極座標変数の直交座標微分

極座標変数の直交座標微分 (81) の導出。

微分を定義通り計算すると

$$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial x}x^2=\frac{1}{2r}2x=\frac{x}{r}=\sin\theta\cos\phi$$

$$\frac{\partial}{\partial x}\cos\theta=\frac{\partial}{\partial x}\frac{z}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial\theta}{\partial x}\frac{\partial\cos\theta}{\partial\theta}=z\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial\theta}{\partial x}(-\sin\theta)=z(-\frac{1}{2})(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}x^2$$

$$-\sin\theta\frac{\partial\theta}{\partial x}=-\frac{1}{2}z(x^2+y^2+z^2)^{-3/2}2x$$

$$-\sin\theta\frac{\partial\theta}{\partial x}=-\frac{xz}{r^3}$$

$$\frac{\partial\theta}{\partial x}=\frac{xz}{r^3\sin\theta}=\frac{r\sin\theta\cos\phi\cdot r\cos\theta}{r^3\sin\theta}=\frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial}{\partial x}\tan\phi=\frac{\partial}{\partial x}\frac{y}{x}$$

$$\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\tan\phi=y\frac{\partial}{\partial x}x^{-1}$$

$$\frac{\partial\phi}{\partial x}\frac{1}{\cos^2\phi}=y(-x^{-2})$$

$$\frac{\partial\phi}{\partial x}=-\cos^2\phi\frac{y}{x^2}=-\cos^2\phi\frac{r\sin\theta\sin\phi}{(r\sin\theta\cos\phi)^2}=-\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial r}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial y}y^2=\frac{1}{2r}2y=\frac{y}{r}=\sin\theta\sin\phi$$

$$\frac{\partial}{\partial y}\cos\theta=\frac{\partial}{\partial y}\frac{z}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial\theta}{\partial y}\frac{\partial\cos\theta}{\partial\theta}=z\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial\theta}{\partial y}(-\sin\theta)=z(-\frac{1}{2})(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial y}y^2$$

$$-\sin\theta\frac{\partial\theta}{\partial y}=-\frac{1}{2}z(x^2+y^2+z^2)^{-3/2}2y$$

$$-\sin\theta\frac{\partial\theta}{\partial y}=-\frac{yz}{r^3}$$

$$\frac{\partial\theta}{\partial y}=\frac{yz}{r^3\sin\theta}=\frac{r\sin\theta\sin\phi\cdot r\cos\theta}{r^3\sin\theta}=\frac{\cos\theta\sin\phi}{r}$$

$$\frac{\partial}{\partial y}\tan\phi=\frac{\partial}{\partial y}\frac{y}{x}$$

$$\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\tan\phi=\frac{1}{x}$$

$$\frac{\partial\phi}{\partial y}\frac{1}{\cos^2\phi} = \frac{1}{x}$$

$$\frac{\partial\phi}{\partial y} = \frac{\cos^2\phi}{r\sin\theta\cos\phi} = \frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial z}z^2 = \frac{1}{2r}2z = \frac{z}{r} = \cos\theta$$

$$\frac{\partial}{\partial z}\cos\theta = \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial\theta}{\partial z}\frac{\partial\cos\theta}{\partial\theta} = (x^2+y^2+z^2)^{-1/2} + z\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial\theta}{\partial z}(-\sin\theta) = \frac{1}{r} + z(-\frac{1}{2})(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial z}z^2$$

$$-\sin\theta\frac{\partial\theta}{\partial z} = \frac{r^2}{r^3} - \frac{1}{2}z(x^2+y^2+z^2)^{-3/2}2z$$

$$-\sin\theta\frac{\partial\theta}{\partial z} = -\frac{-r^2+z^2}{r^3}$$

$$\frac{\partial\theta}{\partial z} = \frac{z^2-r^2}{r^3\sin\theta} = \frac{r^2\cos^2\theta-r^2}{r^3\sin\theta} = \frac{\cos^2\theta-1}{r\sin\theta} = \frac{-\sin^2\theta}{r\sin\theta} = -\frac{\sin\theta}{r}$$

$$\frac{\partial}{\partial z}\tan\phi = \frac{\partial}{\partial z}\frac{y}{x}$$

$$\frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\tan\phi = 0$$

$$\frac{\partial\phi}{\partial z} = 0$$

以上をまとめると (81) を得る。

別解）直交座標変数 (x, y, z) の極座標微分は簡単に計算でき

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{C3}$$

である。求めたい変換はこれの逆変換なので

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{pmatrix}$$

と逆行列を計算すれば良い。定義通り余因子行列を計算すると

$$
\begin{pmatrix}
\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\
r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\
-r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0
\end{pmatrix}^{-1}
$$

$$
= \frac{1}{\Delta}
\begin{pmatrix}
r^2\sin^2\theta\cos\phi & -(-r\sin\theta\cos\theta\cos\phi) & -r(\sin^2\theta+\cos^2\theta)\sin\phi \\
-(-r^2\sin^2\theta\sin\phi) & r\sin\theta\cos\theta\sin\phi & -(-r(\sin^2\theta+\cos^2\theta)\cos\phi) \\
r^2\sin\theta\cos\theta(\cos^2\phi+\sin^2\phi) & -(r\sin^2\theta(\cos^2\phi+\sin^2\phi)) & 0
\end{pmatrix}
$$

$$
= \frac{1}{\Delta}
\begin{pmatrix}
r^2\sin^2\theta\cos\phi & r\sin\theta\cos\theta\cos\phi & -r\sin\phi \\
r^2\sin^2\theta\sin\phi & r\sin\theta\cos\theta\sin\phi & r\cos\phi \\
r^2\sin\theta\cos\theta & -r\sin^2\theta & 0
\end{pmatrix}
$$

で行列式 Δ は変数変換のヤコビアンなので $r^2\sin\theta$、よって

$$
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
= \frac{1}{r^2\sin\theta}
\begin{pmatrix}
r^2\sin^2\theta\cos\phi & r\sin\theta\cos\theta\cos\phi & -r\sin\phi \\
r^2\sin^2\theta\sin\phi & r\sin\theta\cos\theta\sin\phi & r\cos\phi \\
r^2\sin\theta\cos\theta & -r\sin^2\theta & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \phi}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\sin\theta\cos\phi & \frac{\cos\theta\cos\phi}{r} & -\frac{\sin\phi}{r\sin\theta} \\
\sin\theta\sin\phi & \frac{\cos\theta\sin\phi}{r} & \frac{\cos\phi}{r\sin\theta} \\
\cos\theta & -\frac{\sin\theta}{r} & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \phi}
\end{pmatrix}
$$

この各成分が以下のように対応するので、式 (81) を得る。

$$
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \phi}{\partial x} \\
\frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \phi}{\partial y} \\
\frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \phi}{\partial z}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \phi}
\end{pmatrix}
$$

となる。

別解の別解）極座標の $\boldsymbol{\nabla}$ が

$$
\boldsymbol{\nabla} = \begin{pmatrix} \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{pmatrix}
$$

であることに注意し、極座標に変換した後の微分演算子を $\boldsymbol{\nabla}$ に揃えて表記すると、最初の変換は

$$
\begin{pmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \phi}
\end{pmatrix}
=
\begin{pmatrix}
\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\
r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\
-r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{\partial}{\partial r} \\
\frac{1}{r}\frac{\partial}{\partial \theta} \\
\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}
\end{pmatrix}
=
\begin{pmatrix}
\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\
\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
$$

と書くことができる。ここでの変換行列は $R^{-1} = R^t$ を満たす直交行列なので、逆行列は転置で与えられ、

$$
\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}
&= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}^{-1}
\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}
\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\cos\phi & \frac{\cos\theta\cos\phi}{r} & -\frac{\sin\phi}{r\sin\theta} \\ \sin\theta\sin\phi & \frac{\cos\theta\sin\phi}{r} & \frac{\cos\phi}{r\sin\theta} \\ \cos\theta & -\frac{\sin\theta}{r} & 0 \end{pmatrix}
\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}
\end{aligned}
$$

となる。

極座標のラプラシアン（直接計算）

極座標のラプラシアン

$$\boldsymbol{\nabla}^2\psi = \frac{\partial^2}{\partial r^2}\psi + \frac{2}{r}\frac{\partial}{\partial r}\psi + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi \tag{82}$$

の導出。

まず、関数 $f(r,\theta,\phi)$ の x,y,z による 2 階微分を計算する。$(\partial f/\partial x)$ を一つの関数とみなして偏微分の座標変換の式を使うと、

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}f &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\\
&= \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial x}\right)\\
&= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right) - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial x}\right) && \text{(C4)}\\
\frac{\partial^2}{\partial y^2}f &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\\
&= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right) + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial y}\right) && \text{(C5)}\\
\frac{\partial^2}{\partial z^2}f &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\\
&= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial z}\right) && \text{(C6)}
\end{aligned}$$

以下、この各項を評価する。式 (C4) 第 1 項は

$$\begin{aligned}
\sin\theta\cos\phi\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) &= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial}{\partial r}f + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}f - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}f\right)\\
&= \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2}f\\
&\quad + \sin\theta\cos\theta\cos^2\phi\frac{-1}{r^2}\frac{\partial}{\partial\theta}f + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f\\
&\quad - \sin\phi\cos\phi\frac{-1}{r^2}\frac{\partial}{\partial\phi}f - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f\\
&= \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2}f - \frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial}{\partial\theta}f + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f\\
&\quad + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}f - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f && \text{(C7)}
\end{aligned}$$

式 (C4) 第 2 項は

$$
\begin{aligned}
\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right) &= \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial}{\partial r}f + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}f - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}f\right) \\
&= \frac{\cos\theta\cos^2\phi}{r}\cos\theta\frac{\partial}{\partial r}f + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f \\
&\quad + \frac{\cos\theta\cos^2\phi}{r^2}(-\sin\theta)\frac{\partial}{\partial\theta}f + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2}f \\
&\quad - \frac{\cos\theta\sin\phi\cos\phi}{r^2}\left(\frac{-1}{\sin^2\theta}\cos\theta\right)\frac{\partial}{\partial\phi}f - \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f \\
&= \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial}{\partial r}f + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f \\
&\quad - \frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial}{\partial\theta}f + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2}f \\
&\quad + \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}f - \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f
\end{aligned} \tag{C8}
$$

式 (C4) 第 3 項は

$$
\begin{aligned}
-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial x}\right) &= -\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial}{\partial r}f + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}f - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}f\right) \\
&= -\frac{\sin\phi}{r}(-\sin\phi)\frac{\partial}{\partial r}f - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f \\
&\quad - \frac{\sin\phi\cos\theta}{r^2\sin\theta}(-\sin\phi)\frac{\partial}{\partial\theta}f - \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f \\
&\quad + \frac{\sin\phi}{r^2\sin^2\theta}\cos\phi\frac{\partial}{\partial\phi}f + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}f \\
&= \frac{\sin^2\phi}{r}\frac{\partial}{\partial r}f - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f \\
&\quad + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}f - \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f \\
&\quad + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}f + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}f
\end{aligned} \tag{C9}
$$

となる。次に式 (C5) 第 1 項は

$$
\begin{aligned}
\sin\theta\sin\phi\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) &= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial}{\partial r}f + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}f + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}f\right) \\
&= \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2}f \\
&\quad + \sin\theta\cos\theta\sin^2\phi\frac{-1}{r^2}\frac{\partial}{\partial\theta}f + \frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f \\
&\quad + \sin\phi\cos\phi\frac{-1}{r^2}\frac{\partial}{\partial\phi}f + \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f \\
&= \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2}f - \frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial}{\partial\theta}f + \frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f
\end{aligned}
$$

$$
-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}f+\frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f \tag{C10}
$$

式 (C5) 第 2 項は

$$
\begin{aligned}
\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right) &= \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial}{\partial r}f+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}f+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}f\right)\\
&= \frac{\cos\theta\sin^2\phi}{r}\cos\theta\frac{\partial}{\partial r}f+\frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f\\
&\quad +\frac{\cos\theta\sin^2\phi}{r^2}(-\sin\theta)\frac{\partial}{\partial\theta}f+\frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2}f\\
&\quad +\frac{\cos\theta\sin\phi\cos\phi}{r^2}\left(\frac{-\cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\phi}f+\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f\\
&= \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial}{\partial r}f+\frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}f\\
&\quad -\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial}{\partial\theta}f+\frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2}f\\
&\quad -\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}f+\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f
\end{aligned} \tag{C11}
$$

式 (C5) 第 3 項は

$$
\begin{aligned}
\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial y}\right) &= \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial}{\partial r}f+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}f+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}f\right)\\
&= \frac{\cos\phi}{r}\cos\phi\frac{\partial}{\partial r}f+\frac{\cos\phi\sin\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f\\
&\quad +\frac{\cos\theta\cos\phi}{r^2\sin\theta}\cos\phi\frac{\partial}{\partial\theta}f+\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f\\
&\quad +\frac{\cos\phi}{r^2\sin^2\theta}(-\sin\phi)\frac{\partial}{\partial\phi}f+\frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}f\\
&= \frac{\cos^2\phi}{r}\frac{\partial}{\partial r}f+\frac{\cos\phi\sin\phi}{r}\frac{\partial^2}{\partial r\partial\phi}f\\
&\quad +\frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}f+\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}f\\
&\quad -\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}f+\frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}f
\end{aligned} \tag{C12}
$$

となる。最後に式 (C6) 第 1 項は

$$
\begin{aligned}
\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right) &= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r}f-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}f\right)\\
&= \cos^2\theta\frac{\partial^2}{\partial r^2}f\\
&\quad -\sin\theta\cos\theta\left(\frac{-1}{r^2}\right)\frac{\partial}{\partial\theta}f-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}f\\
&= \cos^2\theta\frac{\partial^2}{\partial r^2}f+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}f-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}f
\end{aligned} \tag{C13}
$$

式 (C6) 第 2 項は

$$
\begin{aligned}
-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial z}\right) &= -\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}f - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}f\right) \\
&= -\frac{\sin\theta}{r}(-\sin\theta)\frac{\partial}{\partial r}f - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}f \\
&\quad + \frac{\sin\theta}{r^2}\cos\theta\frac{\partial}{\partial\theta}f + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}f \\
&= \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}f - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}f + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}f + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}f \qquad \text{(C14)}
\end{aligned}
$$

となる。以上を利用し

$$
\boldsymbol{\nabla}^2\psi = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi
$$

を ψ の極座標微分の各項ごとに調べる。

$$
\begin{aligned}
\frac{\partial^2}{\partial r^2}\psi \text{ の項}: &\sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2}\psi + \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2}\psi + \cos^2\theta\frac{\partial^2}{\partial r^2}\psi \quad \leftarrow \text{(C7), (C10), (C13)} \\
&= \left[\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta\right]\frac{\partial^2}{\partial r^2}\psi \\
&= (\sin^2\theta + \cos^2\theta)\frac{\partial^2}{\partial r^2}\psi \\
&= \frac{\partial^2}{\partial r^2}\psi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial r\partial\theta}\psi \text{ の項}: &\frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}\psi + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}\psi + \frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}\psi \\
&\quad + \frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta}\psi - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}\psi - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}\psi \\
&\qquad \leftarrow \text{(C7), (C8), (C10), (C11), (C13), (C14)} \\
&= \left[2\frac{\sin\theta\cos\theta\cos^2\phi}{r} + 2\frac{\sin\theta\cos\theta\sin^2\phi}{r} - 2\frac{\sin\theta\cos\theta}{r}\right]\frac{\partial^2}{\partial r\partial\theta}\psi \\
&= \left[2\frac{\sin\theta\cos\theta}{r} - 2\frac{\sin\theta\cos\theta}{r}\right]\frac{\partial^2}{\partial r\partial\theta}\psi \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial r\partial\phi}\psi \text{ の項}: &-\frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}\psi - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}\psi + \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi}\psi + \frac{\cos\phi\sin\phi}{r}\frac{\partial^2}{\partial r\partial\phi}\psi \\
&\qquad \leftarrow \text{(C7), (C9), (C10), (C12)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial^2\theta}\psi \text{ の項}: &\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2}\psi + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2}\psi + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}\psi \quad \leftarrow \text{(C8), (C11), (C14)} \\
&= \left[\frac{\cos^2\theta(\cos^2\phi + \sin^2\phi)}{r^2} + \frac{\sin^2\theta}{r^2}\right]\frac{\partial^2}{\partial\theta^2}\psi \\
&= \left[\frac{\cos^2\theta(\cos^2\phi + \sin^2\phi)}{r^2} + \frac{\sin^2\theta}{r^2}\right]\frac{\partial^2}{\partial\theta^2}\psi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\cos^2\theta + \sin^2\theta)}{r^2}\frac{\partial^2}{\partial\theta^2}\psi \\
&= \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial\theta\partial\phi}\psi \text{ の項}: &-\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\psi - \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\psi + \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\psi \\
&+ \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\psi \qquad \leftarrow (\mathrm{C8}), (\mathrm{C9}), (\mathrm{C11}), (\mathrm{C12}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial\phi^2}\psi \text{ の項}: &\frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi \qquad \leftarrow (\mathrm{C9}), (\mathrm{C12}) \\
&= \frac{\sin^2\phi + \cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi \\
&= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial r}\psi \text{ の項}: &\frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial}{\partial r}\psi + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r}\psi + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial}{\partial r}\psi + \frac{\cos^2\phi}{r}\frac{\partial}{\partial r}\psi + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}\psi \\
&\quad \leftarrow (\mathrm{C8}), (\mathrm{C9}), (\mathrm{C11}), (\mathrm{C12}), (\mathrm{C14}) \\
&= \left[\frac{\cos^2\theta(\cos^2\phi + \sin^2\phi)}{r} + \frac{\sin^2\phi + \cos^2\phi}{r} + \frac{\sin^2\theta}{r}\right]\frac{\partial}{\partial r}\psi \\
&= \left[\frac{(\cos^2\theta + \sin^2\theta)}{r} + \frac{1}{r}\right]\frac{\partial}{\partial r}\psi \\
&= \frac{2}{r}\frac{\partial}{\partial r}\psi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\psi \text{ の項}: &-\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial}{\partial\theta}\psi - \frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial}{\partial\theta}\psi + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi \\
&- \frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial}{\partial\theta}\psi - \frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial}{\partial\theta}\psi + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi \\
&+ \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}\psi + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}\psi \qquad \leftarrow (\mathrm{C7}), (\mathrm{C8}), (\mathrm{C9}), (\mathrm{C10}), (\mathrm{C11}), (\mathrm{C12}), (\mathrm{C13}), (\mathrm{C} \\
&= \Bigg[-2\frac{\sin\theta\cos\theta\cos^2\phi}{r^2} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta} \\
&- 2\frac{\sin\theta\cos\theta\sin^2\phi}{r^2} + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta} \\
&+ 2\frac{\sin\theta\cos\theta}{r^2}\Bigg]\frac{\partial}{\partial\theta}\psi \\
&= \left[-2\frac{\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)}{r^2} + 2\frac{\sin\theta\cos\theta}{r^2} + \frac{\cos\theta(\sin^2\phi + \cos^2\phi)}{r^2\sin\theta}\right]\frac{\partial}{\partial\theta}\psi \\
&= \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi
\end{aligned}
$$

$$
\frac{\partial}{\partial\phi}\psi \text{ の項}: \frac{\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}\psi + \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\psi + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\psi - \frac{\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}\psi
$$

$$
\begin{aligned}
&- \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\psi - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\psi \quad \leftarrow (\mathrm{C7}), (\mathrm{C8}), (\mathrm{C9}), (\mathrm{C10}), (\mathrm{C11}), (\mathrm{C12}) \\
&= 0
\end{aligned}
$$

以上まとめると

$$
= \frac{\partial^2}{\partial r^2}\psi + \frac{2}{r}\frac{\partial}{\partial r}\psi + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi
$$

を得る。

極座標のラプラシアン（計量テンソル）

極座標のラプラシアン

$$\boldsymbol{\nabla}^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi\right] \tag{83}$$

の計量テンソルを使った導出。

3 次元の曲線座標でのラプラシアンは一般に

$$\boldsymbol{\nabla}^2\psi = \frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial q^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial q^j}\right)\psi$$

と与えられる（導出は国広悌二「量子力学」(東京図書)p.78、猪木慶治・川合光「量子力学 I」(講談社サイエンティフィク)p.286 などを参照)。計量テンソルなどの定義は

$$g_{ij} = \sum_{k=1}^{3}\frac{\partial x^k}{\partial q^i}\frac{\partial x^k}{\partial q^j}, \quad g = \det g_{ij}, \quad \sum_{k=1}^{3} g^{ik}g_{kj} = \delta^i_j \quad (g^{ij}\text{は } g_{ij}\text{の逆行列})$$

であり、極座標の場合の計算には

$$(q_1, q_2, q_3) = (r, \theta, \phi)$$

とすれば良い。計量テンソル g_{ij} を計算すると[5]

$$\begin{aligned}
g_{11} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} \\
&= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta \\
&= 1 \\
g_{12} = g_{21} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
&= \sin\theta\cos\phi r\cos\theta\cos\phi + \sin\theta\sin\phi r\cos\theta\sin\phi + \cos\theta(-r\sin\theta) \\
&= r\sin\theta\cos\theta\cos^2\phi + r\cos\theta\sin\theta\sin^2\phi - r\sin\theta\cos\theta \\
&= r\sin\theta\cos\theta - r\sin\theta\cos\theta \\
&= 0 \\
g_{13} = g_{31} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} \\
&= \sin\theta\cos\phi(-r\sin\theta\sin\phi) + \sin\theta\sin\phi r\sin\theta\cos\phi + \cos\theta\times 0 \\
&= -r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi
\end{aligned}$$

[5] 極座標での微小線素が $ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$ であること使えば係数から計量テンソルがすぐ得られる。

$$
\begin{aligned}
&= 0 \\
g_{22} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \\
&= r^2 \cos^2\theta \cos^2\phi + r^2 \cos^2\theta \sin^2\phi + r^2 \sin^2\theta \\
&= r^2(\cos^2\theta + \sin^2\theta) \\
&= r^2 \\
g_{23} = g_{32} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} \\
&= r\cos\theta\cos\phi(-r\sin\theta\sin\phi) + r\cos\theta\sin\phi r\sin\theta\cos\phi + \cos\theta \times 0 \\
&= -r^2\sin\theta\cos\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\sin\phi\cos\phi \\
&= 0 \\
g_{33} &= \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \phi} \\
&= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi \\
&= r^2\sin^2\theta
\end{aligned}
$$

つまり

$$
g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}
$$

を得る。これを用いて

$$
\begin{aligned}
\sqrt{g} &= \sqrt{r^4\sin^2\theta} = r^2\sin\theta \\
g^{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}
\end{aligned}
$$

ここで対角行列の逆行列は各成分の逆数をとったもので与えられることを用いた。以上より

$$
\begin{aligned}
\boldsymbol{\nabla}^2\psi &= \frac{1}{r^2\sin\theta}\sum_{ij}\frac{\partial}{\partial q^i}\left(r^2\sin\theta g^{ij}\frac{\partial}{\partial q^j}\right)\psi \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta g^{11}\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(r^2\sin\theta g^{22}\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(r^2\sin\theta g^{33}\frac{\partial}{\partial \phi}\right)\right]\psi \\
&= \left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial}{\partial \phi}\right)\right]\psi \\
&= \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial \phi^2}\right]\psi
\end{aligned}
$$

を得る。第 1 項は

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi = \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right)\psi = \frac{\partial^2}{\partial r^2}\psi + \frac{2}{r}\frac{\partial}{\partial r}\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)
$$

となるので

$$\boldsymbol{\nabla}^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\psi\right]$$

を得る。