MIRROR SYMMETRY AND STRANGE DUALITY

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We give a brief introduction to the two items in the title. We will survey first the basic phenomena of mirror symmetry rather conceptually (Section 1) and next of strange duality (Section 2). We add some discussion about the relation of them in Section 3.[1]

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1. Mirror symmetry

Superstring theory and conformal field theory has developed and was connected with algebraic geometry in the late 80's. The mirror symmetry, which implies a kind of symmetry of moduli space of Calabi-Yau varieties, is one of the most striking blows from superconformal field theory. Since there are numerous papers available in this subject now, we will just follow the story and refer the readers to, for example, [M1][HKT][Si] and papers in [Y] for technical details and references.

1-1 Calabi-Yau manifolds

Our universe is described in the superstring theory as a product of the usual four-dimensional space-time and an unknown extremely small compact manifold of real dimension six. This compact manifold should carry a complex structure and in fact become a Kähler manifold (or orbifold) with trivial canonical bundle. In a somewhat wider sense, such a manifold is called a *Calabi-Yau manifold* or especially *Calabi-Yau threefold*, where the mirror symmetry usually seems to work.

The definition of Calabi-Yau manifold is not settled; sometimes we restrict ourselves to the simply connected ones and even to the manifolds whose holonomy group is the special unitary group $SU(n)$ (see (3) in the next Theorem). The relation of these classes of manifolds is described by the following decomposition theorem corresponding to the decomposition of the holonomy group, which is a corollary of Yau's celebrated solution to the Calabi conjecture (see, for example, [B1][B2]):

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Theorem. *Let M be a compact Kähler manifold with null first Chern class $c_1^{\mathbb{R}}(M) = 0$. Then there exists a finite étale covering $\tilde{M} \to M$ where $\tilde{M}$ is decomposed to a direct product of manifolds of the following three classes:*

(1) *(complex torus) $\mathbb{C}^N/L$ where L is a lattice of rank N,*
(2) *(symplectic manifold) a simply connected Kähler manifold with a nowhere vanishing holomorphic 2-form whose powers generate all holomorphic forms on it, and*
(3) *(special unitary manifold) a simply connected projective manifold X with a trivial canonical class which satisfies $h^i(X) = 0$ for $0 < i < \dim X$.*

For the sake of uniqueness, we usually restrict $\dim X \geq 3$ in (3) since in dimension two, there is an isomorphism $SU(2) = Sp(1)$.

We remark that in the context of the classification theory of compact complex varieties, these varieties are quite important 'building blocks' for Iitaka fibration and Albanese fibration (see e.g.[U][Ka]). Following the philosophy of classification theory, at least in the case of complex dimension three, the natural category of varieties to be called 'Calabi-Yau varieties' should includes varieties with mild singularities. For example, Wilson [Wl] used threefolds with only canonical Gorenstein singularities which has a special unitary threefold as a crepant resolution of singularities, and called them *Calabi-Yau models*. Another candidate is a simply connected projective variety with, possibly numerically, trivial canonical bundle with only $\mathbb{Q}$-factorial terminal singularities; they are minimal models of some varieties of Kodaira dimension zero.

We list here some examples to construct Calabi-Yau threefolds in the sense of special unitary threefolds.

(1) complete intersection in a Fano variety, e.g. a weighted projective space[Og],
(2) branched double covering of a Fano threefold,
(3) crepant resolution of singular threefold of type (1)[CLS] or (2)[H],
(4) generalized Kummer variety [U §16](see an example below)
(5) Weierstrass model[Ny],
(6) polarized K3 surface over $\mathbb{P}^1$ [Kb],
(7) fiber product of elliptic surfaces over $\mathbb{P}^1$ [So],
(8) smoothing of a compact Kähler normal crossing variety [KN].

1-2 naïve mirror

The first striking and persuasive observation of the mirror phenomena was the following computer-aided experiment.

Observation. [CLS] We denote by $\mathbb{P}(a_0, \cdots, a_4)$ a weighted 4-space which is the quotient of $\mathbb{P}^4$ by the diagonal action of $\mathbb{Z}/a_0\mathbb{Z} \times \cdots \times \mathbb{Z}/a_4\mathbb{Z}$. We write $X_0, \cdots, X_4$ for the homogeneous coordinates. Let $\bar{M}$ be a weighted hypersurface defined by a Fermat-like polynomial of degree $\sum_{i=0}^4 a_i$. An easiest example is the Fermat quintic $\{X_0^5 + \cdots + X_4^5 = 0\}$ in $\mathbb{P}^4$. In fact this polynomial should be chosen so that the locus $\bar{M}$ is transverse to the singular locus of the weighted 4-space. This $\bar{M}$ is Gorenstein and has a crepant resolution M with trivial canonical bundle. They calculated the Hodge numbers $h^{1,1}$ and $h^{2,1}$ of M using so called 'Vafa's formula'. They did that calculation for various weights and equations, in fact for more than six thousand cases, and plotted the (sum and difference of) Hodge numbers

in a graph, which unexpectedly, at least mathematically, became almost symmetric with respect to the exchange of $h^{1,1}$ and $h^{2,1}$; for almost every M, there exists a W in this set of varieties that satisfies $h^{1,1}(M) = h^{2,1}(W)$ and vice versa.

We call such a pair of M and W naïvely a 'mirror pair'. Can you always find a mirror partner for a given Calabi-Yau threefold? An apparent counter example is the existence of rigid Calabi-Yau threefolds; the first example in the literature seems to be the following manifold due to Ueno[U Ex16.12][B2 2.a)].

Example. Let E to be the elliptic curve which has an automorphism of order 3 with three fixed points. Take a direct product $E \times E \times E$ and divide it by the diagonal $\mathbb{Z}/3\mathbb{Z}$-action of the automorphism. The resulting variety has exactly 27 canonical Gorenstein singular points and has a crepant resolution M. Then $h^{2,1}(M) = 0$ which means that M has no infinitesimal deformation of the complex structure.

This M does not form a mirror pair in the above sense, since the partner W should be Kähler, that means $h^{1,1}(W) > 0$. There are some proposals for the mirror pair of rigid Calabi-Yau varieties. For example, if we follow the original Greene-Plesser construction[GP] of mirror symmetry of Gepner models[Ge], the mirror pair of Ueno's manifold above should be a Fermat cubic sevenfold in $\mathbb{P}^8$. This manifold is Fano and not Calabi-Yau, but still you can find a subspace of its cohomology group which seems to be mirror-symmetric[CDP][BB]. There is also an approach using supermanifolds[Se].

1-3 Mirror symmetry

There are much more than exchanging the Hodge numbers in mirror symmetry. We should explain the usual sense of 'mirror pair' for Calabi-Yau threefolds now.

In the level of representation theory of $N = 2$ superconformal algebra, it is fairly easy to describe rigorously the mirror map [LVW][AL]. But as for the connection with Calabi-Yau threefolds, the mathematical description is still in the darkness.

For each Calabi-Yau orbifold you can associate the quantum cohomology ring, which is a deformation of usual de Rham cohomology ring with some parameter. This part is being justified mathematically using symplectic geometry by [Ru][RT][KM][Kn][Gi].

Certain $N = 2$ superconformal field theory can be modified to at least two different topological field theories: namely, A-model and B-model[Wt]. A-model concerns with $\oplus_p H^{p,p}$ of some polarized Calabi-Yau threefold M and B-model with $\oplus_p H^{p,3-p}$, that is, the usual variation of Hodge structures of another polarized Calabi-Yau threefold W There is expected an isomorphism of the cohomology groups of M and W such that $H^{p,q}(M) \cong H^{3-p,q}(W)$ which exchanges quantum (A-model) multiplicative structure and VHS (B-model) multiplicative structure. This M and W are, together with their Kähler classes, said to form a 'mirror pair', since their Hodge diamonds $\{H^{p,q}\}$ are mirror-symmetric with respect to the line $p = 3/2$. We remark that this pairing is not one-to-one.

The physical quantities such as three-point function in the original $N = 2$ superconformal field theory can be translated to both models. In the famous paper [CdOGP] by Candelas et al., they calculated the so-called *Yukawa coupling* from the B-model of the mirror pair of the Fermat quintic threefold in $\mathbb{P}^4$; by translating it to the A-model of the quintic, they predicted the 'numbers' of the rational curves in the general quintic, which

is, if it is literally the number, very much difficult to count algebro-geometrically but for very small degrees. Let us review their computation.

Let M_ψ be the quintic hypersurface in $\mathbb{P}^4$ defined by $\sum_{i=0}^4 X_i^5 - 5\psi \prod_{i=0}^4 X_i = 0$, where ψ is a parameter in $\mathbb{P}^1$ and $X_0, \cdots, X_4$ are a system of homogeneouscoordinates in $\mathbb{P}^4$. This M_ψ is smooth if $\psi \notin \{0, 1, \infty\}$ and henceforth is a Calabi-Yau threefold with $h^{1,1}$=1 and $h^{2,1}$=101. Take the diagonal subgroup G of $PSL(5, \mathbb{C})$ which leaves M_ψ invariant. Let $\bar{W}_\psi$ be the quotient of M_ψ by G. For $\psi \notin \{0, 1, \infty\}$, this $\bar{W}_\psi$ has only toric canonical Gorenstein singularities and there exists a crepant resolution W_ψ which is also a Calabi-Yau threefold. This W_ψ has $h^{1,1}$=101 and $h^{2,1}$=1. There is an apparent isomorphism $W_\psi \cong W_{\alpha\psi}$ ($\alpha^5 = 1$). One should take a good parameter t which is centered at $\psi = \infty$ where the monodromy is maximally unipotent. Then one has a one-parameter Kuranishi family of the new Calabi-Yau threefolds, which, by abuse of notation, is denoted by W_t. For a holomorphic three form ω of W_t, that is an element of F^3 of VHS, the periods $\int \omega$ of W_t satisfies a Picard-Fuchs equation of order four. From this differential equation, one can calculate the '(normalized) Yukawa coupling' $\kappa_{ttt} = \int \omega \wedge \partial_t^3 \omega$ for an appropriate choice of ω. This is a formal series in t, or rather in q=$e^{2\pi i t}$.

By reading the coefficient of κ_{ttt} from the point of view of A-model of the σ-model, they calculated a series of numbers n_d which is claimed to be the number of the rational curves of degree d in a general quintic threefold in $\mathbb{P}^4$. We cite the expansion here; $\kappa_{ttt} = 5 + \sum_{d=1}^\infty \frac{n_d d^3 q^d}{1-q^d}$. In fact, for $d = 1$, 2 and 3, the predicted numbers agree with what is calculated mathematically (with the aid of computer in the case of $d = 3$)[K1][M1][ES][K3]. The generality was necessary to guarantee the finiteness of the rational curves of given (low) degrees, but no more(e.g.[Ru][Gi]); one can consider n_d as the Euler number of certain vector bundle.

That was a striking computation using mirror symmetry. The calculation itself is now very much simplified (e.g.[M2]). Since then many similar predictions and confirmation of them are done for the rational curves on complete intersections of weighted projective spaces (see [K3][HKT] and its references) and even on higher dimensional varieties[GMP][K2]. Especially, it has been possible to compute the Yukawa coupling in the case where the moduli number of W is greater than one. There are exact results also for curves of higher genera[BCOV].

Their predictions of mirror pair is mostly due to the following Batyrev's construction.

1-4 Batyrev's toric construction

We will quickly review Batyrev's toric mirror construction [Ba] for toric hypersurfaces in Gorenstein toric Fano varieties.

Let M be a free abelian group of rank d and N be its dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We write $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ for the tensor product $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$, respectively. We denote the extension of the natural pairing by $\langle , \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

Let Δ be a convex hull of a finite subset of M in $M_{\mathbb{R}}$, which is not contained in any hyperplanes of $M_{\mathbb{R}}$. Such a set is called a *d-dimensional rational strongly convex polytope*. We will omit 'strongly convex' from now on since all polytopes here will be assumed so. This Δ is a kind of Newton polytope.

We can associate to Δ a graded ring $S_\Delta = \bigoplus_{k=0}^\infty \langle k\Delta \cap M \rangle_{\mathbb{C}}$ and a polarized toric complex variety $\mathbb{P}_\Delta = \mathrm{Proj}\, S_\Delta$ with the natural polarization; here, $\langle T \rangle_{\mathbb{C}}$ denotes the $\mathbb{C}$-

subvector space generated by the vertex set (basis) T in the group ring $\mathbb{C}[M]$, and the multiplication in S_Δ corresponds to the addition in M.

We restrict ourselves to the case that $0 \in M$ is contained in the interior of Δ, in which case we can think the *polar dual* $\Delta^* = \{y \in N_{\mathbb{R}} | \langle x, y\rangle \geq -1 \, \forall x \in \Delta\}$. We say Δ is *reflexive* if its polar dual Δ^* is also a d-dimensional rational polytope in $N_{\mathbb{R}}$. The reflexivity of Δ is equivalent to guarantee that $\mathbb{P}_\Delta$ is a Gorenstein Fano variety with canonical bundle $\mathcal{O}(-1)$.

For a given reflexive polytope Δ, take an irreducible anticanonical member $\bar{\mathcal{M}}_\Delta$ with canonical Gorenstein singularities of $\mathbb{P}_\Delta$. $\bar{\mathcal{M}}_\Delta$ has a maximal crepant desingularization $\mathcal{M}_\Delta$ which is nonsingular in codimension three. In Batyrev's theory, $\mathcal{M}_\Delta$ and $\mathcal{M}_{\Delta^*}$ form a mirror pair. He proved that the Hodge numbers $h^{1,1}$ and $h^{d-2,1}$ interchanges for $\mathcal{M}_\Delta$ and $\mathcal{M}_{\Delta^*}$, that means the hypersurfaces are at least naïve mirror pair.

There are not a even physical verification, except for the case of the Gepner model of the minimal series, that this Batyrev construction really corresponds to the original mirror symmetry in the context of quantum cohomology. However, all calculations, including the numbers of rational curves, using his claim have worked well. Batyrev and Borisov[BB] expanded this method to the complete intersections in toric Fano varieties.

We should mention that the first result of toric mirror symmetry is due to Roan[R1]. He showed Vafa's formula and proved that for Fermat weighted hypersurface in weighted $\mathbb{P}^4$, the Hodge numbers of mirror pair with respect to the dual group in $PSL(5, \mathbb{C})$ are exchanged as expected.

There is yet another construction of mirror manifolds apart from toric method for the product of K3 surface and elliptic curve using involutions[V][Bo].

2. Strange duality

2-1 Arnold's classification of hypersurface singularities

Arnold studied in his series of papers[A2] the germs of a critical point of an analytic function whose essential deformation space is finite-dimensional and classified them in the case where the dimension is low.

Definition. For a given critical point X with a germ of finite-dimensional Kuranishi space S and a versal family $\pi : \mathcal{X} \to S$, *modality* of X is defined to be: $m := \min\{\dim S' | S'$ is an analytic subset of a germ S, $\forall s \in S \, \exists s' \in S' \, \pi^{-1}(s) \cong \pi^{-1}(s')\}$. A critical point of modality zero, one and two is called *simple*, *unimodal* (or *unimodular*) and *bimodal*, respectively. Two germs of analytic functions with one critical point at zero are *stably equivalent* if they are isomorphic after adding the appropriate number of square terms.

Here we cite the very first of Arnold's list, namely of modality less than two.

Theorem. [A1][AGV]

(1) *A simple critical point is stably equivalent to :* A_k*:* x^{k+1} $(k \geq 1)$, D_k*:* $x^2y + y^{k-1}$ $(k \geq 4)$, E_6*:* $x^3 + y^4$, E_7*:* $x^3 + xy^3$ *or* E_8*:* $x^3 + y^5$.

(2) *The germ of a unimodal critical point is stably equivalent to one of the followings:*

(a) $T_{p,q,r}$: $x^p + y^q + z^r + axyz$ ($\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, $a \neq 0$),
(b) $T_{3,3,3} = \tilde{E}_6$: $x^3 + y^3 + z^3 + axyz$ ($a^3 + 27 \neq 0$),
$T_{2,4,4} = \tilde{E}_7$: $x^2 + y^4 + z^4 + ay^2z^2$ ($a^2 \neq 4$),
$T_{2,3,6} = \tilde{E}_8$: $x^2 + y^3 + z^6 + ay^2z^2$ ($4a^3 + 27 \neq 0$) or
(c) *one of the fourteen 'exceptional' critical points (see table 1 below).*

The exceptional critical points are all weighted homogeneous and we have the corresponding surface singularities with $\mathbb{C}^*$-action.

In the Table 1, we followed the notation in [AGV] for the classes. The subscript of the class shows its Milnor number. the second and third column show the weights of the variables and the equation, respectively. a in the equations is a deformation parameter corresponding to unimodality. *Gab*=Gabrielov numbers; *Dol*=Dolgachev numbers; $\Delta = |\det(L)|$ (L is the Milnor lattice or the exceptional lattice); $s = -Z^2$ (Z is the fundamental cycle); t=type. (for their meanings, see below and [P][Nm]).

Table 1. The 14 exceptional unimodal singularities

class	x y z	eq	equation	*Gab*	*Dol*	Δ	s	t
E_{12}	6 14 21	42	$x^7 + y^3 + z^2 + ax^5y$	2 3 7	2 3 7	1	1	1
E_{13}	4 10 15	30	$x^5y + y^3 + z^2 + ax^8$	2 3 8	2 4 5	2	1	2
Z_{11}	6 8 15	30	$x^5 + xy^3 + z^2 + ax^4y$	2 4 5	2 3 8	2	2	1
E_{14}	3 8 12	24	$x^8 + y^3 + z^2 + ax^6y$	2 3 9	3 3 4	3	1	3
Q_{10}	6 8 9	24	$x^4 + y^3 + xz^2 + ax^3y$	3 3 4	2 3 9	3	3	1
Z_{12}	4 6 11	22	$x^4y + xy^3 + z^2 + ax^3y^2$	2 4 6	2 4 6	4	2	2
W_{12}	4 5 10	20	$x^5 + y^4 + z^2 + ax^3y^2$	2 5 5	2 5 5	5	2	2
Z_{13}	3 5 9	18	$x^6 + xy^3 + z^2 + ax^5y$	2 4 7	3 3 5	6	2	3
Q_{11}	4 6 7	18	$x^3y + y^3 + xz^2 + ax^5$	3 3 5	2 4 7	6	3	2
W_{13}	3 4 8	16	$x^4y + y^4 + z^2 + ax^6$	2 5 6	3 4 4	8	2	3
S_{11}	4 5 6	16	$x^4 + xz^2 + y^2z + ax^3z$	3 4 4	2 5 6	8	3	2
Q_{12}	3 5 6	15	$x^5 + y^3 + xz^2 + ax^4y$	3 3 6	3 3 6	9	3	3
S_{12}	3 4 5	13	$x^3y + xz^2 + y^2z + ax^5$	3 4 5	3 4 5	13	3	3
U_{12}	3 4 4	12	$x^4 + y^3 + z^3 + ax^2yz$	4 4 4	4 4 4	16	3	3

2-2 Strange duality

Dolgachev[D] studied the resolution of the exceptional singularities through his study of quotient-conical singularities and found that the exceptional set of the minimal good resolution consists of four $\mathbb{P}^1$, three of which are disjoint with self intersection $-c_1$, $-c_2$, $-c_3$ ($c_1 \leq c_2 \leq c_3$) and intersect the other (-1)-curve at distinct points. We denote *Dol*=(c_1, c_2, c_3) and call them the *Dolgachev numbers*.

On the other hand, Gabrielov[Ga] computed the Milnor lattices and found that it is a direct product $L \oplus H$, where L corresponds to a intersection matrix T_{b_1,b_2,b_3} of a star of (-2)-curves with three branches, whose lengths including the central vertex are b_1, b_2, b_3 ($b_1 \leq b_2 \leq b_3$), and H is a rank two lattice with intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We denote $Gab = (b_1, b_2, b_3)$ to be the *Gabrielov numbers*.

Arnold[A1] observed some pairing among the 14 singularities which interchanges Dolgachev numbers and Gabrielov numbers(see Table 1). He called that phenomenon as *strange duality*. Six of the singularities are self-dual. After that, there have been found several other invariants which behave similarly(s and t), coincide (Δ) or whose sum is constant (Milnor number) for the claimed pair.

Dolgachev[DN] and Pinkham[P] independently proposed a beautiful geometric framework for it; each exceptional unimodal singularity is weighted homogeneous and its Jacobian ring has exactly one term whose weight is bigger than that of the equation (by two), which corresponds to unimodality. We remark that $f(x,y,z)+1=0$ gives smooth affine surface. You can add one variable w of weight one and take the homogeneous subfamily of codimension one in the versal deformation family of the original equation. This family can be compactified in a weighted projective space and the general fiber is a normal K3 surface with exactly three A-type singularities at the infinity $\{w=0\}$, which can be resolved simultaneously. Thus the Milnor fiber can be compactified in a K3 surface S. We denote the algebraic exceptional lattice at boundary by L', whose intersection matrix is T_{c_1,c_2,c_3}. Then the K3 lattice $H^2(S,\mathbb{Z})$ decomposes as $E_8^{\oplus 2}\oplus H^{\oplus 3}\cong L\oplus H\oplus L'$, so the strange duality appears as an exchange of L and L'.

There are several generalizations of this kind of duality[Nm][EW].

Recently, Saito[Sa] gave a combinatorial way to reproduce the duality of weights.

3. Relations

In the context of Landau-Ginzburg theory, the defining equation and its deformation of some hypersurface singularities has been studied[VW]. Arnold's strange duality appeared in one of the equivalence of the superpotentials in [LS], and became understood later as a phenomenon of the mirror symmetry for K3 surfaces.

Roan[R2] applied his toric method in this context and constructed a pairing by a 'non-linear coordinate change' for four pairs, which inspired our present work.

On the other hand, as we have reviewed above, Batyrev[Ba] proposed a purely mathematical way to construct mirror varieties for toric hypersurfaces, which we will apply to K3 surfaces. Borcea[Bo] got a partial result in this direction.

Note that for K3 surfaces, the mirror map exchanges $H^{p,q}$ to $H^{2-p,q}$ and thus the mirror for Hodge numbers is trivial. Moreover, the tangent spaces of moduli of complexified Kähler structure and complex structure are both mixed in $H^{1,1}$, so we should do things carefully[AM].

In general for a d-dimensional reflexive polytope Δ, we denote its standard volume by $vol(\Delta)$. Since $d!vol(\Delta)$ is equal to the degree $(\mathcal{O}_{\mathbb{P}}(1))^d$ of the polarization and $\#\{\Delta\cap M\}$ is the dimension $h^0(\mathcal{O}_{\mathbb{P}}(1))$, thus we have $d!vol(\Delta)=2|\{\Delta\cap M\}|-2d$ using Riemann-Roch theorem[Od]. We define $V(\Delta):=d!vol(\Delta)+d!vol(\Delta^*)$.

First we will see the case when $d=2$. In this case, Batyrev classified all the reflexive polygons. One can show easily the following, with or without the classification.

Proposition. $V(\Delta)=12$ *for any reflexive polygon* Δ.

In the case of dimension three, $V(\Delta)$ is not constant at all, but we can show the following result by direct calculation (Table 2).

Let X be one of the exceptional singularities and its weight system $(a,b,c;h)$[Sa]. Note that all of them satisfy a relation $h=a+b+c+1$. Let Δ=Δ_X to be a polyhedron defined from $(\mathbb{P}(a,b,c,1),\mathcal{O}(h))$, which is unique up to $\mathrm{Aut}(\mathbb{Z}^{\oplus 3})$. In fact, $(\mathbb{P}_\Delta,\mathcal{O}(1))$ is nothing but the image of the rational map $\Phi_{|\mathcal{O}_{\mathbb{P}(a,b,c,1)}(h)|}$ with the very ample line bundle.

Theorem. *Let X be one of the exceptional unimodal singularities and X' be its strange dual partner. Then $V(\Delta_X)=V(\Delta_{X'})$*

This shows that regardless the result of [Bo], there is still something in the polytope method, which reproduces whole strange duality and is applicable also to other weight systems. This will appear elsewhere.

Table 2. Δ and Δ^*.

class	vertices of Δ	v	v^*	sum	n	n^*	sum
E_{12}	w^{42},x^7,y^3,z^2	42	42	84	24	24	48
E_{13}	$w^{30},w^2x^7,x^5y,y^3,z^2$	44	34	78	25	20	45
Z_{11}	$w^{30},w^6y^3,x^5,xy^3,z^2$	36	42	78	21	24	45
E_{14}	w^{24},x^8,y^3,z^2	48	24	72	27	15	42
Q_{10}	$w^{24},w^6z^2,x^4,y^3,xz^2$	30	42	72	18	24	42
Z_{12}	$w^{22},w^2x^5,w^4y^3,x^4y,xy^3,z^2$	38	34	72	22	20	42
W_{12}	w^{20},x^5,y^4,z^2	40	20	60	23	13	36
Z_{13}	$w^{18},w^3y^3,x^6,xy^3,z^2$	42	24	66	24	15	39
Q_{11}	$w^{18},w^4x^2,w^2z^4,x^3y,y^3,xz^2$	32	34	66	19	20	39
W_{13}	w^{16},wx^5,x^4y,y^4,z^2	42	18	60	24	12	36
S_{11}	$w^{16},wy^3,w^4z^2,x^4,y^2z,xz^2$	32	28	60	19	17	36
Q_{12}	$w^{15},w^3z^2,x^5,y^3,xz^2$	36	24	60	21	15	36
S_{12}	$w^{13},wx^4,wy^3,w^3z^2,x^3y,y^2z,xz^2$	34	24	58	20	15	35
U_{12}	w^{12},x^4,y^3,z^3	36	12	48	21	9	30

In Table 2, $v=3!vol(\Delta)$, $v^*=3!vol(\Delta^*)$, $n=h^0(\mathbb{P}_\Delta,\mathcal{O}(h))=\#\{\Delta\cap M\}$, $n^*=h^0(\mathbb{P}_{\Delta^*},\mathcal{O}(h))=\#\{\Delta^*\cap M\}$.

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