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準楕円曲面のファイバー積から得られるCalabi-Yau多様体について

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（瀧 真語 記
（名大多元））

Some Calabi-Yau 3-folds in positive characteristic

Joint work with M. Hirokado
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（まとめて H.I.S.）

以下

$k=\bar{k}$, $\mathrm{char}(k)=p>0$

正標数特有の性質をもったものを構成したい.

- non-liftable ($\nexists \mathcal{X}$ over DVR w res. field k. s.t. $\mathcal{X} \otimes k \simeq X$)
- "Strange" fibration structure.
  (ex. total sp. smooth, general fiber has sing)

1. Construction

Schoen's construction ($/\mathbb{C}$) 1988

$\varphi_i : Y_i \longrightarrow \mathbb{P}^1$ $(i=1,2)$ (semistable rational elliptic surf.)

$$\begin{tikzcd} & Y_1 \times_{\mathbb{P}^1} Y_2 \subset Y_1 \times_k Y_2 \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[dr, "\varphi_1"'] & & Y_2 \arrow[dl, "\varphi_2"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

$Y_1 \times_{\mathbb{P}^1} Y_2 \in |-K_{Y_1 \times_k Y_2}|$

$\therefore K_{Y_1 \times_{\mathbb{P}^1} Y_2} \sim 0.$

準楕円曲面 $\varphi : Y \longrightarrow C$

一般ファイバー : rational curve with one ordinary cusp

(exist only in char 2.3)

Hirokado (2001)

$\begin{cases} \varphi_1 : Y_1 \longrightarrow \mathbb{P}^1 & \text{elliptic surface} \\ \varphi_2 : Y_2 \longrightarrow \mathbb{P}^1 & \text{quasi-elliptic surface.} \end{cases}$

Main Objects C.Y. 3-folds constracted by Schoen's method using two rational quasi-elliptic surfaces $\varphi_i : Y_i \longrightarrow \mathbb{P}^1$ with sections

Difficulties : $\mathrm{Sing}(Y_1 \times_{\mathbb{P}^1} Y_2)$: complicated

Calculation of invariants.

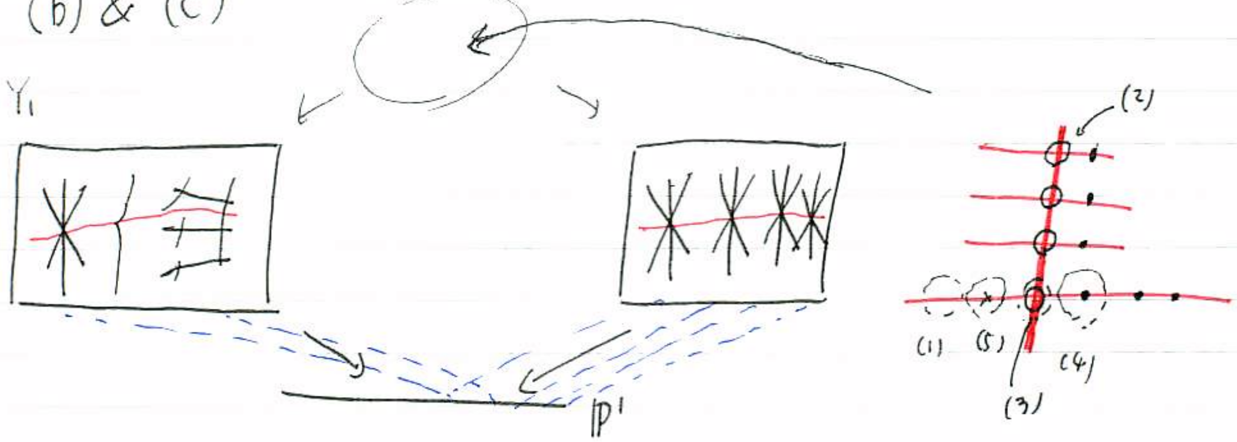
<u>Thm</u> Classification of rational quasi-elliptic surfaces w. sections

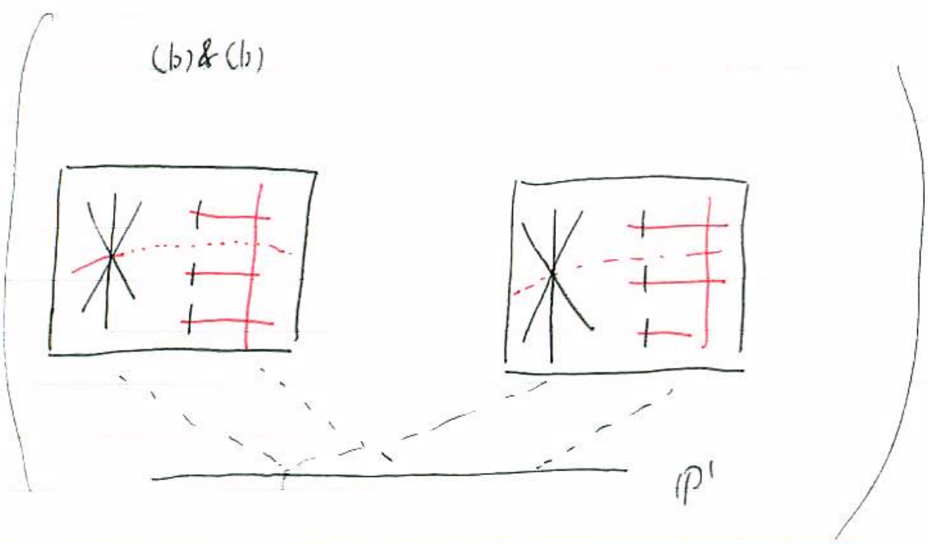
($p=3$)

	Weierstrass form	deg. fibers	
(a)	$y^2 = x^3 + t$	II^*	
(b)	$y^2 = x^3 + t^2$	IV IV^*	
(c)	$y^2 = x^3 + t^4 + t^2$	Four IV's	

($p=2$ は たいへん.)
(a) ~ (g)

(b) & (c)





Prop (1) [Bombieri-Mumford 76]

$f : Y \longrightarrow C$ q. elliptic in char 3

t: local parameter on C

If the fiber over $t=0$ is general, then $\exists$ formal coordinates

x, y on Y s.t. $t = y^2 + x^3$ $\longleftarrow$ $t = u(y^2+x^3)$

正標数!!

(2) If : is of type IV, $t = xy^2 - x^3$

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(1) $y^2 + x^3 = z^2 + w^3 \longrightarrow x^3 + y^2 + z^2 = 0$

(5) $xy^2 - x^3 = zw^2 - z^3 \rightsquigarrow x^3 + xy^2 + y^2 z + zw^2 = 0$

(4) $y^2 + x^3 = zw^2 - z^3 \rightsquigarrow x^3 + y^2 + z^2 w = 0.$

(2) $x^3 + y^2 + z^3 w = 0$ ← おもしろい性質有.

(3) $x^3 + y^2 + z^3 w^2 = 0$

(1)

$$\begin{cases} x = x \\ y = xy \\ z = xz \end{cases}$$

$$x^2[x+y^2+z^2]=0$$

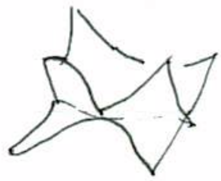
$$E=\{x=0,\ y^2+z^2=0\}$$

(4)

$$\begin{cases} x = x \\ x = xy \\ z = xz \end{cases}$$

$$x^2[x+y^2+z^2w]=0$$

$$E=\{x=0,\ y^2+z^2w=0\}$$



(5) $x^3+xy^2+y^2w+z^2w=0$

$$\begin{cases} x = x \\ y = xy \\ z = xz \end{cases}$$

$$x^2[x+xy^2+wy^2+z^2w]=0$$

$$d \hookrightarrow (1+y^2)dx+2(xy+wy)dy$$
$$+2zw\,dz+(y^2+z^2)dw$$

$$E=\{x=0,\ w(y^2+z^2)=0\}$$

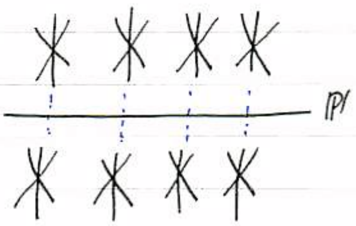
ODP 6E

2. Calculation of invariants

(b_3 を知りたい。そのため.
Euler 数や Picard 数を調べる)
たいへん.

$$\begin{tikzcd}
 & & & X \arrow[d] & \\
\tilde{\tilde{Z}} \arrow[r] \arrow[d] & \tilde{Z} \arrow[rr, "p.i."] \arrow[dd] & & Y_1 \times_{\mathbb{P}^1} Y_2 \arrow[dl] \arrow[dr] & \\
(\mathbb{P}^1\times\mathbb{P}^1)\text{-bl} \arrow[dr] & & Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\
 & \mathbb{P}^1 \arrow[rr, "F"] & & \mathbb{P}^1 &
\end{tikzcd}$$

	Case (b) & (b)	Case (c) & (c)
$e(X)$	72	84
$\rho(X)$	35	41



$$\Rightarrow b_3(X) = 0 \qquad \left(\text{c.f. In char } 0.\ b_3 = \sum_{i+j=3} h^i(\Omega^j) > 0\right)$$

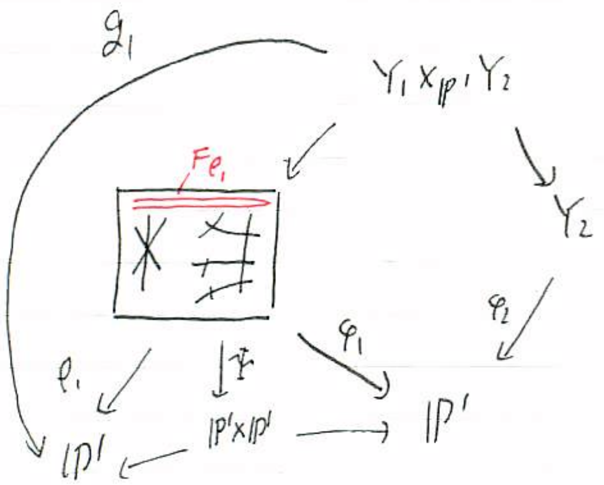
最初に書くつもりだった.

For C.Y. 3-folds $\exists$ non-liftable C.Y. in $p=2,3$ (Hirokado '99, Schröer '04)

(c.f. K3 surface : liftable (in char p) (Deligne, 1981))

3 Supersingular K3 fibrations

$$\begin{tikzcd}
 & & X \arrow[d] & & \\
 & & Y_1 \times_{\mathbb{P}^1} Y_2 \arrow[dl] \arrow[dr] & & \\
 & Y_1 \arrow[dl, "\rho_1"'] \arrow[dr, "\varphi_1"] & & Y_2 \arrow[dl, "\varphi_2"'] \arrow[dr, "\rho_2"] & \\
\mathbb{P}^1 & & \mathbb{P}^1 & & \mathbb{P}^1
\end{tikzcd}$$



canonical bundle formula

$$K_{Y_1} \sim -F_{\varphi_1}$$

$$\Rightarrow F_{\varphi_1} \cdot F_{\varphi_2} = 2$$

$$\begin{tikzcd}
F_{g_1} \arrow[r] \arrow[d] & Y_2 \arrow[d] \\
F_{\rho_1} \arrow[r, "\varphi_1|F_{\rho_1}", "2:1"'] & \mathbb{P}^1
\end{tikzcd}$$

$Y_2 \times F_{\rho_1}$

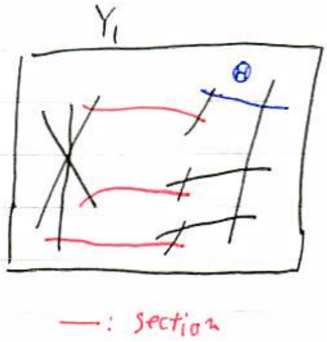
$$F_{\rho_1} \in |-K_{Y_2} \boxtimes K_{\mathbb{P}^1}|$$

OHP.

double cover $\varphi_1|_{F_{\rho_1}} : F_{\rho_1} \longrightarrow \mathbb{P}^1$ is ramified at 2pts (← Hurwitz)

$\Rightarrow$ $Y_1 \longrightarrow \mathbb{P}^1\times\mathbb{P}^1$ has ramification locus

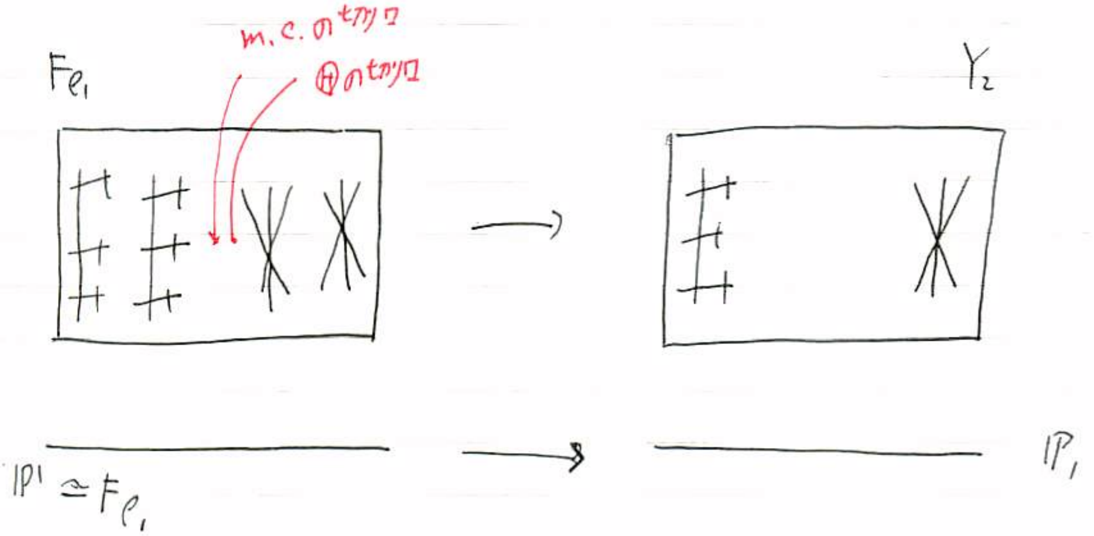
Φ^*(fiber of $pr : \mathbb{P}^1\times\mathbb{P}^1 \longrightarrow \mathbb{P}^1$) $= 2\Theta$



$\varphi_1|_{F_{\rho_1}} : F_{\rho_1} \longrightarrow F$: double cover $\Rightarrow F_{\rho_1}\cdot\Theta = 1$

This means Θ is ram. locus

moving cusp is also .



Crepant resolution $X \longrightarrow Y_1 \times_{\mathbb{P}^1} Y_2$

$\Rightarrow$ fiber of $\bar{g}_1 : X \longrightarrow \mathbb{P}^1$ is a smooth super singular K3

$$\begin{tikzcd}
& & X \arrow[d] \arrow[ddll, bend right, "\bar{g}_1"'] & & \\
& & Y_1\times Y_2 \arrow[dl, "\varphi_1"'] \arrow[dr] & & \\
& Y_1 \arrow[dl, "\rho_1"'] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\
\mathbb{P}_1 & & \mathbb{P}^1 & & \mathbb{P}^1
\end{tikzcd}$$