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トーリックのおもしろい例.

X: Smooth compact

(I) Voskresenskij - Klyachko mfd

(II) Ewald mfd

(III) Miyake oda

(IV) Fujino * Payne

(V) non quasi toric toric mfd

§1. 準備
§2 Fano
- surface で遊ぶ
- Hirzebruch surf

§3. non-proj

§1. 準備

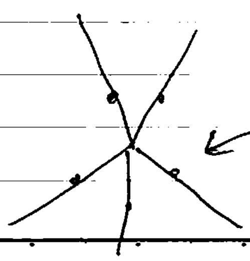
$X = X_\Sigma$ smooth. cop. mfd. d-fold

Σ: fan in $N = \mathbb{Z}^d$ $M, N_\mathbb{R}, \langle , \rangle, \cdots$

記号 $\Sigma(i) := \{ i$次元の cone $\}$

$G(\Sigma) := \{$ 1-dim cone の prim. gener. $\}$

$\sigma \in \Sigma$
$G(\sigma) := \sigma \cap G(\Sigma)$



$G(\Sigma)$

※ smooth $\Leftrightarrow$ $G(\sigma) \subset \mathbb{Z}$-basis $\forall \sigma \in \Sigma$

<u>Blow-up</u>

$\Sigma(i) \overset{1:1}{\longleftrightarrow}$ codim i の T-orbit

~~$\sigma \in \Sigma \Longleftrightarrow$~~

$\sigma \in \Sigma$ に対する T-inv. subvar $V(\sigma)$

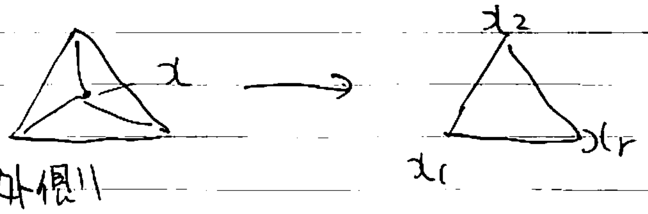
$\tilde{X} \to X$: blow-up along $V(\sigma)$

$\| $

$X_{\tilde{\Sigma}}$

$G(\sigma) = \{x_1, \dots, x_r\}$ $\quad x := x_1 + \dots + x_r$

$\tilde{\Sigma} :=$ σ を x で star-subdiv したもの



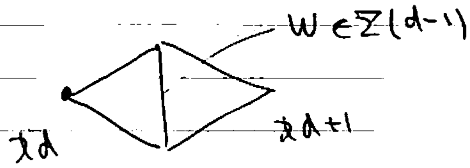
外側

toric 森理論

$$A_1(X) = \{ (a_x)_{x \in G(\Sigma)} \in \mathbb{Z}^{d(\Sigma)} \mid \sum_{x \in G(\Sigma)} a_x x = 0 \}$$

∘ wall

$w \in \Sigma(d-1) \longleftrightarrow$ toric curve



$$a_1 x_1 + \dots + a_{d-1} x_{d-1} + x_d + x_{d+1} = 0$$

∘ primitive relation

<u>Def</u> $P \subset G(\Sigma)$

P = prim collection

def i) P は Σ の cone を生成しない ($P \neq G(\sigma)$)

ii). P の proper subset は生成する
($P \setminus \{x\} = G(\sigma)$)

minimal non-face

P = p.c. $= \{x_1, \cdots, x_n\}$

$\exists!\ \sigma(P) \in \Sigma$ st. $x_1 + \cdots + x_m \in \mathrm{Relint}\,\sigma(P)$

$G(\sigma(P)) = \{y_1, \cdots, y_m\}$ とすると.

$x_1 + \cdots + x_m - (a_1 y_1 + \cdots + a_n y_n) = 0$
$(a_1, \cdots, a_n \in \mathbb{Z}_{>0})$

prim. relation という。

対応する 1-cycle を $r(P)$ と書く.
$\in A_1(X)$

$D = \sum_{x \in G(\Sigma)} h_x D_x$ toric divisor
($D_x \leftrightarrow$ 生成 x)

$\deg_D P := (D \cdot r(P)) = h_{x_1} + \cdots + h_{x_n} - (a_1 h_{y_1} + \cdots + a_n h_{y_n})$

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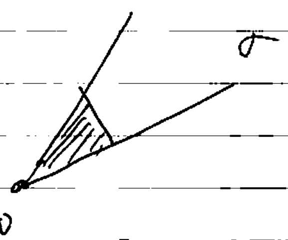
Thm. (Batyrev, Casagrande, Reid)

$$\underset{\text{森コーン}}{\overline{NE}(X)} = \sum_{w \in \Sigma(d-1)} \mathbb{R}_{\geq 0} w V(w)$$

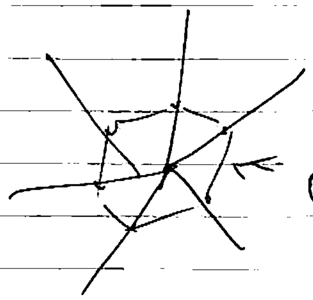
$$= \sum_{P : \text{P.C.}} \mathbb{R}_{\geq 0} r(P)$$

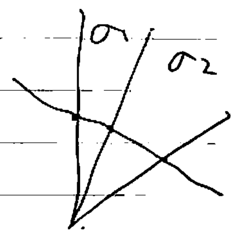
~~Def~~. Prop. (Fano)
X_Σ : fano ($\Leftrightarrow$ $-K_X$: ample)

$\Leftrightarrow$ $\sigma \in \Sigma$
$\Gamma_\sigma := \mathrm{Conv}(G(\sigma) \cup \{0\})$



$\bigcup_{\sigma \in \Sigma} \Gamma_\sigma$ が strictly convex





§1. Fano

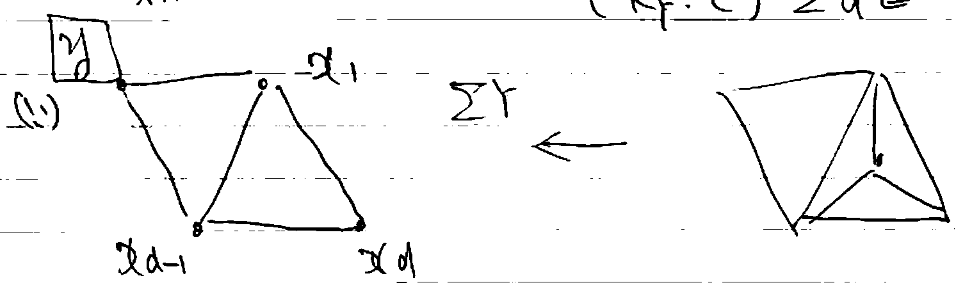
○ del Pezzo surf の分類
toric

$\Sigma_X \quad \Sigma_Y$

Prop. $X \longrightarrow Y$ toric b-u. at a pt P.
$\uparrow$ Fano　　　　$\uparrow$ σ とする.

X: Fano

$\Longleftrightarrow$ ${}^{\forall}C$ = toric curve $C \ni P$ に対して
s.t. $(-K_Y . C) \geq d$ ← 次元



$x := x_1 + \cdots + x_d$

$W = \langle x_1, \cdots, x_{d-1} \rangle$

$\updownarrow$

$y + x_d + a_1 x_1 + \cdots + a_{d-1} x_{d-1} = 0$

$W' =$

$y + x = y + x_1 + \cdots + x_d$

$= x_1 + \cdots + x_{d-1} - (a_1 x_1 + \cdots + a_{d-1} x_{d-1})$

$y + x + (a_1 - 1) x_1 + \cdots + (a_{d-1} - 1) x_{d-1} = 0$

$(-K \cdot V(w)) = \underbrace{2 + a_1 + \cdots + a_{d-1}}_{\geq d} - (d-1) > 0$

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$-K_X = X \setminus T = \sum_{i \in \Sigma(1)} D_i$

$\exists$ $p \rightsquigarrow$ subvar. V

p を通る $\rightsquigarrow$ $V \cap C \neq \emptyset$

$V \not\supset C$

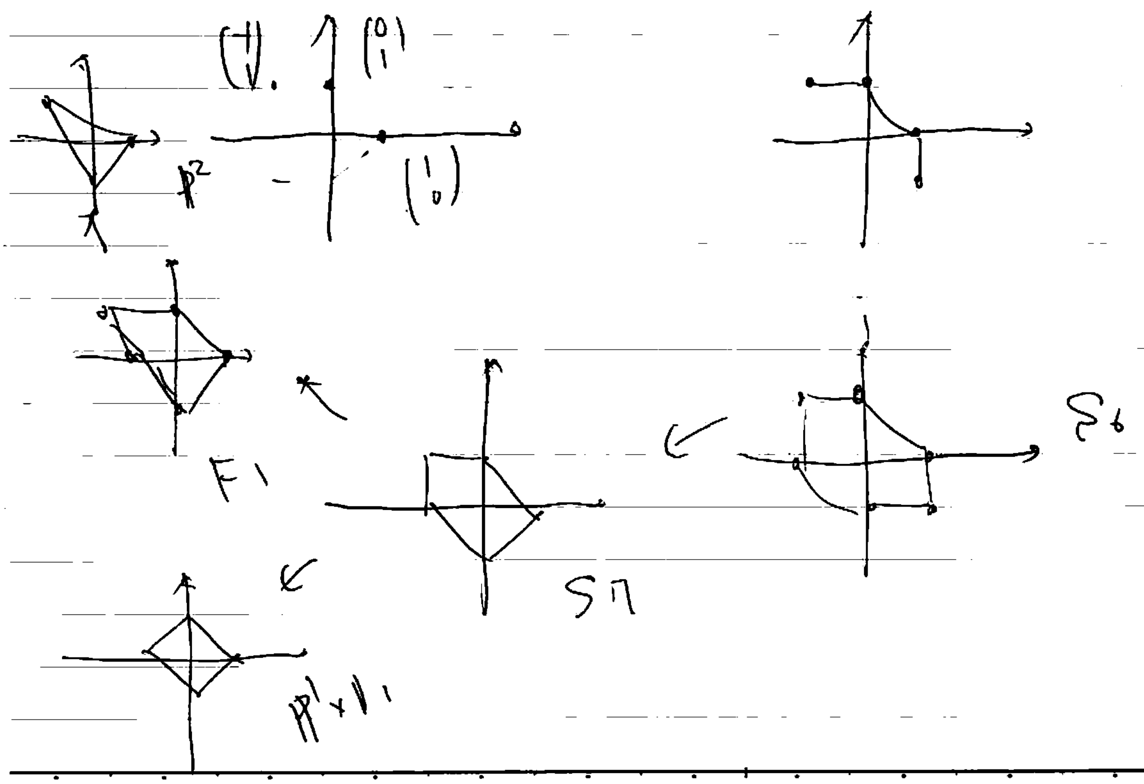
$(-K_X . C) \geq \operatorname{codim} V$ としたら どうか??

$Y \xrightarrow{\text{b.u.}} X$: sm. toric dP surf.

Y ~~X~~ が Fano にならない

$\Leftrightarrow$ $\forall p$: toric pt. $\exists C$: toric curve C. s.t. $(-K_X . C) < 2$

max dP を求める。



maximal $\underset{\text{Easy}}{\Longrightarrow}$ A は a Fano

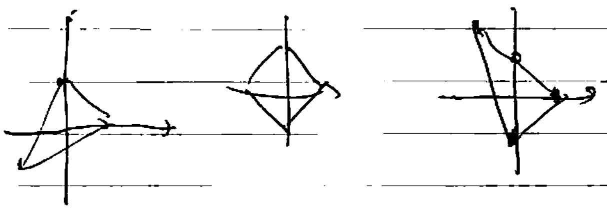
Q2 3-dim で $\nexists$ max $\Longrightarrow$ Fano
で分類せよ

X: Gor dP surf の分類

$\uparrow$ crepant resol

$\tilde{X}$: sm, week dP surf.
$\left(\bigcup_{\sigma \in \Sigma} \Gamma_\sigma : \text{convex} \right)$

これは、$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_2$ を blow-up していく.



$\Rightarrow$ 16個

Q3 $\exists$ 3-dim Gor. toric Fano var. の分類 (4319個)

$\Uparrow$ crepant resol

Sm, toric w.F 3-fold の分類

これを実行しろ!!

ex. ray を求めよ!!

4-dim でも. o.k. (1億個).

Hirzebruch 曲面

$F_a := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(a))$

$$\begin{tikzcd}[column sep=small]
& \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & S_7 \arrow[dl] \arrow[dr] & \\
F_5 & & F_4 & & F_3 & & F_2 & & F_1 & & \mathbb{P}_1 \times \mathbb{P}_1
\end{tikzcd}$$

$\underbrace{F_1 \quad \mathbb{P}_1 \times \mathbb{P}_1}_{\text{Fano 経由.}}$

これを拡張しよう!!

Def. $X = X_\Sigma$: proj

Σ : splitting fan

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall P_1 \neq P_2$: P.C. に対して. $P_1 \cap P_2 = \phi$

$\Longleftrightarrow$ $\exists X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n = X$

$\mathbb{P}^? {}^{\prime\prime}$

$X_i \leftarrow X_{i+1}$
toric $\mathbb{P}^?$ - bdl

Thm. Σ : splitting fan $X = X_\Sigma$

$\exists$

$$\begin{tikzcd}[column sep=small]
& \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & \cdots & & \cdot \arrow[dl] \arrow[dr] & \\
X \arrow[r, phantom] & & X_1 & & X_2 & & X_{n-1} & & X_n = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}
\end{tikzcd}$$

blow up. $X_1, \cdots, X_n$: splitting fan

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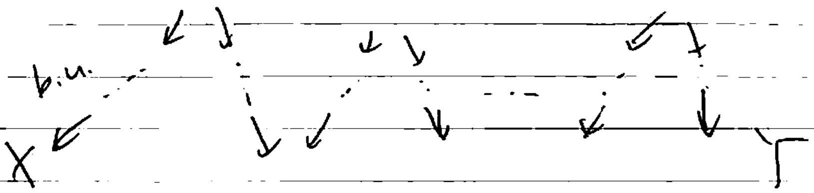
Cor $\rho(X) \le 3$. , X : Fano
Picard

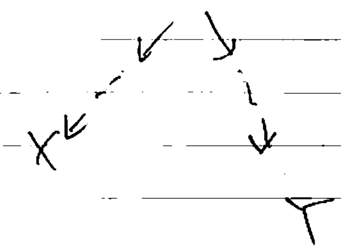
$\Rightarrow$ Fano 経由で 上がって下り, 下がって下り 行ける。
(w.F)

~~Rmk~~. X, Y — toric — d-fold



weak factorization

∂4 $\exists$?



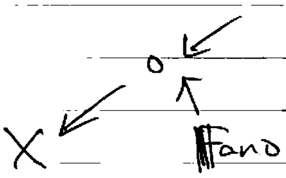
strong factorization

(Morelli $\Rightarrow$ Matsuki ‥‥

$\Rightarrow$ ○ $\Rightarrow$?).

remk. X : toric Fano d-fold.

$d \le 3 \Rightarrow \exists$



$\mathbb{P}^3$

$d = 4 \Rightarrow$ 唯一つの例外を除いて as above

$d = 5 \Rightarrow$?? $\#\{\mathbb{P}^5$ に行ける Fano$\} = 828$

∂5 $\mathbb{P}^6$ に行ける Fano を全て求めよ!!

Ex.

(I) Voskresenskij – Klyachko – mfd.

Thm. $X = X_\Sigma$: toric Fano

Δ : Σ に ass polytope

$= (\bigcup F_\alpha)$

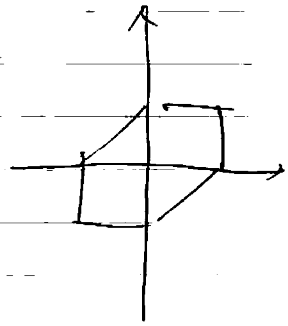
Δ が -1倍で不変 $\iff$ symmetric

$\iff X = (\mathbb{P}^1)^s \times V^{2k_1} \times \cdots \times V^{2k_m}$

V^{2k} : $2k$-dim V-K mfd $\qquad d = 2k$

$V^{2k} \overset{def}{\iff} G(\Sigma) = \{\pm e_1, \dots, \pm e_d, \pm(e_1 + \cdots + e_d)\}$

$d = 2$



max dP surf

$\mathbb{P}^1 \times \mathbb{P}^1$ を 2点 b-u.

Question

i) $k \geq 2$ でも V^{2k} は $(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$ の $2k$ 点 b-u か??

ii) V^d d = odd はどうしてダメか??

(II) Ewald mfd

$\exists\, \sigma_1, \sigma_2 \in \Sigma(d)$

s.t. $\sigma_1 = -\sigma_2$ $\qquad$ pseudo-symmetric

$\iff X = (\mathbb{P}^1)^r \times \tilde{V}^{2k_1} \times \cdots \times \tilde{V}^{2k_n} \times V^{2l_1} \times \cdots \times V^{2l_m}$

$\tilde{V}^{2k}$: $2k$-dim Ewald mfd

Σ

~~def~~ $\Leftrightarrow$ $G(\Sigma) = \{\pm e_1, \dots, \pm e_d, e_1 + \dots + e_d\}$

例) Oda – Miyake

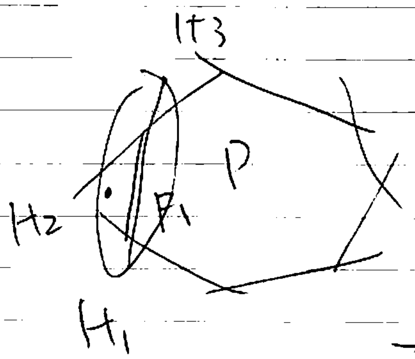
non projective sm. cpt. toric 3-fold で
一番かんたんな mfd

$\rho = 4$ の例　　　　Prop $\rho(X) \leq 3 \Rightarrow X$: proj.

X の fan を作る.

Schlegel 図式

$P \subset \mathbb{R}^n$ n-dim polytope
$\parallel$
$H_1^+ \cap H_2^+ \cap \dots \cap H_m^+$



F_1 を fix $(d-1)$ dim face

$p \in \mathrm{Int}(H_1 \cap H_2^+ \cap \dots \cap H_m^+)$

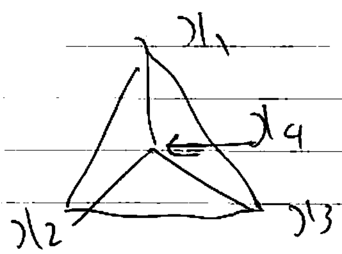
p を基点として F_1 に他の面を proj とする

$\Longrightarrow$ Schlegel 図式

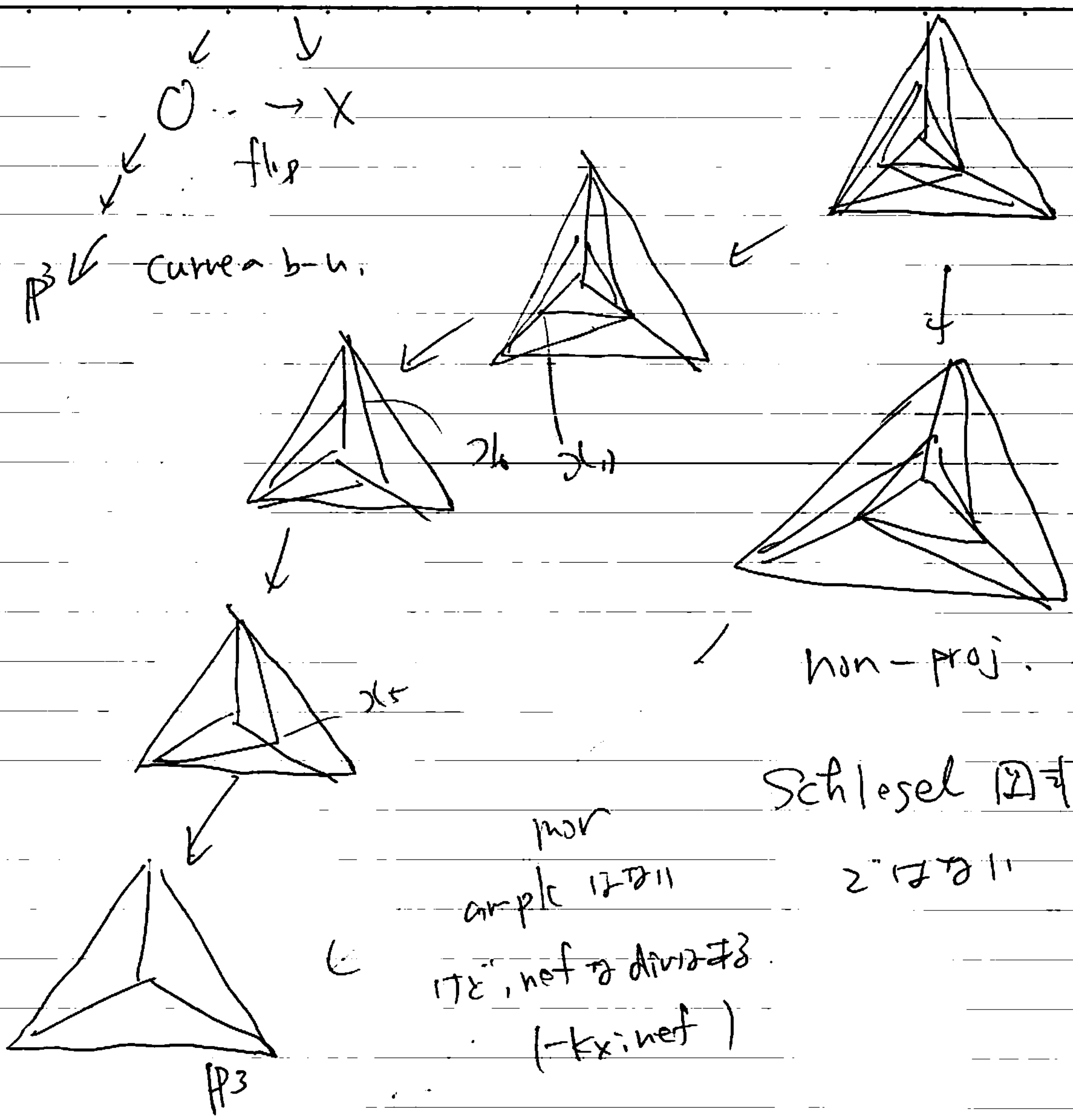
F の $(d-1)$ 図式
$\Leftrightarrow$ F_1 の ~~polytope~~ polyhedral $\mathcal{D}$
s.t. $F \subset F$. face　$F \in \mathcal{D}$

(ii) $\partial F \cap P$　　F の面

$\mathbb{P}^3 \longleftrightarrow$



$G(\Sigma)$
$= \{e_1, e_2, e_3, -(e_1 + \dots + e_3)\}$



X の non proj を 2つの方法で見る.

① Prop (Batyrev)
X: proj d-fold

$\Rightarrow x_1 + \cdots + x_m = 0$ なる p.r. が存在する.

Question
X について p.r. を全て求めて $\cdots = 0$ なる p.r. が無い事を示せ!!

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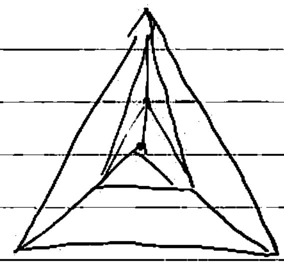
② ···· →

flop ← extray にassにする

X: proj.

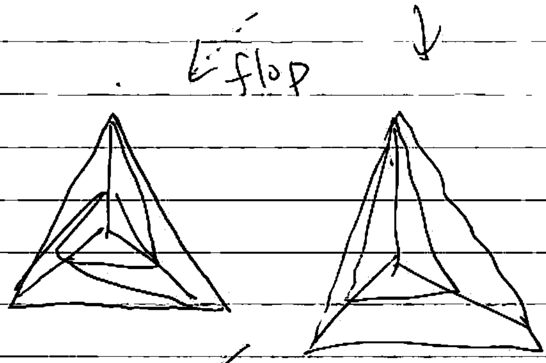


$P: x_3 + x_7 - (x_2 + x_5) = 0$

pr.

$P_1: x_1 + x_7 = (x_2 + x_6) = 0$

flop



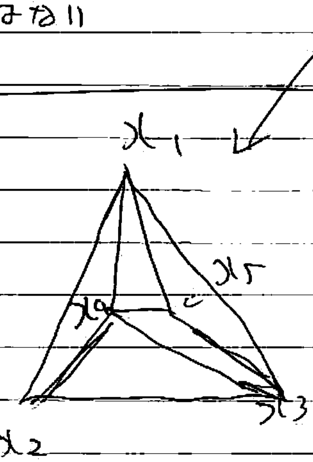
$P_2: x_3 + x_4 - (x_1 + x_5) = 0$

$r(P) = r(P_1) + r(P_2)$

①

$A_1(-)$ extremal ではない

$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$



$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_5 = x_3 + x_4 \end{cases}$$

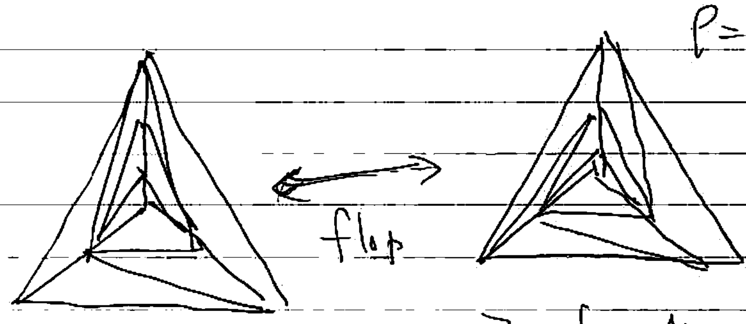
(IV) Fujino Payne

nef div が無い sm. cpt. toric

(non trivial)

$\rho=5$

$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(2))\oplus\mathcal{O}(2))$



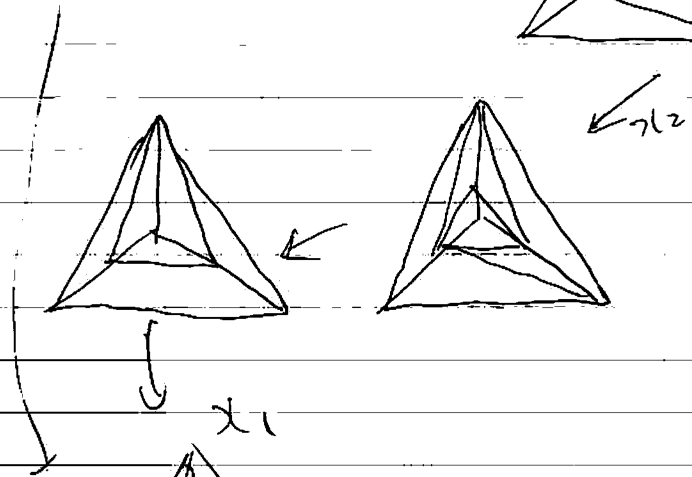
∃nef dim

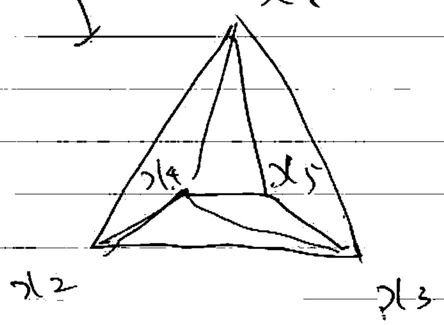
⇕

p.r. を求めて,

$\deg_{\phi} P$ を計算する.

$\overset{?}{\text{nef}}$ ⇒ 0 しか ない.





$x_1+x_3+x_4=0$

$x_2+x_5=2x_3+2x_4$

∂ $\rho=4 \Rightarrow \exists$nef ??

[例] non quasi toric toric mfd

"Def" M: 2d-dim manifold

$(S^1)^k \curvearrowright M$

$M/(S^1)^k$ = simple polytope な時.

M: quasi-toric mfd

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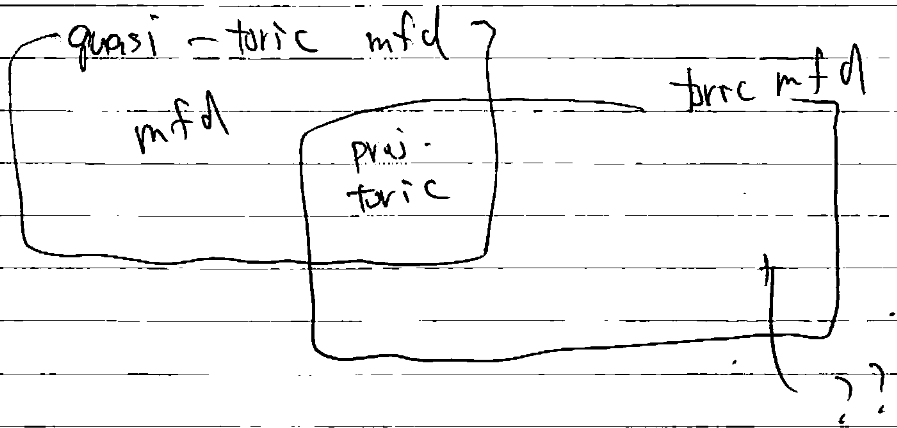
M : toric $\Rightarrow (\mathbb{C}^{\times})^d = M/(S^1)^d$

$\quad \| \qquad\qquad\qquad = \text{tropical toric}$

$(\mathbb{R}_{\geq 0} \times S^1)^d$

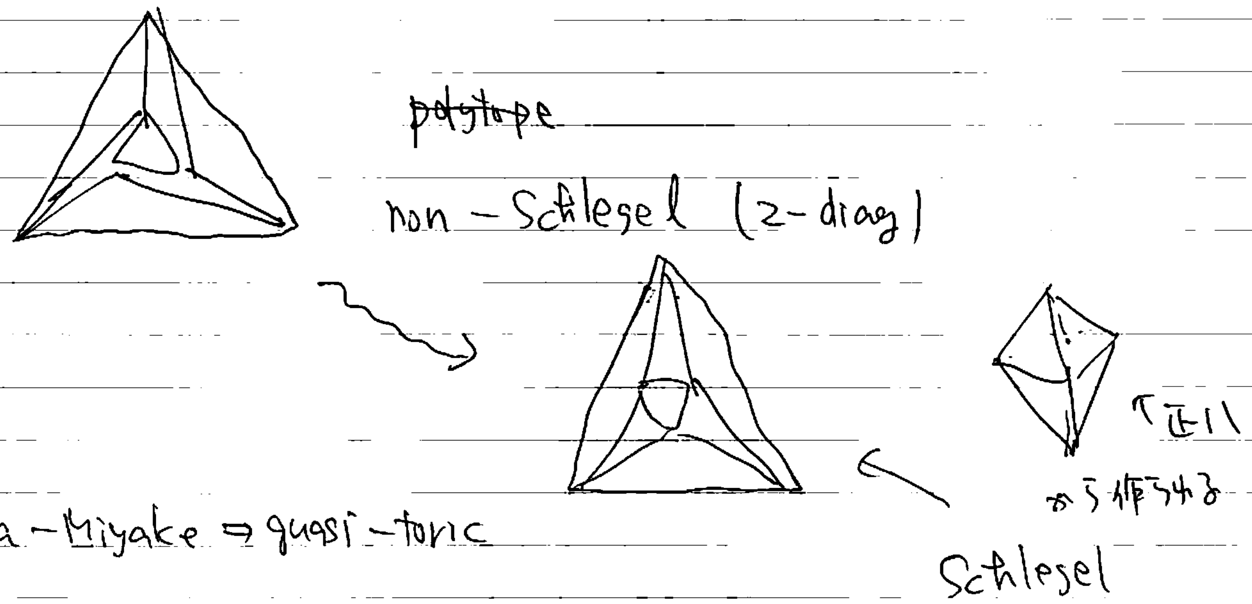


X : cpt toric mfd

$X = X_\Sigma$: proj $\Leftrightarrow$ Σはある polytope の面を伸ばして出来たもの.

$X = X_\Sigma$. quasi-toric

$\Leftrightarrow$ Σは polytope から出来るものに, 組み合わせ同値.



Oda-Miyake $\Rightarrow$ quasi-toric

Fujino-Payne $\Rightarrow$ quasi-toric

Prop. 2-diag $\Rightarrow$ Schlegel

3-dim toric $\Rightarrow$ quasi toric

Schlegel に同値でない

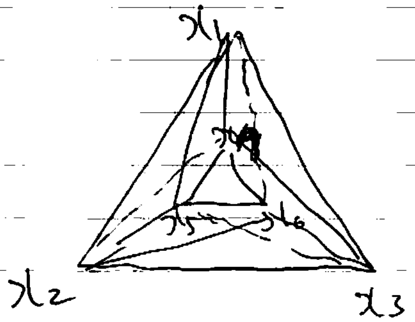
4次元 (Civan)

Ex. まずは、non-polytopal な 3-diagram を用意

Barnette sphere

$\subset \mathbb{R}^4$ 3-diag の base とする.

x_1, x_2, x_3, x_4
で張られる 4-simplex



この中に x_8 をとって simplicial に細分

non-polytopal

$\Leftrightarrow$ ~~F~~ $\{x_1, x_2, x_5, x_7\}$ simplex $= F$

F を base にして、3-diagram は作れない!!

つまり、Barnette sphere は non polytopal

① B. Sphere $\Rightarrow$ fan
??

② resol する. $\Rightarrow$ 出来る!!

Q ちゃんと smooth な fan を作れ!!

Q もっと例を作れ!!

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