2 3 6
$Z_{1,0}$	=	Z_{15}	xy^3	x^3y^2	z^2	2 4 7	14	2 4 7
$J_{3,0}$	=	J_{16}	y^3	x^3y^2	z^2	2 6 9	18	2 6 9

The * signifies that the dual weight is not strongly dual, hence the singularity corresponding to the smaller triangle is reducible. Other weight systems are strongly dual.

(3) Minimal elliptic singularities with $a_0 > 1$.

cl.	C			a_1	a_2	a_3	h	dual		
V'_{18}	-			3	3	4	12	-		
U_{16}	x^5	y^2z	yz^2	3	5	5	15	3	5	5
S_{16}	x^4y	xz^2	y^2z	3	5	7	17	3	5	7
W_{17}	x^5y	z^2	y^2z	3	5	10	20	-	-	-
Q_{16}	x^4z	y^3	xz^2	3	7	9	21	3	7	9
Z_{17}	x^4z	xy^3	z^2	3	7	12	24	-	-	-
E_{18}	x^5z	y^3	z^2	3	10	15	30	-	-	-
${}_2V^*_{18}$	xy^3	x^3z	yz^2	4	5	7	19	4	5	7
${}_1V^*_{18}$	x^3z	y^4	xz^2	4	5	8	20	4	5	8
N_{19}	x^3z	xy^4	z^2	4	5	12	24	-	-	-
S_{17}	x^6	xz^2	y^2z	4	7	10	24	-	-	-
W_{18}	x^7	y^2z	z^2	4	7	14	28	4	7	14
Q_{17}	x^5y	y^3	xz^2	4	10	13	30	-	-	-
Z_{18}	x^6y	xy^3	z^2	4	10	17	34	4	10	17
E_{19}	x^7y	y^3	z^2	4	14	21	42	-	-	-
${}_3V^*_{19}$	y^4	x^3z	yz^2	5	6	9	24	-	-	-
${}_2N_{20}$	y^5	x^3z	z^2	5	6	15	30	-	-	-
V'_{20}	x^3z	z^3	xy^3	6	7	9	27	6	7	9
${}_2V^*_{19}$	x^5	xy^3	yz^2	6	8	11	30	-	-	-
${}_1V^*_{19}$	x^4y	y^4	xz^2	6	8	13	32	4	7	16
${}_1N_{20}$	x^5y	xy^4	z^2	6	8	19	38	6	8	19
Q_{18}	x^8	y^3	xz^2	6	16	21	48	3	16	24
Z_{19}	x^9	xy^3	z^2	6	16	27	54	4	18	27
E_{20}	x^{11}	y^3	z^2	6	22	33	66	6	22	33
V'_{21}	z^3	y^4	x^3z	8	9	12	36	8	9	12
V^*_{20}	y^4	x^5	yz^2	8	10	15	40	5	8	20
N_{21}	y^5	x^5y	z^2	8	10	25	50	8	10	25

□

Remark 2.4.5. The four weight systems in (3): $(6,8,13;32)$, $(6,16,21;48)$, $(6,16,27;54)$ and $(8,10,15;40)$ are not in the list of Reid[28][17], hence do not correspond to K3 surface with cyclic quotient singularities.

For the weights with some C but without a dual weight (e.g. W_{17}), C does not satisfy primitivity for the dual weight. On the contrary, they enjoy a dual correspondence defined in Chapter 1, except $(5,6,15;30)$.

All the dual weights are strongly dual.

Remark 2.4.6. There are 41 weight systems with $a_0 = 1$ in Reid's list and 19 of them are not minimally elliptic. A similar calculation shows that all the duality for them are as follows; $(1,1,1;4)$, $(1,1,2;5)$, $(1,2,2;6)$, $(1,2,3;7)$ and $(1,2,4;8)$ are self-dual; $(1,1,3;6)$ and $(1,2,2;6)$ are dual; $(1,2,4;8)$, $(1,3,4;9)$, $(1,3,5;10)$ and $(1,4,6;12)$ have dual weight systems which are minimally elliptic as in Proposition 2.4.4 (2); and all the ten weight systems with $a_1 > 1$ do not have dual weight systems.

3. Relation of duality of weight systems and polar duality.

We shall show in this chapter that the duality of weight for $a_0 = 1$ is a special case of dual correspondence defined in Chapter 1. We will always choose one coordinate for the compactification parameter.

Let $W_a = (a_1, \ldots, a_n; h)$ be a weight system with $h \in \sum_{i=1}^n \mathbb{N}a_i$ and $a_0 := h - \sum_{i=1}^n a_i$ as usual. We assume $a_0 > 0$ in this section.

Let $\mathbb{P}(a)$ be the weighted projective space of weight $(a_0, a_1, \ldots, a_n)$ and $X_0, \ldots, X_n$ be the natural homogeneous coordinates of $\mathbb{P}(a)$ induced by the definition. Then the n-dimensional complex torus $\mathbb{T}$ is embedded equivariantly in $\mathbb{P}(a)$ as the locus $\{F_0 := \prod_{i=0}^n X_i \neq 0\}$.

Let $M(W_a)$ be the group defined in Section 3. $M(W_a)$ is nothing but the abelian group of exponents of rational monomials, which appears in the context of toric construction. For a point $P = (\alpha_1, \ldots, \alpha_n)$ on $M(W_a)$, we write the corresponding monic monomial by $F_P = \prod_{i=0}^n X_i^{\alpha_i+1}$, where α_0 is determined by $\sum_{i=0}^n a_i\alpha_i = 0$.

Definition 3.1. Let $\overline{\Delta}(a)$ be the n-simplex in $M(W_a)_{\mathbb{Q}}$ which is the convex hull of the following vertices: $(-1+h/a_1, -1, \ldots, -1), \ldots, (-1, \ldots, -1, -1+h/a_n)$ and $(-1, \ldots, -1)$.

$(\overline{\Delta}, \overline{M}) = (\overline{\Delta}(a), M(W_a))$ is an element of $\mathcal{P}_n$ with $\mathbb{P}_{\overline{\Delta},\overline{M}} \cong \mathbb{P}(a)$.

Lemma 3.2. *Let $(\overline{\Delta}, M)$ be as above. Then its dual $(\overline{\Delta}^*, M^*)$ is integral, homothetic to $\overline{\Delta}$ and the vertices generate M^*.*

Proof. For the sake of simplicity, we assume that W_a is reduced. Recall that $M = \{(\alpha_1, \ldots, \alpha_n) | \sum_{i=1}^n a_i\alpha_i \equiv 0 \mod a_0\}$. Let $M' := \mathbb{Z}^{\oplus n}$, which contains M as a $\mathbb{Z}$-submodule of index a_0. By simple computation, the dual of $(\overline{\Delta}, M')$ is an n-simplex with vertices $v_1 := (1, 0, \ldots, 0), \ldots, v_n := (0, \ldots, 0, 1)$ and $v_0 := (-a_1/a_0, \ldots, -a_n/a_0)$ in $(M')^*_{\mathbb{Q}}$, with a relation $\sum_i a_i v_i = 0$ which shows the required homothety. Thus $v_1, \ldots, v_n$ generates $(M')^*$. Since $(M')^*$ is a submodule of M^*, Those n vertices are also contained in M^*. On the other hand, v_0 also belongs to M^* since $\sum_{i=1}^n (-a_i/a_0)\alpha_i \in \mathbb{Z}$ for any $(\alpha_1, \ldots, \alpha_n) \in M$.

The order of v_0 in $M^*/(M')^*$ is exactly a_0 since $(a_1, \ldots, a_n)$ is reduced. Thus the last statement follows. □

Remark 3.3. On the contrary, an n-simplex satisfying the conclusion of Lemma 3.2 is isomorphic to $\overline{\Delta}^*$.

By the previous proof, the submodule $(M')^*$ generated by $v_1, \ldots, v_n$ has index a_0 in M.

Let $(\overline{\nabla}, \overline{N}) = (\overline{\Delta}(b), M(W_b))$ be another pair. We assume W_b is also reduced for the sake of simplicity.

We restrict here the correspondence σ in Chapter 1 as follows: (*)

(1) σ sends the vertex of $(\overline{\Delta})^*$ which corresponds to the divisor $\{X_0 = 0\}$ to the vertex of $\overline{\nabla}$ which corresponds to the point $(1 : 0 : \cdots : 0)$ (i.e., $X_1 = \cdots = X_n = 0$).
(2) same for ${}^t\sigma$.

Proposition 3.4. *Assume that $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ has a dual correspondence σ which satisfies the condition (*) above.*

Let $(c_{i1} - 1, \ldots, c_{in} - 1)$ $(1 \le i \le n)$ be the vertices except $(-1, \ldots, -1)$ of $\sigma(\overline{\nabla}^*)$ in $\overline{M}$. Let B be the $n \times n$-matrix $(c_{ij} - 1)$.

Then

(1) *$a_0|h$, $b_0|h$ and $|\det B| = a_0 b_0$,*
(2) *If $a_0 = b_0 = 1$, W_a and W_b are dual weight systems.*

Proof. (1) By $\sigma(\overline{N}^*) = \overline{M}$ and $\sigma(-b_1/b_0, \ldots, -b_n/b_0) = (-1, \ldots, -1)$, $(-1, \ldots, -1) \in \overline{M}$. Thus $a_0|h$. Similarly for $b_0|h$. Since $\sigma(1, 0, \ldots, 0), \ldots, \sigma(0, \ldots, 0, 1)$ generates a submodule of index b_0 in $\overline{M}$, $|\det B| = a_0 b_0$.

(2) Let C be (c_{ij}). By condition (*), C is a weighted magic square of weight $(W_a; W_b)$ and primitive by $|\det B| = 1$ using Lemma 2.3.5. □

Thus we have

Theorem 3.5. *Assume two weight systems W_a and W_b with $a_0 = b_0 = 1$ are dual. Then*

(1) *there exists an identification $M = (\overline{N})^* = \overline{M}$ such that $\overline{\nabla}^*$ is an integral simplex and a subcone of $\overline{\Delta}$ with a common apex $(-1, \ldots, -1)$ and a common plane $\pi = \{\sum_{i=1}^n a_i \alpha_i = a_0\}$ of facet. The vertices of $\overline{\nabla}^*$ on π generates M.*
Similarly for the induced identification $N = (\overline{M})^* = \overline{N}$.

(2) *If a linear subsystem of anticanonical divisor of $\mathbb{P}(a)$ corresponds to some reflexive polytope Δ such that $\overline{\nabla}^* \subset \Delta \subset \overline{\Delta}$, then the family corresponds to Δ^* is a linear subsystem of anticanonical divisor of $\mathbb{P}(b)$. This relation is inclusion-reversing with respect to polytopes.*

Remark 3.6. The facet above is the Newton polytope for an $(n-1)$-dimensional hypersurface singularity X_0 with $\mathbb{C}^*$-action. $\bar{Z}$ is the toric compactification in $\mathbb{P}(a)$ of a deformation Z of X_0.

Remark 3.7. Let Δ be the *full Newton polytope* of $\mathbb{P}(a)$, that is the convex hull of all vertices corresponding to the monomial anticanonical divisors. We have the associated projective variety $\mathbb{P}_\Delta$ as usual. In general, Δ does not contain all the generators of the ring of regular functions of $\mathbb{P}(a)$. Thus $\mathbb{P}(a)$ and $\mathbb{P}_\Delta$ are not isomorphic in general.

Remark 3.8. By [5 Theorem 5.4.5, Corollary 5.4.6], for a reflexive simplex $\Delta = \overline{\Delta}$, that is, for the case $a_i | h$ for all i, $\mathbb{P}_\Delta$ is a weighted projective variety whose weight is the coefficients $(a_0, \ldots, a_n)$ of the unique linear relation $\sum_{i=0}^n a_i P_i = 0$ ($a_i \in \mathbb{Z}_+$, $\gcd(a_0, \ldots, a_n) = 1$) among the vertices $\{P_i\}$. Moreover, the polar dual family of deformation of Fermat-type hypersurfaces is the quotient of a subfamily by $\pi_1(\Delta, M)$ [5 Corollary 5.5.6].

Remark 3.9. Borcea's construction[8] does not assume the property (*) which is coming from singularity theory. As he has pointed out, for a given reflexive polytope, its realization as a family of weighted hypersurface has some choice. Nevertheless, for the fourteen exceptional singularities, our duality of weights is unique and compatible with that of usual singularity theory.

4. Application for $n = 3$.

4.1. K3 surfaces.

We refer the reader [6] for more complete reference about K3 surfaces and [3] for general arguments about string theory on K3 surfaces.

Let X be K3 surface, that is, a compact complex surface with trivial canonical bundle and irregularity $h^1(\mathcal{O}_X) = 0$. It is well known that any K3 surface is a simply connected, Kähler and symplectic manifold. The only nonzero Hodge numbers of X are $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$ and $h^{1,1} = 20$.

First we recall here some notions about lattices.

Definition 4.1.1. Let L be a free $\mathbb{Z}$-module of finite rank with a symmetric bilinear $\mathbb{Z}$-valued pairing $(\ ,\ )$. Such an L is called a *lattice*. L is *even* if for each $x \in L$, (x, x) is an even integer, and *unimodular* if the discriminant is ± 1. The *index* of L is the triplet $(\lambda_+, \lambda_0, \lambda_-)$ of the numbers of positive, null and negative eigenvalues of the corresponding matrix. L is said to be *positive definite*, *negative definite*, *definite*, *indefinite*, respectively, if the corresponding matrix is so.

Since $\pi_1(X) = \{1\}$, we can identify $H^2(X,\mathbb{Z})$ and its natural image in $H^2(X,\mathbb{C})$. $H^2(X,\mathbb{Z})$ is naturally regarded as a lattice by the cup product, and it is an even unimodular lattice of index $(3,0,-19)$, so is isomorphic to $L := (-E_8)^{\oplus 2} \oplus U^{\oplus 3}$. Here, $-E_8$ is the even unimodular negative definite lattice of rank 8, whose matrix is (-1) times that of the lattice corresponding to the Dynkin diagram E_8. U is the even unimodular indefinite lattice of rank 2, whose matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In general, the algebraic lattice S_X and the transcendental lattice T_X of a compact complex surface X are defined as sublattices in (the image of) $H^2(X,\mathbb{Z})$ as $S_X := H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ and $T_X := S_X^{\perp}$ in $H^2(X,\mathbb{Z})$. Here $\perp$ denotes the orthogonal complement. The rank of S_X is called the *Picard number* of X and denoted by $\rho(X)$.

Ruan and Tian have constructed the *quantum cohomology ring* for semipositive symplectic manifolds in [30]. As a graded module, the quantum cohomology ring is same as the classical cohomology ring, but in general, the multiplicative structures are different. The mirror map μ is usually thought to be an involution of moduli of Calabi-Yau manifolds with Kähler structure which induces an isomorphism between the cohomology groups of the Calabi-Yau manifolds exchanged by μ, interchanging so-called the A-model intersection number described by the quantum cohomology ring and the B-model intersection number coming from Gauss-Manin connection.

The following is an easy modification of [30 Example 8.4].

Proposition 4.1.2. *Let X be a compact Kähler surface with trivial canonical bundle. Then the quantum cohomology ring of X is isomorphic to the classical cohomology ring of X.*

Thus for a K3 surface or a 2-dimensional complex torus, the A-model intersection number is the classical intersection product. On the other hand, B-model intersection number for those surfaces are also the intersection product by Hodge theory. Thus we may assume that a mirror map preserves the usual cup product.

4.2. Fixed set of the mirror map.

First we shall prove some finiteness result of a special automorphism group of marked K3 surface.

Theorem 4.2.1. *Let X be a K3 surface with a Kähler class ω, L be the second integral cohomology lattice $H^2(X,\mathbb{Z})$ and G be a subgroup of $\mathrm{Aut}X$. Assume that the induced action of G on L fixes the cohomology class of a nonzero holomorphic 2-form and ω. Then*

(1) *The fixed lattice L^G contains T_X.*
(2) *The orthogonal complement of $(L^G)^{\perp}$ in L is a negative definite sublattice of S_X.*
(3) *G is a finite group.*

Remark 4.2.2. If the algebraic dimension $a(X)$ of X is not one, it is essentially proved in [24].

Remark 4.2.3. ω can be assumed to be complexified Kähler class, since the action of automorphism preserves the real structure of L.

First we note the following elementary fact.

Lemma 4.2.4. *Let L be a lattice and K be a subset of L. Let ϕ be an isometry on L.*

(1) *If $\phi(K) = K$ then $\phi(K^\perp) = K^\perp$.*

(2) *If ϕ acts as identity on K, then for any $x \in L$, $\phi(x) - x \in K^\perp$.*

Corollary 4.2.5. *Let X be a compact Kähler surface and $g \in \mathrm{Aut}X$. Then $g^*(S_X) = S_X$ and $g^*(T_X) = T_X$.*

Proof of Theorem. First note that the kernel of the multiplication of a nonzero holomorphic two form Ω on $H^2(X, \mathbb{R})$ is $H^{1,1} \cap H^2(X, \mathbb{R})$.

Take an arbitrary element $x \in T_X$ and $g \in G$. Then $y := g^*x - x \in T_X$. By the previous lemma, $y \in T_X \cap \Omega^\perp \cap \omega^\perp = T_X \cap S_X \cap \omega^\perp$. Since $y \in T_X \cap S_X$, we have $(y, y) = 0$. On the other hand, since $S_X \cap \omega^\perp$ is negative definite by the signature theorem, one has $y = 0$. Thus $g^*x = x$. This proves (1).

(2) follows from $(L^G)^\perp \subset (T_X)^\perp \cap \Omega^\perp \cap \omega^\perp = S_X \cap \omega^\perp$.

Next we shall prove (3).

Since $(L^G)^\perp$ is negative definite and finitely generated, the isometry group $O((L^G)^\perp)$ is a finite group.

Let $j : G \to O((L^G)^\perp)$ be the natural group homomorphism. We claim that j is injective.

Suppose the induced action of an element $g \in G$ on $(L^G)^\perp$ is identity. For an arbitrary $x \in L$, $y := g^*x - x$ belongs to $(L^G)^\perp$. Thus $g^*y = y$. Then for an integer n, we have $(g^n)^*x = x + ny$ and since g^* is isometry, $((g^n)^*x, (g^n)^*x) = (x, x)$. This leads to $y = 0$ by definiteness of $(L^G)^\perp$. Thus g^* is the identity on L.

Since g^* is, by definition, an effective Hodge isometry on L, g is the identity element in $\mathrm{Aut}X$ by Torelli's theorem. □

Remark 4.2.6. The structure of G can be found in the list of [24] once one knows that G is a finite algebraic automorphism group. Also, $(L^G)^\perp$ does not contain square (-2)-elements.

Corollary 4.2.7. *$L^G + (L^G)^\perp$ is a sublattice in L of finite index and $L_{\mathbb{Q}} = (L^G)_{\mathbb{Q}} \oplus (L^G)^\perp_{\mathbb{Q}}$.*

Proof. For any $x \in L$, there is an orthogonal decomposition whose coefficients are in $\frac{1}{|G|}\mathbb{Z}$:

$$x = \frac{1}{|G|}(x_1 \oplus x_2),\ x_1 = \sum_{g \in G} g^*x \in (L^G),\ x_2 = \sum_{g \in G}(x - g^*x) \in (L^G)^\perp. \quad \square$$

Take an arbitrary $h \in O((L^G)^\perp)$. For arbitrary element x in L, we have a decomposition $x = x_1 + x_2$ in $L_{\mathbb{Q}}$, where $x_1 \in L^G$ and $x_2 \in (L^G)^\perp$. We define an action of $O((L^G)^\perp)$ on $L_{\mathbb{Q}}$ by $h(x) := x_1 + h(x_2)$.

Proposition 4.2.8. *If $h(L) = L$ holds, there exists a unique $g \in \mathrm{Aut}X$ which satisfies: $g^*\omega = \omega$, $g^*\Omega = \Omega$ and $h = g^*$.*

Proof. Since ω and Ω belong to $(L^G)_{\mathbb{C}}$, h is effective Hodge isometry on L. Thus the result follows from Torelli's theorem.

Remark 4.2.9. For an arbitrary choice of ω and Ω in period domain, there exists a unique K3 surface X with complexified Kähler class ω by the surjectivity of period map. The

marking is unique up to the finite automorphism group of X fixing ω, since G is canonically isomorphic to the subgroup of $O(L)$ which fixes ω and Ω.

Next we define a sublattice $\langle\omega\rangle$ such that ω is 'general' in $(\langle\omega\rangle)_{\mathbb{C}}$.

Definition 4.2.10. For a fixed $\alpha \in L_{\mathbb{C}}$, we denote by $\langle\alpha\rangle$ the minimum primitive sublattice of L such that $(\langle\alpha\rangle)_{\mathbb{C}}$ contains α.

Remark 4.2.11. In fact, $\langle\alpha\rangle$ is determined by the set of all rational equalities which the coefficients of α satisfy, for an appropriate basis of L. When $\alpha \in L$, $\langle\alpha\rangle$ is nothing but $\mathbb{Z}\alpha$. This definition works for an arbitrary torsion-free $\mathbb{Z}$-module L and its extension of coefficients by any field of characteristic 0.

$\langle\omega\rangle$ is independent of the choice of the complex structure X.

Lemma 4.2.12.

(1) $\langle\alpha+\beta\rangle \subset \langle\alpha\rangle + \langle\beta\rangle$.
(2) $\langle c\alpha\rangle = \langle\alpha\rangle$ *for* $c \in \mathbb{C}^*$.
(3) $\langle\alpha + i\beta\rangle = \langle\alpha, \beta\rangle$ *for* $\alpha, \beta \in \mathbb{R}$.
(4) *Let* $\{x_i\}$ *be a basis of* $\langle\alpha\rangle$. *If* $\alpha = \sum_i a_i x_i$ $(a_i \in \mathbb{C})$ *then* a_i*'s are linearly independent over* $\mathbb{Q}$.
(5) $\alpha^{\perp}$ *coincides with* $\langle\alpha\rangle^{\perp}$ *in* L.

Proof. Immediate from the definition.

Proposition 4.2.13. $\alpha \in L^G$ *if and only if* $\langle\alpha\rangle$ *is contained in* L^G.

Proof. Take a $\mathbb{Z}$-basis $\{x_i\}$ of $\langle\alpha\rangle$ and write α as an $\mathbb{R}$-linear combination $\sum_i a_i x_i$. From the minimality of $\langle\alpha\rangle$, a_i's are linearly independent over $\mathbb{Q}$. Let g be an element of G. Then we have $\alpha = \sum_i a_i g^* x_i$. The lattice generated by $\{g^* x_i\}$ is also primitive, thus it coincides with $\langle\alpha\rangle$ from the minimality. We can write $g^* x_i$ as a $\mathbb{Z}$-combination $\sum_j b_{ij} x_j$. From the equality $\sum_i a_i g^* x_i = \alpha = \sum_j a_j x_j$ and torsion-freeness of L, we get $b_{ij} = \delta_{ij}$. Thus $g^* x_i = x_i$ and we are done.

Remark 4.2.14. Thus we have $L^G \supset \langle\omega, \Omega\rangle$. By adding ω a small general element in $V := (L^G)_{\mathbb{C}} \cap ((\langle\omega, \Omega\rangle)^{\perp})_{\mathbb{C}}$, one can arrange as $L^G = \langle\omega, \Omega\rangle$ holds, since V is rationally defined and $V \cap L$ is negative definite by index theorem.

Remark 4.2.15. The lattice L_D in the next section is nothing but $\langle\omega\rangle$ when ω is the restriction of the cohomology class of a 'general' ample $\mathbb{R}$-divisor of an embedded variety, and the lattice L_0 is $\langle\omega\rangle^{\perp} \cap \langle\Omega\rangle^{\perp}$ in L, and in fact, $(L^G)^{\perp}$ in L.

When ω is general in S_X, $\langle\omega\rangle = S_X$ and $L_0 = 0$.

4.3. Mirror symmetry for K3 toric hypersurfaces.

In this section, we will treat the algebraic K3 surfaces which are the minimal resolutions of toric hypersurfaces.

Definition 4.3.1. A *polyhedron* is a polytope of dimension three.

For a given reflexive polyhedron Δ, we use Θ and Γ for 2-and 1-dimensional faces of Δ. The dual face Γ^* of Γ is a 1-dimensional face of Δ^*, so the summation over Γ is same as that over Γ^*.

We denote by Z the intersection of a Δ-regular anticanonical section of $\mathbb{P}_\Delta$ and the n-dimensional orbit of the torus, and by $\bar{Z}$ its closure, namely the section itself. $\bar{Z}$ has only canonical Gorenstein singularities by Theorem 1.1.8 due to Batyrev. Let $\tilde{Z}$ be the minimal resolution of $\bar{Z}$. $X = \tilde{Z}$ is an algebraic K3 surface.

We define the following sublattices of $H^2(X, \mathbb{Z})$ for toric hypersurface K3 surface X.

Definition 4.3.2.

(1) $L_D := \mathrm{Im}(H^{1,1}(\tilde{\mathbb{P}}_\Delta) \to H^{1,1}(X)) \cap H^2(X, \mathbb{Z})$,
(2) $L_0 := L_D^\perp$ in S_X,

where $\tilde{\mathbb{P}}_\Delta$ is a toric resolution of $\mathbb{P}_\Delta$.

From a result of [12][4][2], we get

Proposition 4.3.3. *Let Δ be three-dimensional reflexive polytope and $\tilde{Z}$ be the K3 surface which is the minimal resolution of a Δ-regular anticanonical section of $\mathbb{P}_\Delta$. Then*

(1) $L_D(\tilde{Z}) = \mathbb{Z}[\Delta^{*[1]} \cap N]/N$,
(2) $L_0(\tilde{Z}) = \bigoplus_{\dim \Gamma = 1} \mathbb{Z}(\mathrm{Int}\,\Gamma \cap M) \times (\mathrm{Int}\,\Gamma^* \cap N)$,
(3) $\mathrm{rk}\, T_{\tilde{Z}} - 2 = l(\Delta^{[1]}) - 3$.

Especially,

(4) $\mathrm{rk}\, L_D(\tilde{Z}) = l(\Delta^{*[1]}) - 3$,
(5) $\mathrm{rk}\, L_0(\tilde{Z}) = \sum_{\dim \Gamma = 1} l^*(\Gamma^*) l^*(\Gamma)$.

Proof. By definition we can follow the same argument of calculation of $H^{1,1}_{toric}$ in [2], one can show that $L_D(\tilde{Z}) = \mathbb{Z}[\Delta^{*[1]} \cap N]/N$ and in particular, $\mathrm{rk}\, L_D(\tilde{Z}) = l(\Delta^{*[1]}) - 3$.

By [12 5.11], we have $\dim H^2(Z) = l(\Delta^{[1]}) - 1$, which is generated by vanishing cycles whose image is nothing but $\mathrm{rk}\, T_{\tilde{Z}}$.

The toric compactification $\mathbb{T} \to \mathbb{P}_\Delta$ adds a divisor for each vertice of Δ^*.

A toric resolution of the ambient space produces the divisors corresponding to the integral points P of $\Delta^* - (\Delta^{*(0)} \cup \{0\})$. We denote by C_P the support of the center of blowing up in $\mathbb{P}_\Delta$ corresponding to P. If P is in some interior of a 2-dimensional face of Δ^*, C_P is a point which is disjoint from $\bar{Z}$, since $\bar{Z}$ is Δ-regular.

If P is in some $\mathrm{Int}\,\Gamma^*$, C_P is a curve in $\mathbb{P}_\Delta$ which intersects normally with $\bar{Z}$ at $(l^*(\Gamma)+1)$ points. Since each toric blowing-up corresponding to P creates one ambient divisor, its orthogonal complement increases L_0 by $l^*(\Gamma)$ divisors. Thus we have the description of L_0 and $\mathrm{rk}\, L_0(\tilde{Z}) = \sum_\Gamma l^*(\Gamma) l^*(\Gamma^*)$. □

Remark 4.3.4. In fact, we can easily determine the lattices L_D and L_0 from the data (Δ, M). For instance, the dual graph of L_D is obtained from $\Delta^{*[1]}$ by removing three integral points which consist a basis of N. Each remaining vertex is a (-2)-curve and each remaining integral point on a edge Γ is a sum of $(l^*(\Gamma^*) + 1)$ disjoint (-2)-curves. Each edge in Γ joining integral points shows the intersection number of the two end-points is $(l^*(\Gamma^*) + 1)$.

We conjecture that the same rule of intersection product holds for L_G and $\Delta^{[1]}$.

Corollary 4.3.5. *For a fixed reflexive polyhedron Δ, $\mathrm{rk}\, T_{\tilde{Z}}$, $L_D(\tilde{Z})$ and $L_0(\tilde{Z})$ are independent of the choice of Δ-regular hypersurface Z.*

We will thus write them as $L_D(\Delta)$ and $L_0(\Delta)$, respectively.

Corollary 4.3.6. $\operatorname{rk} T(\Delta) - 2 = \operatorname{rk} L_D(\Delta^*)$, $L_0(\Delta) = L_0(\Delta^*)$.

Remark 4.3.7. Since the rank of H^2 is 22, $\operatorname{rk} T(\Delta) + \operatorname{rk} L_D(\Delta) + \operatorname{rk} L_0(\Delta) = 22$. That formula gives a relation $\sum_\Gamma (l^*(\Gamma)+1)(l^*(\Gamma^*)+1) = 24$.

We have $\operatorname{rk} T(\Delta) - 2 = l(\Delta^{[1]}) - 3 = \operatorname{rk}(\Delta^{[1]} \cap M)/M$, which is the expected dimension for $H^{1,1}_{poly}$ in [2].

Remark 4.3.8. In the case of Arnold's strange duality, the lattice structure of $L_G(\Delta)$ is isomorphic to that of $L_D(\Delta^*)$.

In general, $L_G \oplus U$ is not isomorphic as a lattice over $\mathbb{Z}$ to the Milnor lattice but it seems so over $\frac{1}{|G|}\mathbb{Z}$. This may be for a future investigation.

Theorem 4.3.9. *Let X_0 be one of the fourteen exceptional unimodal isolated hypersurface singularities and $W_a = (a_1, a_2, a_3; h)$ be the corresponding weight system.*

(1) *There exists a unique dual weight system W_b.*
(2) *W_b coincides with the weight system of the dual singularity in the sense of Arnold.*
(3) *The full Newton polyhedron, namely the convex hull of $\overline{\Delta}(a) \cap M(W_a)$ in $M(W_a)_{\mathbb{Q}}$ is a reflexive polyhedron containing $(\overline{\nabla})^*$ with the dual correspondence above.*
(4) *Let Δ be any reflexive polyhedron in $\overline{\Delta}(a)$ which contains $(\overline{\nabla})^*$. The intersection with Δ and the plane $\sum_{i=1}^3 a_i\alpha_i = 1$ is a facet which determines the equivalent singularity to X_0.*
(5) *Δ^* is a reflexive polyhedron in $\overline{\nabla}$ containing $(\overline{\Delta})^*$.*
(6) *There exists a Δ such that $L_0(\Delta) = 0$.*

Proof. (1)(2)(5) is done in previous chapters. (3) is well-known and (4) is easy case by case check. Here we show an example of choices of Δ with $L_0 = 0$. For dual singularities, these reflexive polytopes are dual to each other. For self-dual singularities, these examples satisfy $\Delta^* \cong \Delta$. These choices of Δ are not unique in general.

class	a_1 a_2 a_3	h	Δ
E_{12}	6 14 21	42	W^{42}, X^7, Y^3, Z^2
E_{13}	4 10 15	30	W^{30}, W^6X^6, X^5Y, Y^3, Z^2
Z_{11}	6 8 15	30	W^{30}, W^6Y^3, X^5, XY^3, Z^2
E_{14}	3 8 12	24	W^{24}, W^6X^6, X^4Z, Y^3, Z^2
Q_{10}	6 8 9	24	W^{24}, W^6Z^2, X^4, Y^3, XZ^2
Z_{12}	4 6 11	22	W^{22}, W^6X^4, W^4Y^3, X^4Y, XY^3, Z^2
W_{12}	4 5 10	20	W^{20}, $W^{10}Y^2$, $W^2X^2Y^2$, X^5, Y^2Z, Z^2
Z_{13}	3 5 9	18	W^{18}, W^6X^4, W^3Y^3, X^3Z, XY^3, Z^2
Q_{11}	4 6 7	18	W^{18}, W^6X^3, W^4Z^2, X^3Y, Y^3, XZ^2
W_{13}	3 4 8	16	W^{16}, W^4X^4, W^4Y^3, X^4Y, Y^4, Z^2
S_{11}	4 5 6	16	W^{16}, W^6Y^2, W^4Z^2, X^4, XZ^2, Y^2Z
Q_{12}	3 5 6	15	W^{15}, W^6X^3, W^3Z^2, X^3Z, XZ^2, Y^3
S_{12}	3 4 5	13	W^{13}, W^4X^3, W^3Z^2, WY^3, X^3Y, XZ^2, Y^2Z
U_{12}	3 4 4	12	W^{12}, W^4Y^2, W^4Z^2, X^4, Y^2Z, YZ^2

□

Remark 4.3.10. One should choose the polyhedron rather carefully, since the restricted Kähler class ω of a general Kähler class of $\tilde{\mathbb{P}}_\Delta$ is sometimes not general in $(S_X)_{\mathbb{R}}$. Namely, there might be some non-zero (non-effective) integral algebraic 2-cocycle α such that $\omega \cup \alpha = 0$. This is where L_0 appears. Thus it is necessary to resort to a subpolytope of the full Newton polytope.

4.4. Examples.

We give some illuminating examples other than exceptional unimodal isolated singularities.

Example 4.4.1. Let us think about a pair of dual weight systems $W_a = (2,3,6;12)$ and $W_b = (2,4,5;12)$. In these cases $a_0 = b_0 = 1$. We will denote the homogeneous coordinates by $W = X_0$, $X = X_1$, $Y = X_2$ and $Z = X_3$ and the inhomogeneous ones by small letters. A polytope is represented by its vertex set (vertices).

The weighted magic square C is $\begin{pmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ whose determinant is (-12). Thus $B = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, $-B^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

We use $(\alpha_1, \ldots, \alpha_n)$ for the coordinates coming from $M(W_a) = \mathbb{Z}^{\oplus n}$ and $[\beta_1, \ldots, \beta_n]$ for those from $M(W_b)$. Then $[\beta_1, \ldots, \beta_n]B = (\alpha_1, \ldots, \alpha_n)$ holds for the vertices in $M_{\mathbb{Q}}$ and the hyperplanes in $N_{\mathbb{Q}}$.

$\overline{\Delta} = \langle W^{12}, X^6, Y^4, Z^2 \rangle$. The coordinates of the vertices are : $(-1,-1,-1)$, $(5,-1,-1)$, $(-1,3,-1)$, $(-1,-1,1)$; $[-2,-3,-5]$, $[-2,2,1]$, $[2,0,-1]$, $[0,0,1]$, respectively. This is a reflexive simplex. In fact, the dual is given by $(\overline{\Delta})^* = \langle W^{12}, Y^3, X^2Y^2, XZ^2 \rangle$.

Let X_0 be the singularity $\{x^6 + y^4 + z^2 = 0\}$ in $\mathbb{C}^3$. This is an isolated hypersurface singularity with Milnor number $\mu = 15$. In the table of Arnold[1], X_0 belongs to class $W_{1,0}$.

$Z = \{x^6 + y^4 + z^2 + 1 = 0\}$ is a smooth surface in $\mathbb{C}^3$ thus a Milnor fiber of X_0.

Ebeling [16] has calculated Milnor lattices of isolated bimodal singularities. The Milnor lattice $H_2(Z)$ is a direct sum of U and the lattice $L_G(\overline{\Delta})$ represented by Fig. 1,

(Fig. 1)

where circles are vanishing cycles.

The resolution graph of $X \to \bar{Z}$ is as in Fig. 2.

(Fig. 2)

where a circle represents a smooth rational curve with self-intersection (-2) and the central curve is $\{W = 0\}$. L_D is generated by the central curve and three (-4)-elements $a+a'$, $b+b'$ and $c+c'$. L_0 is generated by three (-4)-elements $e_1 := a - a'$, $e_2 := b - b'$ and $e_3 := c - c'$. The set of all the (-4)-elements of L_0 are $\{\pm e_1, \pm e_2, \pm e_3, \pm(e_1 + e_2)\}$ and the orthogonal group $O(L_0)$ keeps this set. The elements of $O(L_0)$ which extend to elements of $O(L)$ as in 4.2 are : id, $\sigma(e_1 \mapsto e_1, e_2 \mapsto e_2, e_3 \mapsto -e_3)$, $\tau(e_1 \mapsto -e_1, e_2 \mapsto -e_2, e_3 \mapsto e_3)$ and $\sigma\tau$. Thus $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in fact each isomorphism is realized by a multiplication of

coordinates: $\sigma(W:X:Y:Z) = (iW:X:iY:-Z)$, $\tau(W:X:Y:Z) = (\zeta^2 W:\zeta X:Y:-Z)$ where $i := \exp(\pi i/2)$ and $\zeta := \exp(\pi i/3)$.

Next think about the dual polyhedron $\overline{\Delta}^*$. The singularity $\{y^3 + x^2y^2 + xz^2 = 0\}$ has a singular curve $\{x = 0\}$. We have a Milnor fiber $\{y^3 + x^2y^2 + xz^2 + 1 = 0\}$ and its weighted compactification $\bar{Z} = \{Y^3 + X^2Y^2 + XZ^2 + W^{12} = 0\}$ in $\mathbb{P}(1,2,4,5)$. In this case, the restriction of the divisor $\{X = 0\}$ decomposes as a sum of three divisors $E_i : \{X = Y + \zeta^i W^4 = 0\}$ $(i = 0,1,2)$ where ζ is a nontrivial cubic root of unity. The other intersections with singular 2-dimensional strata of $\mathbb{P}$ are irreducible.

The singular set of $\bar{Z}$ consists of one D_8 at $\{W = Y = Z = 0\}$, one A_4 at $\{W = X = Y = 0\}$ and one A_1 at $\{W = X^2 + Y = Z = 0\}$. The resolution graph at infinity (including the divisor $W = 0$) becomes as in Fig. 3.

(Fig.3)

This graph is one longer than that of the $L_G(\overline{\Delta})$. L_0 is generated by $a - a'$, $E_0 - E_1$ and $E_1 - E_2$. L_D is generated by $a + a'$ and other circles. Change the base as $a + a' \to a' + a + b$ (other circles are same), we have the same graph as the $L_G(\overline{\Delta})$.

Let us take other polyhedra. In our duality, every reflexive polyhedron in $M_{\mathbb{Q}}$ should contain $\overline{\nabla}^* = \langle W^{12}, X^3Y^2, Y^2Z, Z^2 \rangle$, or rather $\nabla^* = \langle W^{12}, Y^2W^6, X^3Y^2, Y^2Z, Z^2 \rangle$, where ∇ is the maximum Newton polyhedron in N, that is the convex hull of $\overline{\nabla} \cap N$.

Take $\Delta = \langle W^{12}, W^3Y^3, W^2X^5, WXY^3, X^3Y^2, X^3Z, Y^2Z, Z^2 \rangle$, which has the same L_G as $\overline{\Delta}$ and $L_0(\Delta) = 0$. Thus Δ represents a Milnor fiber of the singularity of class $W_{1,0}$ and the dual graph of L_D coincides with the resolution graph as above. Its dual Δ^* is $\langle W^{12}, W^7Z, W^6XY, W^5XZ, W^3YZ, X^2Y^2, Y^3, XZ^2 \rangle$.

Example 4.4.2. The weight system $W = (2,3,4;10)$ is self-dual. Take the following six reflexive polyhedra (cf. Fig. 4):

$$\begin{aligned}
\Delta_1 &= \langle W^{10}, W^2X^4, X^3Z, X^2Y^2, XZ^2, Y^2Z, W^2Z^2, WY^3 \rangle, \\
\Delta_2 &= \langle W^{10}, W^4X^3, WX^3Y, X^2Y^2, XZ^2, Y^2Z, W^2Z^2, WY^3 \rangle, \\
\Delta_3 &= \langle W^{10}, W^6X^2, W^2X^2Z, X^2Y^2, XZ^2, Y^2Z, W^2Z^2, WY^3 \rangle, \\
\Delta_4 &= \langle W^{10}, W^6X^2, W^2X^2Z, X^2Y^2, XZ^2, Y^2Z, W^6Z, W^4Y^2 \rangle, \\
\Delta_5 &= \langle W^{10}, W^8X, W^3X^2Y, X^2Y^2, XZ^2, Y^2Z, W^6Z, W^4Y^2 \rangle \text{and} \\
\Delta_6 &= \langle W^{10}, W^4XZ, X^2Y^2, XZ^2, Y^2Z, W^6Z, W^4Y^2 \rangle.
\end{aligned}$$

These polyhedra all satisfy $L_0(\Delta_i) = 0$ and $\Delta_i^* \cong \Delta_{7-i}$. Δ_1 corresponds to the Milnor fiber of the singularity of class $S_{1,0}$.

The dual graphs of $L_D(\Delta_i)$ is as in Fig. 5.

(Fig. 5)

There are inclusions of reflexive polyhedra: $\Delta_i \supset \Delta_j$ for $i < j$ and thus specializations of the families of weighted hypersurfaces. As the family specializes, $2A_1$ singularities at infinity get 'closer' to A_3, D_4, D_5 and finally to D_6 singularities.

Example 4.4.3. For the weight system $W = (2, 6, 9; 18)$, ten reflexive polytopes $\Delta_i = \langle W^{18}, W^{18-2i}X^i, X^3Y^2, Y^3, Z^2 \rangle$ satisfy the relation $\Delta_i^* \cong \Delta_{9-i}$.

Example 4.4.4. Let us think families of quartic surfaces in $\mathbb{P}^3$; $W = (1, 1, 1; 4)$. The full Newton polytope $\overline{\Delta}$ itself is reflexive with $L_0 = 0$ and its dual $\overline{\Delta}^*$ can be embedded in $\overline{\Delta}$ as $\langle W^4, X^2YZ, XY^2Z, XYZ^2 \rangle$. The computation of lattices is in [14].

Let us make a self-dual polyhedron. An easy choice $\Delta = \langle W^4, X^2Y^2, Y^2Z^2, Z^2X^2 \rangle$ is a self-dual polyhedron with the same correspondence σ as in the case of $\overline{\Delta}$, but the rank ρ of algebraic lattice is thirteen. Another choice $\Delta' = \langle W^4, X^4, X^2Y^2, XY^2Z, XYZ^2, Z^2X^2 \rangle$ is self-dual and satisfies $L_0 = 0$ and $\rho = 10$.

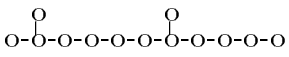

Fig. 1

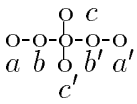

Fig. 2

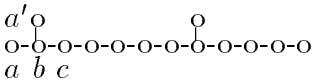

Fig. 3

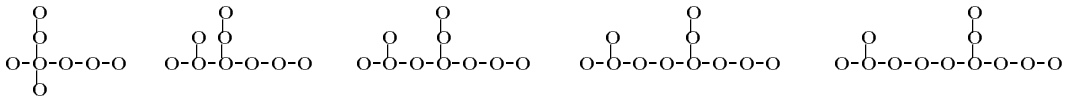

Fig. 5

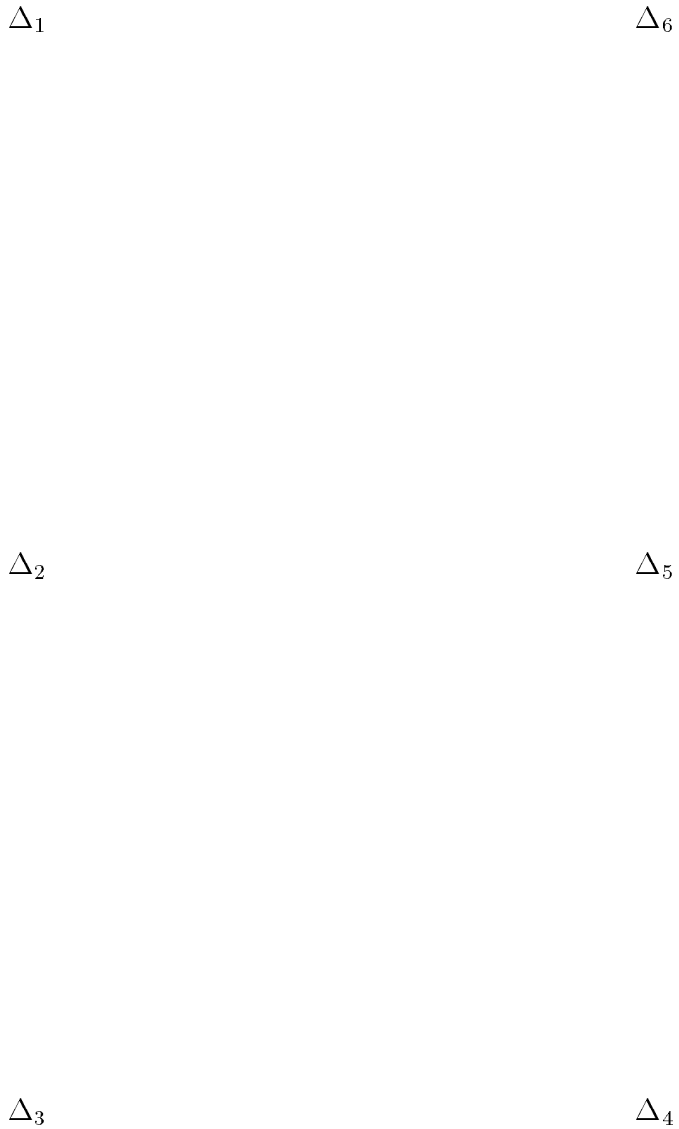

Fig. 4. $W = (2, 3, 4; 10)$. The black circle represents the origin $XYZW$.

References

1. Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of Differentiable Maps, Volume I (Monographs in Mathematics 82) Boston, Basel, Stuttgart: Birkhäuser 1985

2. Aspinwall, P.S., Greene, B.R., Morrison, D.R.: The monomial-divisor mirror map. Internat. Math. Research Notices **72,** 319–337(1993)

3. Aspinwall, P.S., Morrison, D.R.: String theory on K3 surfaces. hep-th/9404151

4. Batyrev, V.V.: Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori. Duke Math. J. **69,** 349–409(1993)

5. Batyrev, V.V.: Dual polyhedra and the mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Alg. Geom. **3,** 493–535(1994)

6. Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces(Ergebnisse der Math. (3) Bd.4) Berlin Heidelberg New York Tokyo:Springer 1984

7. Berglund, P., Katz,S.: Mirror symmetry constructions: a review. hep-th/9406008 (1994)

8. Borcea, C.: K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds. preprint

9. Candelas, P., de la Ossa, X., Katz, S.: Mirror symmetry for Calabi-Yau hypersurfaces in weighted $\mathbb{P}_4$ and extensions of Landau Ginzburg theory. hep-th/9412117 (1994)

10. Candelas, P., Lynker, M., Schimmrigk, R.: Calabi-Yau manifolds in weighted $\mathbb{P}_4$. Nucl. Phys. **B341,** 383–402(1990)

11. Danilov, V.I.: The geometry of toric varieties. Russian Math. Surveys **33,** 97–154(1978)

12. Danilov, V.I., Khovanski, A.G.: Newton polyhedra and an algorithm for computing Hodge-Deligne numbers. Math. USSR Izv. **29,** 279–298(1987)

13. Dolgachev, I.V.: Weighted projective spaces. In:Group Actions and Vector Fields (Lect. Notes Math.,vol. 956, pp. 34–71) Berlin Heidelberg New York: Springer 1982

14. Dolgachev, I.V.: Mirror symmetry for lattice polarized K3-surfaces. alg-geom/9502005 (1995)

15. Dolgachev, I.V., Nikulin, V.V.: Arnold's exceptional singularities and K-3 surfaces. In:The all-union topological conference in Minsk, Minsk 1977

16. Ebeling, W.: Quadratische Formen une Monodromiegruppen von Singularitäten. Math. Ann. **255,** 463–498(1981)

17. Fletcher, A.R.: Working with weighted complete intersections. MPI/89-35; Univ. of Warwick Thesis (1989)

18. Kobayashi, M.: Mirror symmetry and strange duality. To appear in Séminaire de Topologie des Singularités de Surfaces Complexes (1994)

19. Higuchi, T., Yoshinaga, E., Watanabe, K.: Tahensu-Fukuso-Kaiseki Nyumon (Sugaku library vol. 51) Tokyo:Morikita Shuppan 1980 (in Japanese)

20. Lian, B.H., Yau, S.-T.: Arithmetic properties of mirror map and quantum coupling. hep-th/9411234 (1994)

21. Looijenga, E.J.N., Wahl, J.: Quadratic functions and smoothing surface singularities. Topology **25,** 261–291(1986)

22. Mori, S.: On a generalization of complete intersections. J. Math. Kyoto Univ. **15,** 619–646(1975)

23. Morrison, D.R.: Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians. J. Amer. Math. Soc. **6,** 223–247(1993)

24. Nikulin, V.V.: Finite automorphism group of Kähler K3 surfaces. Trans. Moscow Math. Soc. **38,** 71–135(1980)

25. Oda, T.: Convex Bodies and Algebraic Geometry: an Introduction to the Theory of Toric Varieties (Ergebnisse der Math. (3) Bd.15) Berlin, Heidelberg, New-York, London, Paris, Tokyo: Springer 1988

26. Orlik, P., Wagreich, P.: Isolated singularities of algebraic surfaces with $\mathbf{C}^*$-action. Ann. of Math. **93,** 205–228(1971)

27. Pinkham, H.: Singularités exceptionelles, la dualité étrange d'Arnold et les surfaces K-3. C. R. Acad. Sc. Paris **284** (A), 615–618(1977)

28. Reid, M.: Canonical 3-folds. In: Proc. Alg. Geom. Anger 1979 (pp. 273–310) Sijthoff and Nordhoff

29. Roan, S.: Mirror symmetry and Arnold's duality. MPI/92-86 (1992)

30. Ruan, Y., Tian, G.: A Mathematical theory of quantum cohomology. J. Diff. Geom. **42,** 259–367(1995)

31. Saito, K.: Quasihomogene isolierte Singularitäten von Hyperflächen. Invent. Math. **14,** 123–142 (1971)

32. Saito, K.: Regular system of weights and their associated singularities. In: T. Suwa and P. Wagreich (eds.) Complex Analytic Singularities (Adv. Stud. Pure Math., vol. 8 pp. 479–526) Tokyo: Kinokuniya, Amsterdam New York Oxford: North Holland 1986

33. Saito, K.: Algebraic surfaces associated to regular systems of weights. In: Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata (pp. 517–614) Tokyo:Kinokuniya, Amsterdam New York Oxford: North-Holland 1987

34. Saito, K.: On a duality of characteristic polynomials for regular system of weights. preprint (1994)

35. Stanley, R.P.: Combinatorics and Commutative algebra (Progress in Math. vol. 41) Boston, Basel, Stuttgart: Birkhäuser 1983

36. Voisin, C.: Miroirs et involutions sur les surfaces K3. In: Journées de Géométrie Algébrique d'Orsay (Juillet 1992), (Astérisque vol. 218, pp. 273–322) 1993

37. Yau, S.-T. (ed.) Essays on Mirror Manifolds, Hong Kong: International Press 1992

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro, Tokyo 152 Japan;

E-mail address: masanori@math.titech.ac.jp