How many Nambu-Goldstone modes?

H.B. Nielsen and S. Chadha, Nucl. Phys. B105, 445 (1976)

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Nambu-Goldstone theorem

Spontaneous symmetry breaking <--> massless particles

Y. Nambu, G. Jona-Lasinio, Phys. Rev. 122, 345, (1961)
J. Goldstone, Nuovo Cim. 19, 154 (1961)
J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. 127, 965 (1962)

With Lorentz invariance,

- # of NG bosons = # of broken generators
- Ex.) $SU(N_f)_R \otimes SU(N_f)_L \to SU(N_f)_V$: π (K, η) in QCD

Without Lorentz invariance,

- # of NG modes ≤ # of broken generators
- condensed matter physics
- QCD at finite density
- Ex.) SU(2) -> U(1): magnon(s) in (anti-)ferromagnetism

How many NG modes in general?

Theorem

G: **symmetry group of a Lagrangian** L **with** n **parameters.**
Q_a: **conserved charges** (a=1,...,n)

- **There exist** m **fields** Φ_i **and a vacuum state** |0>, **s.t.,**

$$\det\langle 0|[\Phi_i, Q_a]|0\rangle \neq 0, \quad i, a = 1, ..., m$$

- **For local operators** A **and** B **(no long range force),**

$$|\boldsymbol{x}| \to \infty : |\langle 0|[A(\boldsymbol{x}, t), B(0)]|0\rangle| \to e^{-\tau|\boldsymbol{x}|}, \quad \tau > 0$$

- **Translational invariance is not entirely broken.**

Then,

$$n_I + 2n_{II} \geq m$$

- m: **# of broken generators**
- n_I: **# of type I NG modes** (E ~ **odd powers of** p)
- n_{II}: **# of type II NG modes** (E ~ **even powers of** p)

H.B. Nielsen, S. Chadha, Nucl. Phys. B105, 445-453, (1976)

Three steps of the proof

(i) minimal # of NG modes

$$m/2 \leq l$$

(ii) existence of type II NG modes

$$\text{If } m/2 \leq l < m, \text{ then } n_{II} > 0$$

(Key notion: linear dependence of NG modes)

(iii) # of type II NG modes

$$n_{II} = m - p$$

- **m: # of broken generators**
- **l: total # of NG (massless) modes** ($l = n_I + n_{II}$)
- **p: rank of matrix** $\underline{\nu}$

From Eq.(12),

$$l - p \geq 0$$

$$l + (m - p) \geq m$$

$$[l - (m - p)] + 2(m - p) \geq m$$

$$n_I + 2n_{II} \geq m$$

(If Eq.(12) is equality, so is the final result.)

Step (i)

Spectral decomposition of order parameter

Consider the following quantity (m×m matrix):

$$M_{ia} \equiv \langle 0 | [\Phi_i, Q_a] | 0 \rangle$$

- Q_a: **spatial integration of current** j_a^0
- **Insert complete set**
- **translational invariance of vacuum**

$$= \sum_{n=1}^{l} \left\{ e^{-iE_{\boldsymbol{k}}t} \langle 0 | \Phi_i | n_{\boldsymbol{k}} \rangle \langle n_{\boldsymbol{k}} | j_a^0 | 0 \rangle - e^{iE_{-\boldsymbol{k}}t} \langle 0 | j_a^0 | n_{-\boldsymbol{k}} \rangle \langle n_{-\boldsymbol{k}} | \Phi_i | 0 \rangle \right\} \Bigg|_{\boldsymbol{k}=0} \quad (6)$$

- |n>: **only l massless (NG) modes** **<-- time derivative of (6)**

$$= \sum_{n=1}^{l} \left\{ \langle 0 | \Phi_i | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | j_a^0 | 0 \rangle - \langle 0 | j_a^0 | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | \Phi_i | 0 \rangle \right\}$$

$$= 2i \ \mathrm{Im} \left(\sum_{n=1}^{l} \langle 0 | \Phi_i | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | j_a^0 | 0 \rangle \right) \equiv 2i \ \mathrm{Im} \ \underline{\nu}_{ia} \quad (8)$$

Broken symmetry condition: $\det M \neq 0$

$$\det \left(\mathrm{Im} \ \underline{\nu} \right) \neq 0 \quad (9), \qquad \mathrm{rank} \left(\mathrm{Im} \ \underline{\nu} \right) = m \quad (10)$$

Step (i)

Rank of $\underline{\nu}$ and Im $\underline{\nu}$

The explicit form of the matrix $\underline{\nu}$

$$\underline{\nu} = \begin{pmatrix} \sum_{n=1}^{l} \langle 0 | \Phi_1 | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | j_1^0 | 0 \rangle & \sum_{n=1}^{l} \langle 0 | \Phi_1 | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | j_2^0 | 0 \rangle & \cdots & & \cdots \\ \sum_{n=1}^{l} \langle 0 | \Phi_2 | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | j_1^0 | 0 \rangle & \ddots & & \boldsymbol{v}_a & \\ \vdots & & & & \end{pmatrix}$$

a-th column vector of $\underline{\nu}$ (a=1,...,m)

$$\boldsymbol{v}_a = \sum_{n=1}^{l} \gamma_{an} \boldsymbol{A}_n \quad (13), \qquad \gamma_{an} = \langle n_{\mathbf{0}} | j_a^0 | 0 \rangle, \qquad \boldsymbol{A}_n = \begin{pmatrix} \langle 0 | \Phi_1 | n_{\mathbf{0}} \rangle \\ \vdots \\ \langle 0 | \Phi_m | n_{\mathbf{0}} \rangle \end{pmatrix} \quad (11)$$

is expressible by linear combination of A_n (n=1,...,l).

rank: # of linear independent column vectors

$$\text{rank } \underline{\nu} \equiv p \leq l \quad (12)$$

(Equality <-- all A_n are independent.)

Imaginary part of $\underline{\nu}$

$$\text{Im } \boldsymbol{v}_a = \sum_{n=1}^{l} \text{Re } \gamma_{an} \text{ Im } \boldsymbol{A}_n + \sum_{n=1}^{l} \text{Im } \gamma_{an} \text{ Re } \boldsymbol{A}_n \quad (14)$$

Rank of Im $\underline{\nu}$ ≤ 2l --> m/2 ≤ l

Linear relations

What happens if l < m**?**

$$\text{rank}\ \underline{\nu} = p \leq l < m$$

Only p **vectors are linearly independent among** $\mathbf{v}_a$ (a=1,...,m)

--> There are m-p **linear relations:**

$$\sum_{a=1}^{m} C_a^{\alpha} \boldsymbol{v}_a = \mathbf{0}, \quad \alpha = 1, \ldots, m-p \quad (15)$$

On the other hand, for any α**, there exist at least one** i **s.t.**

$$\langle\, 0\, |[\Phi_i, \sum_{a=1}^{m} C_a^{\alpha} Q_a]|\, 0\,\rangle = \sum_{a=1}^{m} C_a^{\alpha} M_{ia} \neq 0$$

<-- if not, rank M < m

Spectral decomposition: there must be at least one massless state which couples to $\Sigma C_a^{\alpha} j_a^0$

What dispersion?

Step (ii)

Dispersion of the NG mode

Assumption: commutator exponentially vanishes

$$|\langle 0 | [A(\boldsymbol{x},t), B(0)] | 0 \rangle| \to e^{-\tau|\boldsymbol{x}|} \quad \text{for} \quad |\boldsymbol{x}| \to \infty$$

--> Fourier transform is analytic (differentiable)

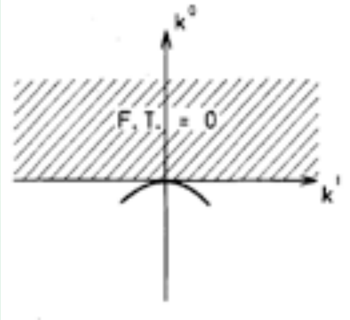
Consider 4-d Fourier transform of $\langle 0 | [\Phi_i, \sum_a C_a^\alpha j_a^0(x)] | 0 \rangle$

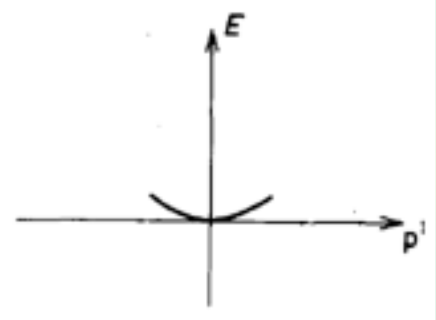
- (15) and analyticity
- small momentum

$$\mathrm{FT}(\boldsymbol{k} \simeq 0) = -2\pi \sum_{n=1}^{l} \left[\delta(k^0 + E_{\boldsymbol{k}}) \sum_{a=1}^{m} C_a^\alpha \langle 0 | j_a^0 | n_{-\boldsymbol{k}} \rangle \langle n_{-\boldsymbol{k}} | \Phi_i | 0 \rangle \right]_{\boldsymbol{k} \simeq 0} \qquad (20)$$

- For $k^0 \neq 0$**, FT is nonzero only for** $k^0 < 0$ **because** $E>0$
- For $k^0 = 0$**,** $\mathbf{k} = \mathbf{0}$ **(massless mode)**
- analytic in k

Dispersion is quadratic!





Comments: Lorentz invariant case

In the textbook of relativistic quantum field theory:

$$l = m$$

Proof:

If $l<m$**, linear combinations of** Σj_a^μ **do not couple to NG mode.**

--> Some broken charges Q_a **are well defined.**

--> Contradiction to broken symmetry condition!

Nielsen-Chadha:

Q_a **are well-defined, but** Σj_a^0 **couples to a massless mode.**

--> only possible with broken Lorentz invariance.

--> type II NG mode does not couple to Σj_a^μ

Lorentz invariance:

Only linear dispersion ($E \sim p$**) is allowed for massless mode.**

--> no type II NG mode.

--> $l = m$

Step (iii)

Number of type II NG modes

How many type II NG modes?

$$\boldsymbol{\rho}^\alpha = \sum_{a=1}^{m} C_a^{\alpha *} \boldsymbol{v}_a, \quad \alpha = 1, ..., m-p$$

1) ρ$^\alpha$ is not a zero vector,
2) {ρ$^\alpha$} are linearly independent, and
3) ρ$^\alpha$ couples to a type II NG mode.

1) Imaginary part of Eq. (15) should vanish:

$$\mathrm{Im} \sum_{a=1}^{m} C_a^\alpha \boldsymbol{v}_\alpha = \sum_{a=1}^{m} \mathrm{Re}\ C_a^\alpha\ \mathrm{Im}\ \boldsymbol{v}_a + \sum_{a=1}^{m} \mathrm{Im}\ C_a^\alpha\ \mathrm{Re}\ \boldsymbol{v}_a = \mathbf{0}$$

$$\Rightarrow \sum_{a=1}^{m} \mathrm{Re}\ C_a^\alpha\ \mathrm{Im}\ \boldsymbol{v}_a = -\sum_{a=1}^{m} \mathrm{Im}\ C_a^\alpha\ \mathrm{Re}\ \boldsymbol{v}_a$$

Imaginary part of ρ$^\alpha$

$$\mathrm{Im}\ \boldsymbol{\rho}^\alpha = \sum_{a=1}^{m} \mathrm{Re}\ C_a^\alpha\ \mathrm{Im}\ \boldsymbol{v}_a - \sum_{a=1}^{m} \mathrm{Im}\ C_a^\alpha\ \mathrm{Re}\ \boldsymbol{v}_a = 2\sum_{a=1}^{m} \mathrm{Re}\ C_a^\alpha\ \mathrm{Im}\ \boldsymbol{v}_a \neq \mathbf{0}$$

--> ρ$^\alpha$ is not a zero vector

Step (iii)

Number of type II NG modes

2) If {ρ$^\alpha$} are linearly dependent, then

$$\begin{aligned}\mathbf{0} &= \sum_{\alpha=1}^{m-p} \beta_\alpha \boldsymbol{\rho}^\alpha - \sum_{\alpha=1}^{m-p} \beta_\alpha^* \boldsymbol{\rho}^{\alpha *} \\ &= \sum_{\alpha=1}^{m-p} \beta_\alpha \sum_{a=1}^{m} C_a^{\alpha *} \boldsymbol{v}_a - \sum_{\alpha=1}^{m-p} \beta_\alpha^* \sum_{a=1}^{m} C_a^{\alpha} \boldsymbol{v}_a^* \\ &= \sum_{a=1}^{m} \left[\sum_{\alpha=1}^{m-p} [\beta_\alpha C_a^{\alpha *} - \beta_\alpha^* C_a^\alpha] \right] \mathrm{Re}\, \boldsymbol{v}_a + \sum_{a=1}^{m} \left[\sum_{\alpha=1}^{m-p} [\beta_\alpha C_a^{\alpha *} + \beta_\alpha^* C_a^\alpha] \right] \mathrm{Im}\, \boldsymbol{v}_a\end{aligned}$$

This contradicts to det im $\underline{v} = 0$ **--> {ρ$^\alpha$}: linearly independent**

3) Spectral decomposition

$$(\boldsymbol{\rho}^\alpha)_i = \sum_{n=1}^{l} \langle 0 | \Phi_i | n_{\mathbf{0}} \rangle \langle n_{\mathbf{0}} | \sum_{a=1}^{m} C_a^{\alpha *} j_a^0 | 0 \rangle \quad (24)$$

--> ρ$^\alpha$ couples to a type II NG mode

There are m-l **type II NG modes.**

Summary of proof

(i) minimal # of NG modes

$$m/2 \leq l$$

(ii) existence of type II NG modes

$$\text{If } m/2 \leq l < m, \text{ then } n_{II} > 0$$

(iii) # of type II NG modes

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(Key notion: linear dependence of NG modes)

- m: # of broken generators
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- p: rank of matrix $\underline{\nu}$

From Eq.(12),

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$$l + (m-p) \geq m$$

$$[l - (m-p)] + 2(m-p) \geq m$$

$$n_I + 2n_{II} \geq m$$

(If Eq.(12) is equality, so is the final result.)

Recent developments

Commutator of conserved charges

T. Schafer, D.T. Son, M.A. Stephanov, D. Toublan, J.J.M. Verbaarshot
Phys. Lett. B522, 67, (2001)

$$\langle 0 | [Q_i, Q_j] | 0 \rangle = 0 \quad \text{for all } i, j \quad \Rightarrow \quad m = l$$

Another inequality

H. Watanabe, T. Brauner, Phys. Rev. D84, 125013 (2011); D85, 085010 (2012)

$$m - l \leq \frac{1}{2} \text{rank} \langle 0 | [Q_i, Q_j] | 0 \rangle$$

Complete proofs

- effective Lagrangian approach

H. Watanabe, H. Murayama, Phys. Rev. Lett. 108, 251602 (2012)

- Mori's projection operator method

Y. Hidaka, arXiv:1203.1494 [hep-th]

$$m - l = \frac{1}{2} \text{rank} \langle 0 | [Q_i, Q_j] | 0 \rangle = n_{II}$$

saturation of inequality!