Masanori KOBAYASHI and Tzee-Char KUO

On Blow-analytic equivalence of embedded curve singularities

Introduction

We consider the case of two real variables. A function $g : \mathbb{R}^2 \to \mathbb{R}$ is blow-analytic if there exists a composition of simple blowings up, each centered at a point, $\beta = \beta_1 \circ \cdots \circ \beta_n : X \to \mathbb{R}^2$ such that $g \circ \beta$ is analytic ([7][4]). If a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ and its inverse, h^{-1}, both have blow-analytic components, we say h is blow-analytic.

S. Koike([5]) was the first to discover a blow-analytic homeomorphism of $\mathbb{R}^3$ with itself which is not Lipschitz (see also [4]). L. Paunescu, in [9], has discovered one, also of $\mathbb{R}^3$, which does not preserve the multiplicity of analytic arcs.

On the other hand, M. Suzuki([11]) and T. Fukui ([3]) have found some blow-analytic invariants which can be used to show, for example, that functions like x, $x^2 - y^3$, $x^3 - y^7$ are not blow-analytically equivalent. A seemingly simple question, raised by Koike, is whether a blow-analytic homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ can carry a line $\{x = 0\}$ to a singular curve such as $\{x^2 = y^3\}$ or $\{x^3 = y^7\}$. (These are as topological spaces; analytic structures are ignored.)

We shall answer the question in the affirmative.

Let C be the germ of a singular curve in $\mathbb{R}^2$, which is unibranched in the sense that its complexification has only one branch ([12]).

We first take a sequence of blowings up to desingularize C to a smooth curve, $\tilde{C}$, transversal to exactly one exceptional curve, we then prove that there is a way to blow down to $\mathbb{R}^2$ again so that the images of $\tilde{C}$, at all stages, remain smooth. This is the content of the Main Theorem (Theorem 6).

Our results and techniques are developped merely for the real analytic case. We no longer have them in the complex case, where things are too rigid.

Embedded real analytic curves

First we shall classify neighbourhoods of a real curve.

Nash' s Theorem[8]. *If two real analytic compact algebraic manifolds are diffeomorphic, then they are also equivalent as algebraic manifolds.*

Thus any compact connected smooth real analytic manifold of dimension one is real analytically equivalent to $\mathbb{R}P^1$.

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Proposition 1. *There are only two real analytic equivalence classes of germs of compact connected smooth real analytic manifold of dimension one embedded in a smooth surface; that is,*

(1) *the direct product of the germ of a one dimensional open segment and* $\mathbb{R}P^1$ *or*
(2) *the tubular neighbourhood of the exceptional divisor of the blowing up of the origin of* $\mathbb{R}^2$.

Topologically (1) is an annulus, (2) is a Möbius band.

Proof. A small tubular neighbourhood of $\mathbb{R}P^1$ in a smooth surface is real analytically isomorphic to that of the zero section in the normal bundle. A normal bundle of $\mathbb{R}P^1$ is either trivial or twisted once. □

We say a neighbourhood of $\mathbb{R}P^1$ is *even* for (1) and *odd* for (2). We also call the central curve *even* and *odd* accordingly.

Let C be a complex projective line embedded in some smooth complex surface which is a complexification of a real projective line embedded in a real analytic manifold of dimension two. The normal bundle is isomorphic to $\mathcal{O}(n)$ for some integer n and the parity of n coincides with the parity of the real slice.

Real exceptional curves

Definition 2. We call a blowing-up of a maximal ideal of a smooth point in a surface a *simple blowing up*.

The exceptional curve is an odd curve. A simple blowing up centered at a point on a curve changes the parity of the curve.

Proposition 3. *Any real analytic curve singularity embedded in a smooth surface can be resolved by a sequence of simple blowings up.*

Proof. A complexification of the singularity can be resolved by a sequence of point blowings up. The relevant points are $\mathbb{R}$-valued. □

Take a sequence of simple blowings-up $\beta = \beta_n \circ \cdots \circ \beta_1 : X = X_n \to X_0 = \mathbb{R}^2$, $\beta_i : X_i \to X_{i-1}$, whose centers are all above the origin of $\mathbb{R}^2$. The whole sequence is just the real slice of a natural composition of point blowings-up of $\mathbb{C}^2$. The exceptional set in X is a connected union of normal crossing real projective lines E_i which is the strict transform of the exceptional divisor of β_i.

We associate to $(X, \{E_i\})$ a matrix whose entries are in $\mathbb{Z}/2\mathbb{Z}$.

Definition 4. The *intersection matrix* of $(X, \{E_i\})$ is an $n \times n$ matrix (a_{ij}) where

(1) a_{ii} is zero or one according to the parity of E_i and
(2) a_{ij} $(i \neq j)$ is one if E_i and E_j intersect or zero otherwise.

This is nothing but the modulo two object of the usual intersection matrix $((E_i.E_j))$ for a complexification. We can speak of the determinant of the intersection matrix, which is defined to be one for $\mathbb{R}^2$.

Proposition 5. *The determinant of the intersection matrix of the exceptional curves is invariant under simple blowings-up and down.*

Proof. An easy case-by-case calculation. □

Thus, the surface made by simple blowings up from $\mathbb{R}^2$ has a tree of $\mathbb{R}P^1$'s whose intersection matrix has determinant one.

In the complex case, the determinant $\det(-(E_i.E_j))$ is a $\mathbb{Z}$-valued invariant.

By performing additional simple blowings up if necessary, one can assume that all components of the exceptional set are odd curves for the sake of simplicity.

Theorem 6. *Let X be a two-dimensional smooth tubular neighbourhood of a tree of odd curves. Assume that the determinant of the intersection matrix is one. Then X can be obtained by a composition of simple blowings up $f : X \to \mathbb{R}^2$, whose exceptional set is the given tree.*

Moreover, choose any noncompact smooth curve C which cuts transversally only one of the odd curves. Then f can be chosen so that $f_{|C}$ is a real analytic isomorphism on every step of simple blowing down.

We use a standard language of dual graphs; an exceptional real projective line corresponds to a vertex, and if two lines intersect at a point, the corresponding vertices are joined by an edge. In the following, a graph means a weighted graph, where each vertex is assigned a parity as its weight. We represent an odd curve by a white circle and an even curve by a black one. We call a white circle a 'cell'.

Remark. By translating the curve configuration into the language of graphs, one loses the information of the cyclic order of the double points on each real projective line. However, this does not affect the proof of the statement.

Note also that ratio of points on $\mathbb{R}P^1$ is not an invariant any more by Cartan's Theorem B [2]. See Also [6].

We introduce the following four procedures, which are all composition of simple blowings-up.

Definition 7. A *space station* is a graph arising from an iterated applications of the following four constructions to the null graph:

(1) *launch*: add a single cell without edge;
(2) *twin extension*: growing two cells from a mother cell;
(3) *rifle extension*: growing three cells in a row;
(4) *tick extension*: growing four cells filed in a row, with the second one being connected to a mother cell.

We use the word 'contraction' for the inverse of 'extension'.

In fact, the centers of simple blowings up corresponding to each case are (*cf. Fig. 1*):

(1) a point not on any curve;
(2) two simple points on a same curve;

(3) a simple point on a curve, a simple point on the exceptional curve, the intersection of the original curve and the first exceptional curve,

(4) a simple point on a curve, the intersection of the original curve and the exceptional curve, the intersection of two exceptional curves, a simple point on the second exceptional curve.

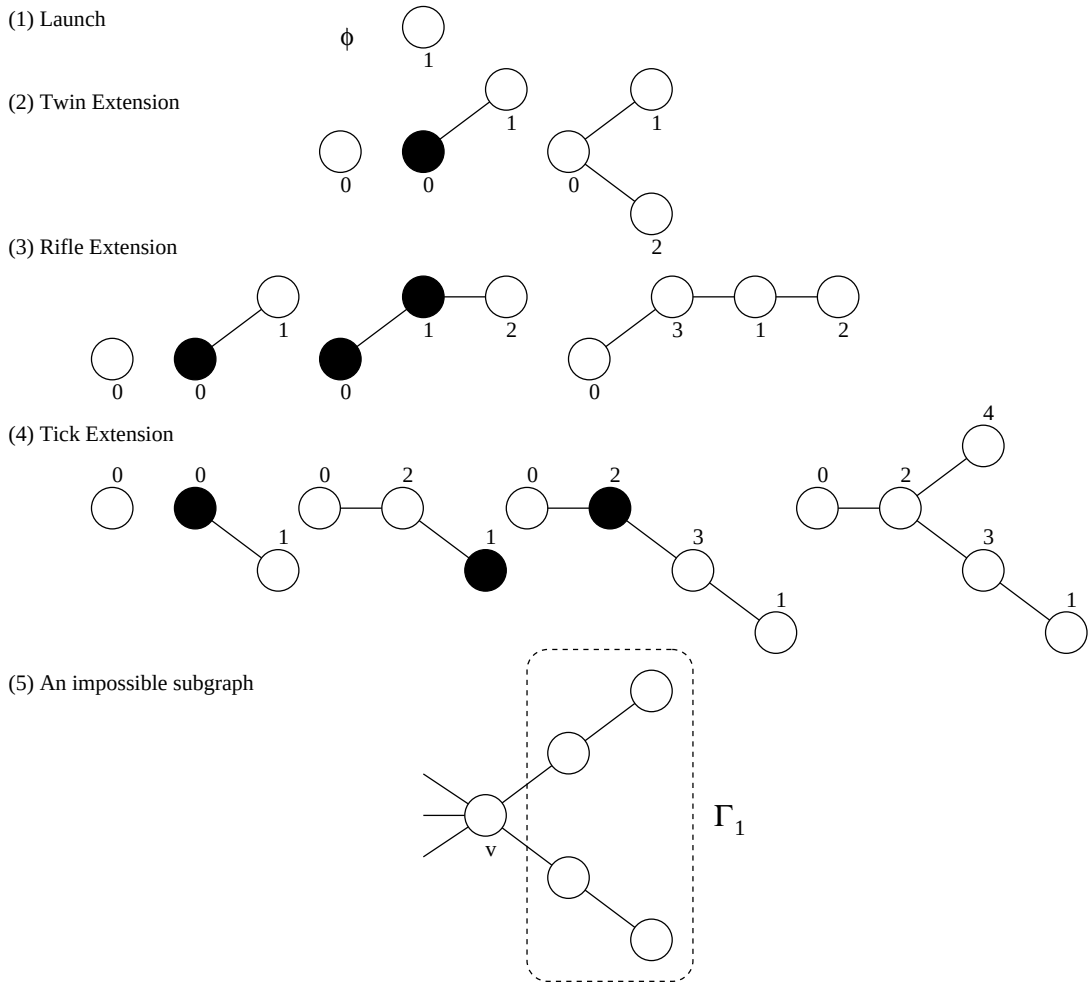

Fig. 1 Possible and impossible extensions

A construction begins with (1). The graph is connected if and only if (1) is never used again. We are merely interested in connected space stations, which is a tree.

In a space station, there is at most one edge between two cells; each cell has weight 1.

We define the *determinant* of the graph to be that of the obvious intersection matrix. The determinant of null graph is defined to be one. Note that the determinant does not depend on the numbering of cells.

Proposition 8.

(1) *Let S be a connected graph consisting of cells. Then S is a space station if*

and only if S is a tree and the determinant is one.

(2) *Let S be a space station and c be any cell of S. Then by contractions (2)(3) and (4), S can become one of the following three space stations: (A) c, (B) $c-o-o$ or (C) $o-c-o-o$, where o's are the other cells.*

A subgraph Γ' of a graph Γ is said to be *full* if all edges E of Γ whose end points are both in Γ' belong to Γ'.

Lemma 9. *Let Γ be a graph, Γ_i $(i=1,2)$ be two full subgraphs and v be a vertex. Let m be the weight of v. Assume*

(1) *v does not belong to neither of Γ_i,*
(2) *every vertex of Γ except v belongs to precisely one of Γ_i and*
(3) *there are no edges between vertices of Γ_1 and Γ_2.*

We denote by $\tilde{\Gamma}_i$ $(i=1,2)$ the full subgraph whose vertex set is the union of that of Γ_i and $\{v\}$.

Then

$$\det(\Gamma) = \det(\tilde{\Gamma}_1)\det(\Gamma_2) + \det(\Gamma_1)\det(\tilde{\Gamma}_2) - m\det(\Gamma_1)\det(\Gamma_2).$$

Proof. Cofactor expansion along the row corresponding to v. □

Corollary 10. *If the graph contains a full subgraph consisting of linear five cells whose intersection with other vertices are only at the mid vertex, then the determinant of the whole graph is zero.*

Proof. Take the mid vertex as v, and the full subgraph consisting of other four vertices as Γ_1 (*cf. Fig. 1*). Both of the determinants of Γ_1 and $\tilde{\Gamma}_1$ are zero. □

Proof of the Proposition 8. First we shall prove (2).

The distance $d(v)$ between a vertex v and c is the number of edges in the unique path joining them.

Assume first that $\max d(v) \geq 3$.

Let v be one of the vertices which attain the maximum of d. By assumption, there exists a unique vertex w which is connected to v. Moreover, $d(w) = d(v) - 1$.

Case 1. If there exists another vertex v' which is adjacent to w and has the property $d(v') = d(v)$, then w is the unique vertex which is connected to v'. Therefore we can perform the twin contraction to v and v' and decrease the number of the vertices. The length of the path has been reduced by one.

Case 2. Otherwise, w is adjacent to v and to another unique vertex, say x. We have $d(x) = d(w) - 1$.

Subcase 2-1. If there is no vertices other than w which is adjacent to x and has bigger distance than x, then we can perform the rifle contraction. The length of the path has been reduced by three.

Subcase 2-2. Otherwise, there is a vertex w' adjacent to x, other than w, such that $d(w') = d(w)$. By twin contraction, we can assume that there is at most one

vertex clinging to w' further. But if there exists one, it contradicts to Corollary 10. Thus the unique adjacent vertex of w' is x. By twin contraction if necessary, one can assume that the number of such w' is at most one. If none, go to Subcase 2-1; otherwise, we can perform the tick contraction, which reduces the length of the path by three.

Thus we can decrease the number of vertices and eventually $\max d(v)$. Now we can assume that the maximum of $d(v)$ is at most two. By twin contraction and Corollary 10, we can further assume that the base vertex is connected with (a) at most one linear subgraph consisting of two cells and (b) at most one single cell. The graph consisting of intersecting two vertex has the even determinant, which is excluded.

Thus we have (2).

For (1), one direction is obvious. For the other direction, choose any cell c in S. By (2), we have to treat only three cases, all of which can be contracted to nothing by launch and possibly one twin or rifle contraction. □

Proof of the Theorem 6. Take the component intersecting the noncompact curve as the cell c in (2) of Proposition 8, we can perform simple blowings down to the three types of configuration with the neighbourhood of the noncompact curve untouched.

The final stage is as follows (*cf. Fig. 2*).

(A) Blow down c.

(B) For the configuration $c - o - o$, first blow down the two curves on both ends; the noncompact curve intersects the remaining odd curve transversally at one point. Thus the blowing down of the odd curve leaves the noncompact curve smooth.

(C) For the configuration $o - c - o - o$, blowing down the third, first, second and fourth curves from the left in this order has the disired property. □

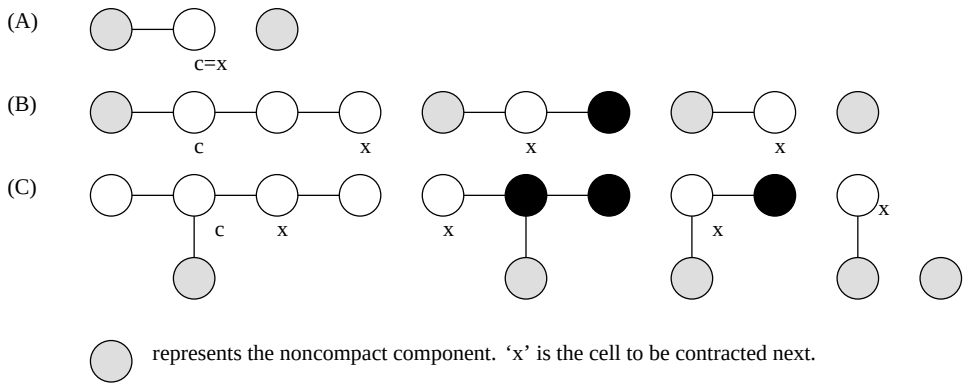

Fig. 2 Final stage of contraction

Blow-analytic equivalence

Theorem 11. *A germ of real unibranched curve in* $\mathbb{R}^2$ *is unique up to blow-analytic homeomorphism.*

Proof. Take any two such germs. For each germ, by Theorem 6, we have a surface which can be blown down by simple blowings up both to the germ and to a germ of a line embedded in $\mathbb{R}^2$. The surface has different complexifications for those two contractions.

Since the two surfaces for two germs are both over a same germ of a line, they can be complexified as two surfaces blown up from a germ of a complex line in $\mathbb{C}^2$. Those two complex surfaces are bimeromorphic and we have a surface which is a resolution of indeterminacy for the bimeromorphic map. A bimeromorphic morphism between two smooth complex surfaces is a composition of blowings up at a smooth point. Taking a real slice, we have a desired smooth real surface which is contractible by simple blowings down to both of the initial real surfaces. □

Example 12. Here we give an explicit answer to Koike's question in the introduction (*cf. Fig. 3*).

(1) Case $\{x^2 = y^3\}$. By three simple blowings up, we have a good resolution of the curve, in the sense that the strict transform of the cuspidal curve is smooth, intersects only one exceptional curve and the intersection is transversal. One of the exceptional curve, which comes from the second blowing up, is even; Thus we perform another simple blowing up at a general point of that curve. The resulting dual graph is nothing but the case (C) in Proposition 8(2), thus we can blow down the strict transform to a line in $\mathbb{R}^2$.

(2) Case $\{x^3 = y^7\}$. By five simple blowings up, we reach a good resolution. The dual graph of the exceptional curves has linear five vertices and only the mid vertex is odd, which the strict transform of the original curve intersects. By two simple blowings up, each at the intersection of an end component and the next one, all the vertex becomes odd. Performing two rifle contractions, we arrive at the case (A).

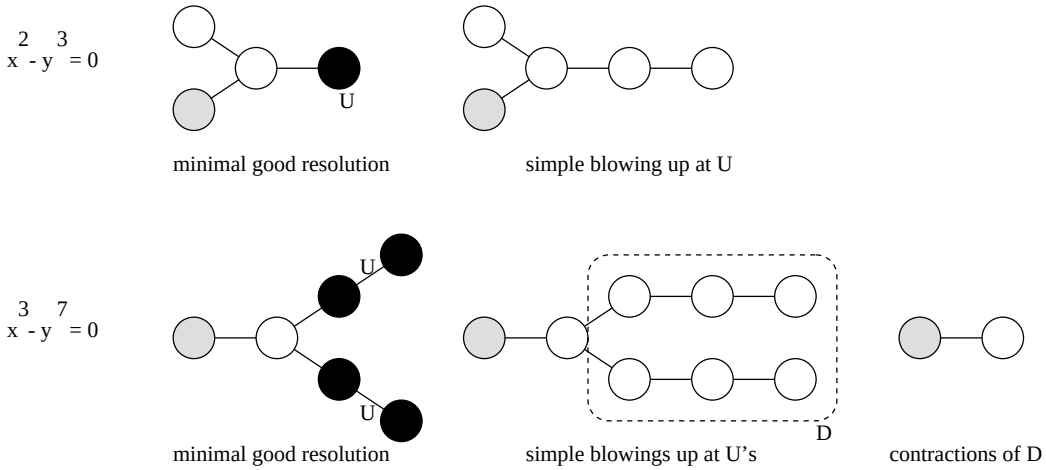

Fig. 3 Example

Thus in each case, the singular curve is mapped to a line in $\mathbb{R}^2$ by a blow-analytic

homeomorphism.

We have the following Paunescu-type result for $\mathbb{R}^2$.

Corollary 13. *A blow-analytic homeomorphism of* $\mathbb{R}^2$ *may not preserve the multiplicity of analytic arcs.*

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Department of Mathematics, Faculty of Science, Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo, 152, Japan

School of Mathematics and Statistics, University of Sydney
Sydney, NSW 2006, Australia