Determination of the a_0-a_2 Pion Scattering Length from $K^+ \to \pi^+\pi^0\pi^0$ Decay

N. Cabibbo, Phys. Rev. Lett. 93, 121801 (2004)
(see also N. Cabibbo and G. Isidori, JHEP 03, 021 (2005))

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Contents

Scattering length

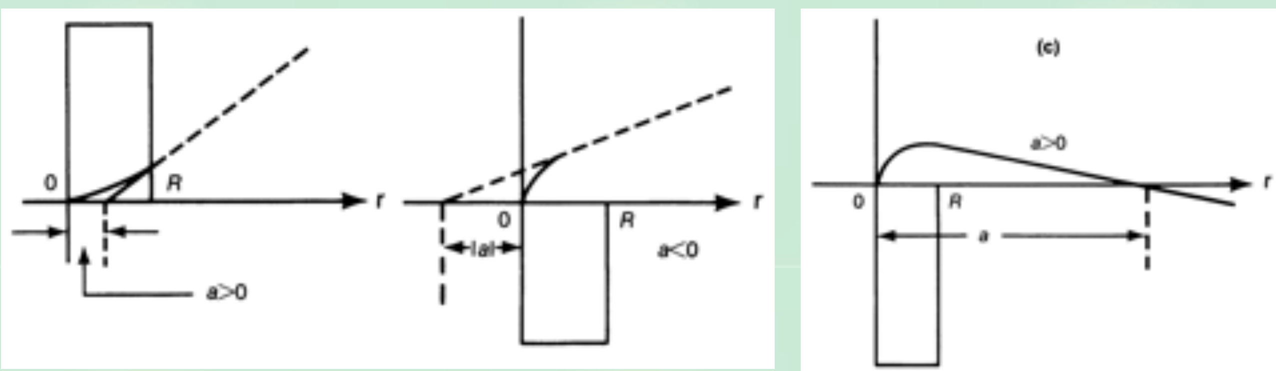
Scattering length: amplitude at threshold

- characterizes the low energy scattering
- changes the sign if there is a bound state

$$a = -f(k)|_{k\to 0}$$

$$\frac{d\sigma}{d\Omega} = |f(k)|^2, \quad \lim_{k\to 0}\sigma(k) = 4\pi a^2$$

$$f = \frac{1}{k\cot\delta_0 - ik}, \quad k\cot\delta_0 = -\frac{1}{a} + r_e\frac{k^2}{2} + \dots$$



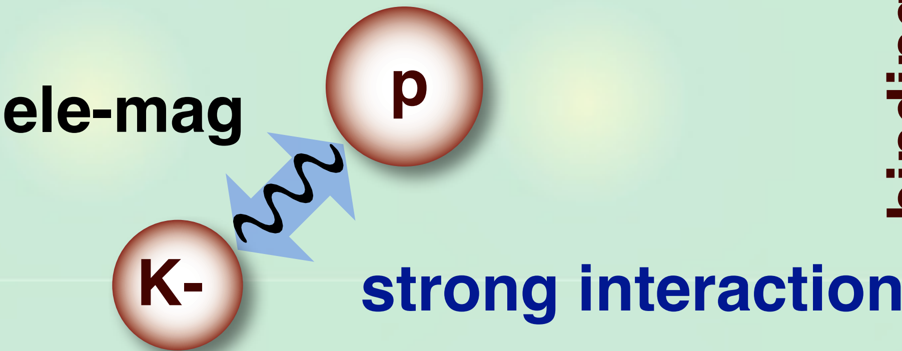
J.J. Sakurai, Modern Quantum Mechanics, p. 415

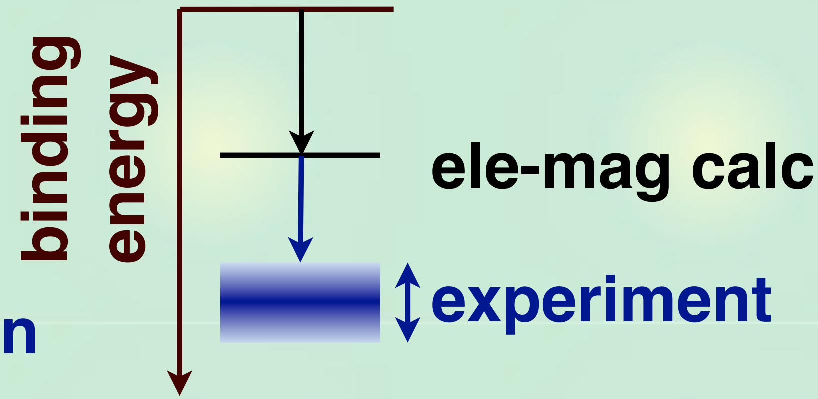
**(In hadron physics we usually adopt the opposite sign
--> positive for attraction, negative for repulsion)**

Hadron scattering length measurement

Extraction of hadron scattering length

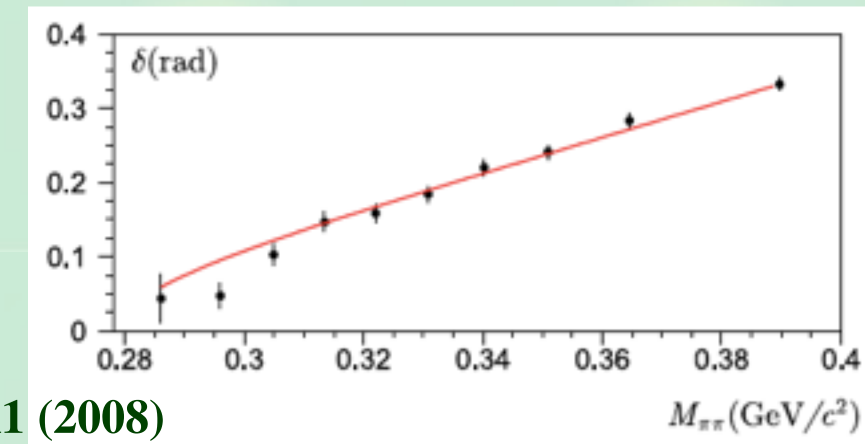
- **shift** and **width** of atomic state (coulomb bound state)
  ex) Kaonic hydrogen





problem: charged state only

- extrapolation of phase shift
  ex) ππ scattering



NA48/2, J.R. Batley et al, Eur. Phys. J. C54, 411 (2008)

problem: uncertainty of the extrapolation

- threshold effect in the decay spectrum --> today's topic

Outline of the paper

Cabibbo's idea for ππ scattering length

- **small isospin violation**

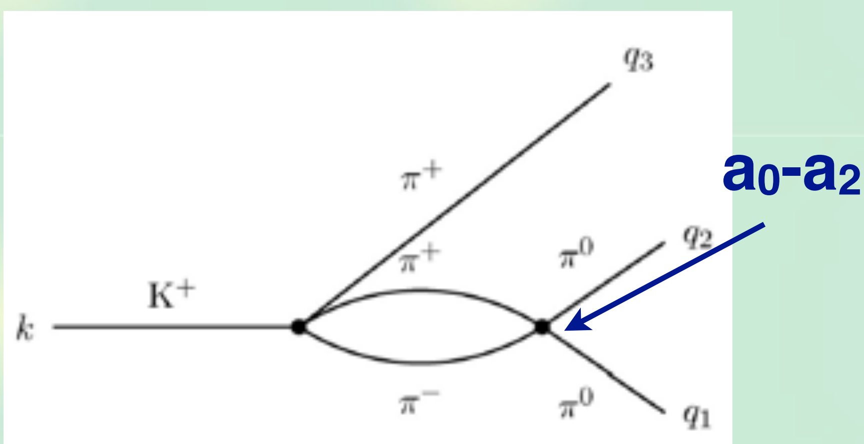
$$m_{\pi^\pm} \sim m_{\pi^0} + 5\ \mathrm{MeV}$$

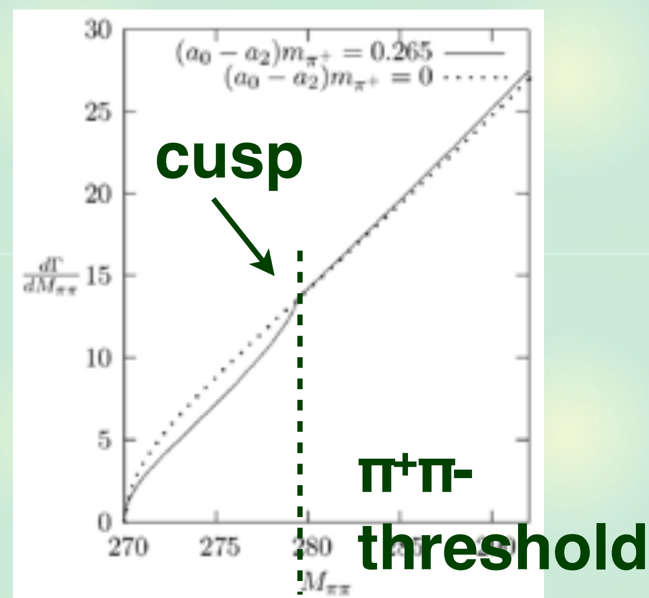
  --> charged ππ state is heavier than the neutral channel

- **threshold cusp effect in the π0π0 spectrum**

  --> cusp appears at a higher energy threshold

- **transition amplitude is proportional to a_0-a_2**

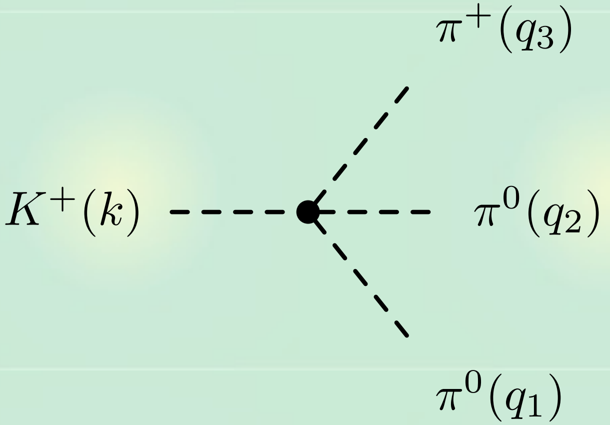


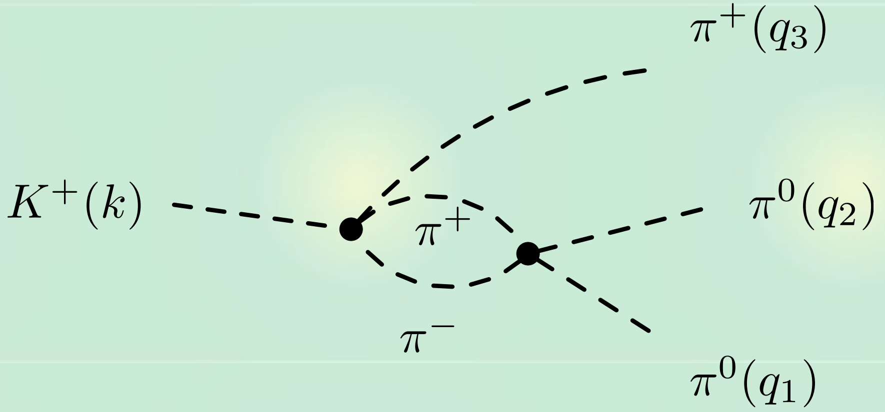


K decay amplitudes

Goal: to show the cusp in the K --> $\pi^+\pi^0\pi^0$ spectrum

There are two (or more) processes:





The rescattering amplitude has an imaginary part for

$$s_{\pi\pi} > 4m_{\pi^\pm}^2$$

$$s_{\pi\pi} = (q_2 + q_1)^2 = (k - q_3)^2$$

So we divide the amplitude into two pieces

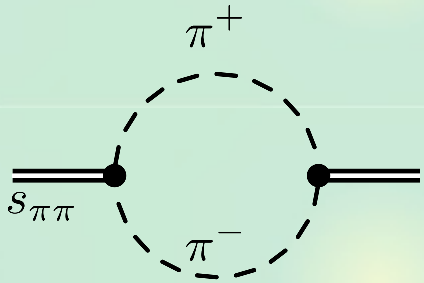
$$\mathcal{M}(K^+ \to \pi^+\pi^0\pi^0) = \mathcal{M}_0 + \mathcal{M}_1$$

such that $\mathcal{M}_1 = \begin{cases} \text{pure imaginary} & s_{\pi\pi} > 4m_{\pi^\pm}^2 \\ 0 & s_{\pi\pi} = 4m_{\pi^\pm}^2 \\ \text{real} & s_{\pi\pi} < 4m_{\pi^\pm}^2 \end{cases}$

Imaginary part of M_1 amplitude

Imaginary part of the loop function

$G(s_{\pi\pi}) \sim$



$$\sim \int_{4m_{\pi^\pm}^2}^{\infty} ds' \frac{\rho(s')}{s_{\pi\pi} - s' + i\epsilon}$$

$$\text{Im } G(s_{\pi\pi}) = -\rho(s_{\pi\pi}) = -\frac{p}{8\pi\sqrt{s_{\pi\pi}}} \propto -v \quad \text{for} \quad s_{\pi\pi} > 4m_{\pi^\pm}^2$$

ρ: phase space, p: three-momentum, v: velocity

$$p = \frac{\sqrt{s_{\pi\pi} - 4m_{\pi^\pm}^2}}{2}, \quad v = \frac{p}{E} = \sqrt{\frac{s_{\pi\pi} - 4m_{\pi^\pm}^2}{s_{\pi\pi}}}, \quad E = \frac{\sqrt{s_{\pi\pi}}}{2}$$

We can then write the amplitude as

$$\mathcal{M}_1 \propto J = \begin{cases} J_+ = -i\pi v & s_{\pi\pi} > 4m_{\pi^\pm}^2 \\ 0 & s_{\pi\pi} = 4m_{\pi^\pm}^2 \\ J_- = \pi\tilde{v} & s_{\pi\pi} < 4m_{\pi^\pm}^2 \end{cases}, \quad \tilde{v} = \sqrt{\frac{4m_{\pi^\pm}^2 - s_{\pi\pi}}{s_{\pi\pi}}}$$

analytic continuation of v

Step1) Threshold cusp effect

Threshold cusp in the spectrum

Amplitude of the process

$$\mathcal{M}(K^+ \to \pi^+\pi^0\pi^0) = \mathcal{M}_0 + \boxed{\mathcal{M}_1}$$

$\mathcal{M}_0$: real

$\mathcal{M}_1$: above threshold: imaginary, below threshold: real

The $\pi^0\pi^0$ invariant mass spectrum $(M_{\pi\pi} = \sqrt{s_{\pi\pi}})$

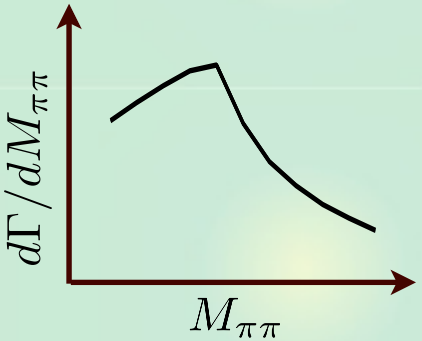
$$\frac{d\Gamma}{dM_{\pi\pi}} \propto |\mathcal{M}|^2 = \begin{cases} (\mathcal{M}_0)^2 + (i\mathcal{M}_1)^2 & s_{\pi\pi} > 4m^2_{\pi^\pm} \\ (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + \underline{2\mathcal{M}_0\mathcal{M}_1} & s_{\pi\pi} < 4m^2_{\pi^\pm} \end{cases}$$

- Spectrum is continuous

$$\lim_{s_{\pi\pi}\to 4m_{\pi^\pm}} \mathcal{M}_1 = 0 \quad \Rightarrow \quad \lim_{+0} |\mathcal{M}|^2 = \lim_{-0} |\mathcal{M}|^2$$

- Derivative of the spectrum is discontinuous

$$\lim_{-0} \frac{d}{dM_{\pi\pi}}|\mathcal{M}|^2 - \lim_{+0} \frac{d}{dM_{\pi\pi}}|\mathcal{M}|^2 = 2\mathcal{M}_0 \frac{d}{dM_{\pi\pi}}\mathcal{M}_1$$



--> threshold cusp

It is purely kinematical effect, independent of the interaction.

Amplitude for the subprocesses

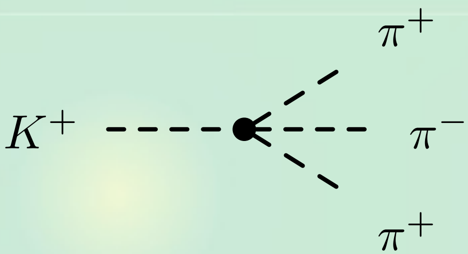
Goal: to construct the M$_1$ amplitude from one-loop graph

- Imaginary part <-- loop function

$$\mathcal{M}_1 \propto J$$

- Model for the K decay (essentially ChPT)

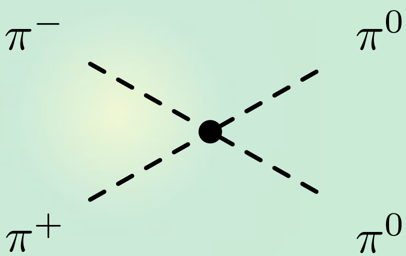
S. Weinberg, Phys. Rev. Lett. 4, 87 (1960); PDG(2008)



$$\sim \mathcal{M}_+ = A^+_{\rm av}\left(1 + \frac{g^+(s_{\pi\pi} - s_0)}{2m^2_{\pi^\pm}}\right) \qquad s_0 = (M_K^3 + 3m_\pi^3)/3$$

parameters

- Rescattering amplitude (around threshold)



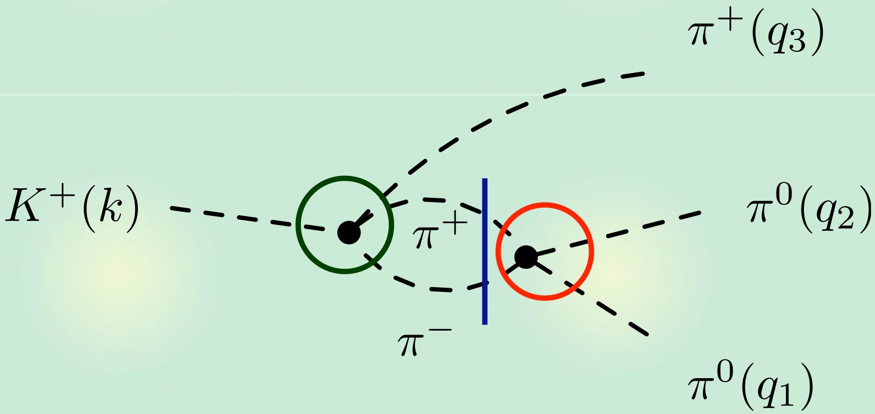
$$\sim \frac{16\pi(a_0 - a_2)m_{\pi^\pm}}{3}$$

isospin relation

$$a_{\pi^+\pi^-\to\pi^0\pi^0} = \frac{a_0 - a_2}{3}$$

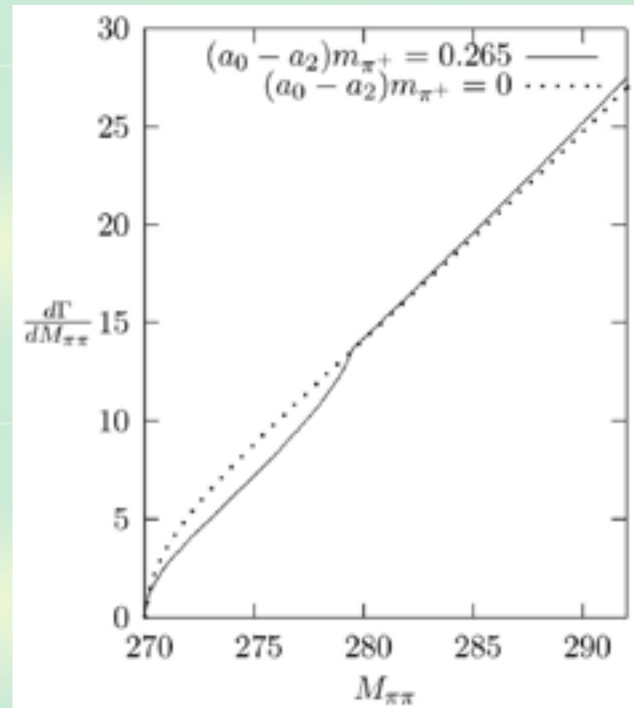
M_1 amplitude

M_1 term from the one-loop amplitude



$$\mathcal{M}_1 = -\frac{2(a_0 - a_2)m_{\pi^\pm}}{3} \cdot J \cdot \mathcal{M}_+$$

rescattering · loop · initial decay



Using a model for M_0 amplitude, we can calculate the mass spectrum.

If a_0-a_2=0, no rescattering (dashed line)
For finite a_0-a_2 (solid), cusp appears.

Experimental determination

Goal: to analyze the experimental spectrum without the model for the M_0 amplitude.

Region around threshold --> momentum expansion

$$|\mathcal{M}|^2 = a + b\delta + c\delta^2 + \mathcal{O}(\delta^3) \equiv F(\delta^2) + \mathcal{O}(\delta^3)$$

$$\delta = \frac{\sqrt{4m_{\pi^\pm}^2 - s_{\pi\pi}}}{2m_\pi} = \frac{p}{m_{\pi^\pm}}$$

Using this expansion and the M_1 amplitude, it is possible to extract the scattering length a_0-a_2.

Formulae:

$$\frac{d\Gamma}{dM_{\pi\pi}} \propto |\mathcal{M}|^2 = \begin{cases} (\mathcal{M}_0)^2 + (i\mathcal{M}_1)^2 & s_{\pi\pi} > 4m_{\pi^\pm}^2 \\ (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1 & s_{\pi\pi} < 4m_{\pi^\pm}^2 \end{cases}$$

$$\mathcal{M}_1 = -\frac{2(a_0 - a_2)m_{\pi^\pm}}{3} \cdot J \cdot \mathcal{M}_+ \qquad \mathcal{M}_+ = A_{\text{av}}^+ \left(1 + \frac{g^+(s_{\pi\pi} - s_0)}{2m_{\pi^\pm}^2}\right)$$

Procedure for experimental analysis

Four steps for the experimental determination of a_0-a_2

1) Determine M_+ by K --> $\pi^+\pi^+\pi^-$ decay spectrum

$$\mathcal{M}_+ = A^+_{\rm av}\left(1 + \frac{g^+(s_{\pi\pi} - s_0)}{2m^2_{\pi^\pm}}\right) \longrightarrow \quad A^+_{\rm av}, g^+ : \text{fixed}$$

2) Fit the $\pi^+\pi^0\pi^0$ spectrum above threshold by $F(\delta^2)$

$$|\mathcal{M}|^2_{\rm above} = (\mathcal{M}_0)^2 - (\mathcal{M}_1)^2 = F(\delta^2) \longrightarrow a, b, c : \text{fixed}$$

3) Extract M_1 from the $\pi^+\pi^0\pi^0$ spectrum below threshold

$$\begin{aligned} |\mathcal{M}|^2_{\rm below} &= (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1 \\ &= (\mathcal{M}_0)^2 - (\mathcal{M}_1)^2 + 2(\mathcal{M}_1)^2 + 2[(\mathcal{M}_0)^2 - (\mathcal{M}_1)^2 + (\mathcal{M}_1)^2]^{1/2}\mathcal{M}_1 \\ &= F(\delta^2) + 2\underline{(\mathcal{M}_1)}^2 + 2[F(\delta^2) + \underline{(\mathcal{M}_1)}^2]^{1/2}\underline{\mathcal{M}_1} \end{aligned}$$

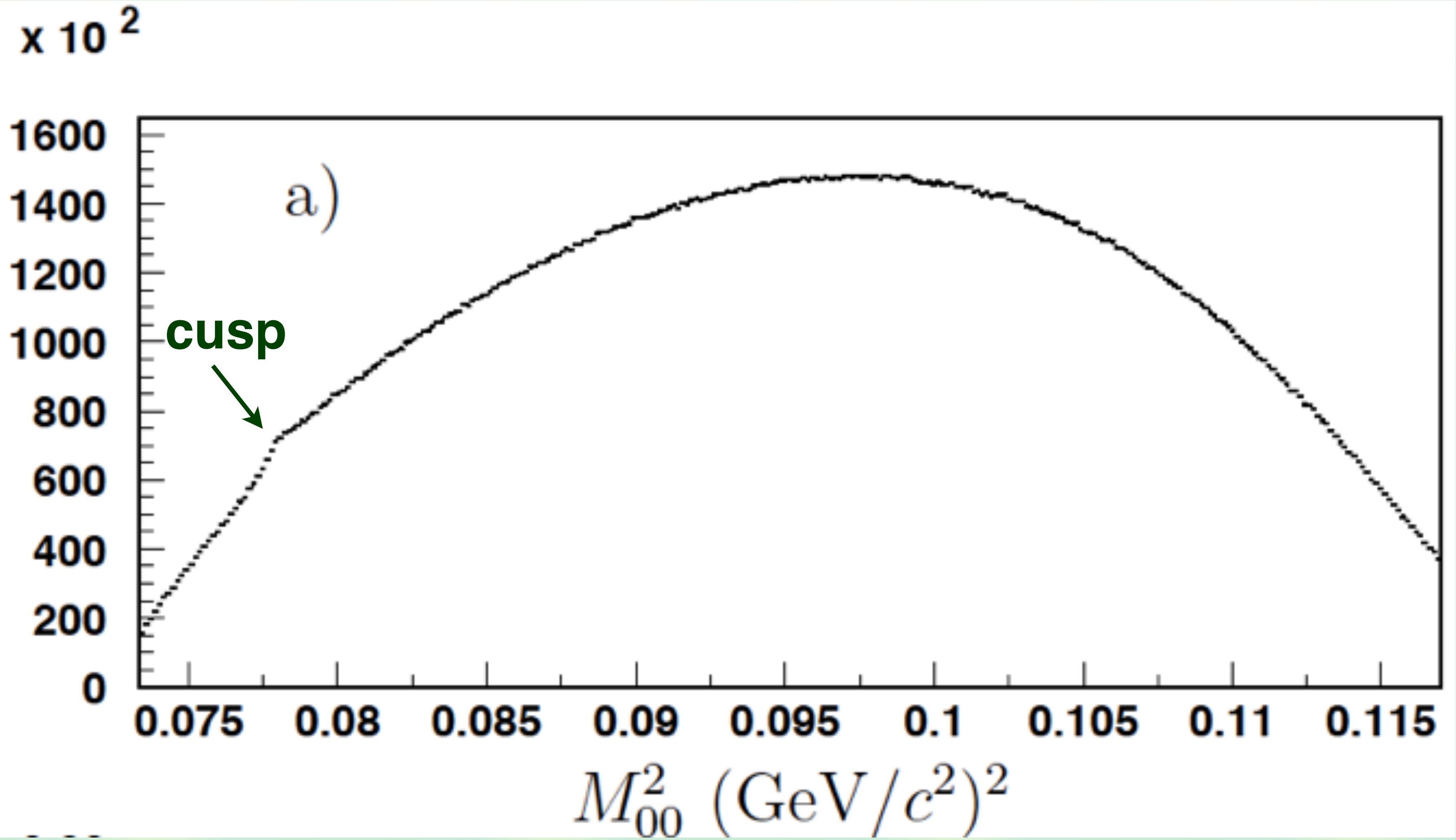
4) Calculate a_0-a_2 from M_+ and J

$$\mathcal{M}_1 = -\frac{2(a_0 - a_2)m_{\pi^\pm}}{3} \cdot J \cdot \mathcal{M}_+$$

This method does not require any model for M_0

Experimental feasibility

Cusp is indeed seen in the experimental spectrum



NA48/2, J.R. Batley et al, Phys. Lett. B686, 101 (2010)

Summary

A method to extract the ππ scattering length from the $K^+ \to \pi^+\pi^0\pi^0$ decay is discussed.

- Isospin violation causes **mass difference** between charged and neutral pions
- Threshold **cusp** at $\pi^+\pi^-$ threshold is proportional to a^0-a^2 **scattering length**
- Experimental determination is possible with momentum expansion

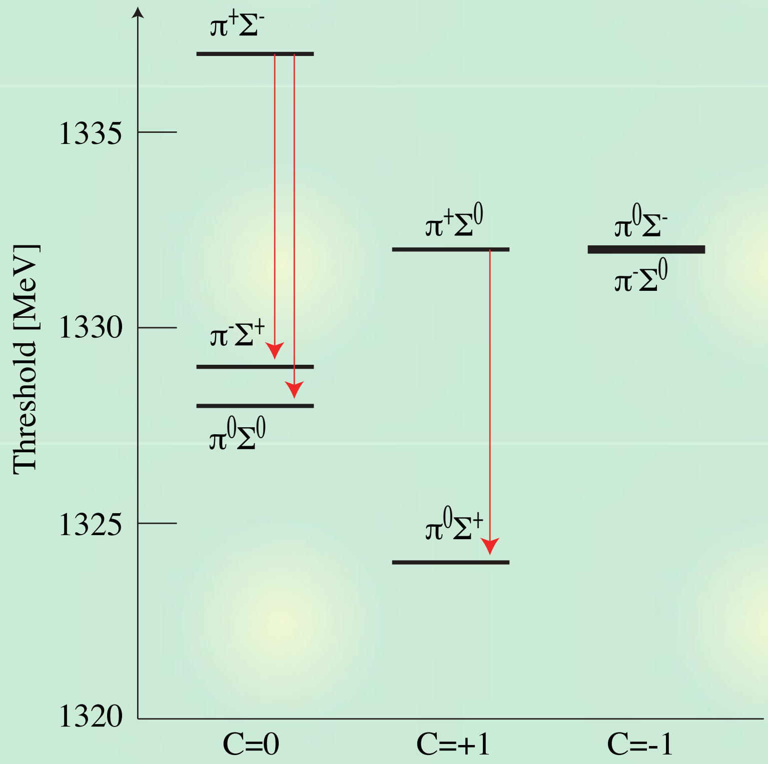
Higher order corrections:

N. Cabibbo and G. Isidori, JHEP 03, 021 (2005)

Determination of πΣ scattering length

Similar approach for πΣ scattering length?

T. Hyodo, M. Oka, work in progress



Σ^+(~uus) < Σ^0(~uds) < Σ^-(~dds)
--> complicated spectrum

Possible decay modes

$$\langle \pi^-\Sigma^+ |T| \pi^+\Sigma^- \rangle|_{\text{threshold}} = \frac{1}{3}a^0 - \frac{1}{2}a^1 + \frac{1}{6}a^2$$

$$\langle \pi^0\Sigma^0 |T| \pi^+\Sigma^- \rangle|_{\text{threshold}} = \frac{1}{3}a^0 - \frac{1}{3}a^2$$

$$\langle \pi^0\Sigma^+ |T| \pi^+\Sigma^0 \rangle|_{\text{threshold}} = -\frac{1}{2}a^1 + \frac{1}{2}a^2$$

Determination of $\pi\Sigma$ scattering length

How to measure?

- Λ_c -> π $\pi\Sigma$

Branching fraction of the Λ_c decay (Γ_i/Γ) in PDG:
$\pi^+\Sigma^-\pi^+$ (1.7%), $\pi^-\Sigma^+\pi^+$ (3.6%), $\pi^0\Sigma^0\pi^+$ (1.8%)

A lot of Λ_c in B decay (Belle, Babar) --> feasible?

Significance?

- Important constraint for KN-$\pi\Sigma$ interaction at low energy
- Lower pole position of the Λ(1405)
  <-- sensitive to the $\pi\Sigma$ scattering length

model	BNW [8]	ORB [5]	HNJH [6]	BMN [9]	virtual state
$a_{\pi\Sigma}$ [fm]	0.517	0.789	0.692	0.770	~ 5
z_1 [MeV]	$1388 - 39i$	$1389 - 64i$	$1400 - 76i$	$1440 - 76i$	1325 (V)

Y. Ikeda, T. Hyodo, D. Jido, in preparation