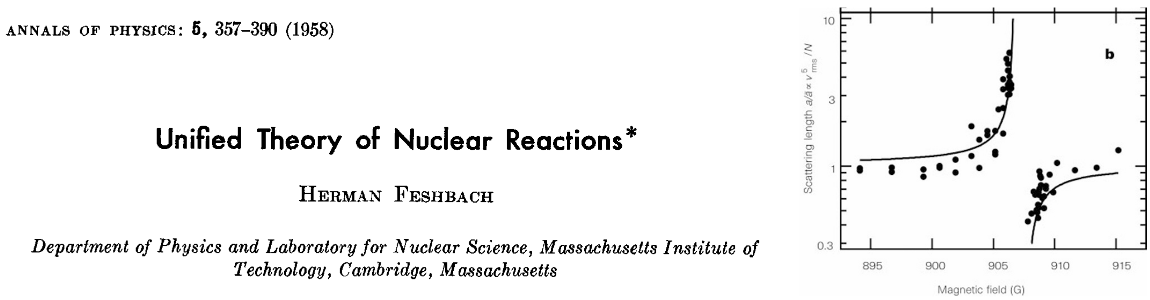
4 Theory of Feshbach resonances

4.1 Overview

- Feshbach resonance : resonance in coupled-channel scattering
- Threshold energy E_{th} and channels
  - open channels ($E > E_{\mathrm{th}}$) : scattering occurs at energy E
  - closed channels ($E < E_{\mathrm{th}}$) : scattering does not occur at energy E
- Original paper by Feshbach [30, 31] : theory of compound nuclear reaction (Fig. 12, left)
- Realization with cold atoms [7] : controlling scattering length by magnetic field (Fig. 12, right)


ANNALS OF PHYSICS: **5**, 357–390 (1958)

Unified Theory of Nuclear Reactions*

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Figure 12: Left : original paper, H. Feshbach, Ann. Phys. **5**, 357 (1958). Right: controlling scattering length of cold atoms by magnetic field, adopted from S. Inouye, Nature (London) **392**, 151 (1998).

4.2 Two-channel Hamiltonian

- Two channels P and Q, setting threshold of P at $E_{\mathrm{th}}(P) = 0$ [32]
- Schrödinger equation in matrix form

$$\hat{H}|\,\psi\,\rangle = E|\,\psi\,\rangle \tag{30}$$

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} = \begin{pmatrix} \dfrac{\hat{\boldsymbol{p}}^2}{2\mu_P} + \hat{V}_P & \hat{V}_t \\ \hat{V}_t & \dfrac{\hat{\boldsymbol{p}}^2}{2\mu_Q} + \Delta + \hat{V}_Q \end{pmatrix}, \quad |\,\psi\,\rangle = \begin{pmatrix} |\,P\,\rangle \\ |\,Q\,\rangle \end{pmatrix}$$

  - $\hat{V}_P, \hat{V}_Q$: potential in each channel (Fig. 4), vanishes at $r \to \infty$
  - $\hat{V}_t$: channel transition potential
  - $\Delta > 0$: threshold energy difference $E_{\mathrm{th}}(Q) - E_{\mathrm{th}}(P)$ (originates in Zeeman splitting of atoms, proportional to magnetic field strength)
  - Energy region $0 < E < \Delta$: P is open (entrance) channel, Q is closed channel

- Projection operators

$$\hat{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}^2 = \hat{P}, \quad \hat{Q}^2 = \hat{Q}, \quad \hat{P}\hat{Q} = \hat{Q}\hat{P} = 0, \quad \hat{P} + \hat{Q} = \hat{I}$$

Each component can be written as $|\, X\,\rangle = \hat{X}|\,\psi\,\rangle$ and $\hat{H}_{XY} = \hat{X}\hat{H}\hat{Y}$

- **Effective Hamiltonian for channel P** : eliminating $|\, Q\,\rangle$

It follows from the lower component of Eq. (30) that

$$\begin{aligned} \hat{H}_{QP}|\, P\,\rangle + \hat{H}_{QQ}|\, Q\,\rangle &= E|\, Q\,\rangle \\ \hat{H}_{QP}|\, P\,\rangle &= (E - \hat{H}_{QQ})|\, Q\,\rangle \\ |\, Q\,\rangle &= (E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}|\, P\,\rangle \end{aligned}$$

Substituting this into the upper component of Eq. (30) :

$$\begin{aligned} \hat{H}_{PP}|\, P\,\rangle + \hat{H}_{PQ}|\, Q\,\rangle &= E|\, P\,\rangle \\ \hat{H}_{PP}|\, P\,\rangle + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}|\, P\,\rangle &= E|\, P\,\rangle \end{aligned}$$

then

$$\begin{aligned} &\hat{H}^{\text{eff}}(E)|\, P\,\rangle = E|\, P\,\rangle, \\ &\quad \hat{H}^{\text{eff}}(E) = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP} \end{aligned} \tag{31}$$

$\hat{H}^{\text{eff}}$ is effective Hamiltonian of P, incorporating the effect of Q

- Eq. (31) is a single-channel (not in matrix form) Schrödinger equation in P
- No approximations $\Rightarrow$ Solution of Eq. (31) is equivalent to $|\, P\,\rangle$ in Eq. (30)
- $\hat{H}^{\text{eff}}(E)$ is energy dependent (Eq. (31) should be solved self-consistently)

4.3 Single-resonance approximation

- Eigenstates of $\hat{H}_{QQ}$ (Fig. 13) : bound states $|\,\phi_i\,\rangle$, continuum states $|\,\phi(\epsilon)\,\rangle$ labeled by energy ϵ

$$\begin{aligned} &\hat{H}_{QQ}|\,\phi_i\,\rangle = \epsilon_i|\,\phi_i\,\rangle, \\ &\hat{H}_{QQ}|\,\phi(\epsilon)\,\rangle = \epsilon|\,\phi(\epsilon)\,\rangle \end{aligned}$$

$|\,\phi\,\rangle$: eigenstates without channel transition ($\hat{V}_t = 0$) but with $\hat{V}_Q \neq 0$ only, $|\,\phi\,\rangle \neq |\, Q\,\rangle$

- Spectral decomposition (continuum starts from $\epsilon = \Delta$)

$$\hat{I} = \sum_i |\,\phi_i\,\rangle\langle\,\phi_i\,| + \int_\Delta^\infty d\epsilon|\,\phi(\epsilon)\,\rangle\langle\,\phi(\epsilon)\,|$$

With this, $\hat{H}^{\text{eff}}$ can be written as

$$\hat{H}^{\text{eff}}(E) = \hat{H}_{PP} + \sum_i \frac{\hat{H}_{PQ}|\,\phi_i\,\rangle\langle\,\phi_i\,|\hat{H}_{QP}}{E - \epsilon_i} + \int_\Delta^\infty d\epsilon \frac{\hat{H}_{PQ}|\,\phi(\epsilon)\,\rangle\langle\,\phi(\epsilon)\,|\hat{H}_{QP}}{E - \epsilon + i0^+} \tag{32}$$

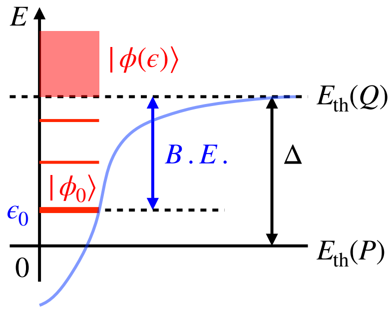


Figure 13: Schematic figure of eigenstates of $\hat{H}_{QQ}$.

- The 3rd term of Eq. (32) has an imaginary part for $E > \Delta$

$$\int dx \frac{f(x)}{x-a+i0^+} = \mathcal{P}\int dx \frac{f(x)}{x-a} - i\pi f(a) \quad \text{(when } x=a \text{ is in the integral range)}$$

  c.f.) When $\Delta < 0$, $\hat{H}^{\rm eff}$ has an imaginary part from $E=0$

- In the 2nd and 3rd terms of Eq. (32), state with the nearest eigenenergy with E is dominant Denoting the state with the smallest ϵ_i as $|\,\phi_0\,\rangle$, at low energy with $E \ll \Delta$,

$$\hat{H}^{\rm eff}(E) \approx \hat{H}_{PP} + \frac{\hat{H}_{PQ}|\,\phi_0\,\rangle\langle\,\phi_0\,|\hat{H}_{QP}}{E-\epsilon_0} \tag{33}$$

  If $\hat{H}_{QQ}$ is confining potential (without continuum) with a single bound state $|\,\phi_0\,\rangle$, then Eq. (33) is exact

- ϵ_0 is measured from threshold of P ($E=0$); binding energy from threshold of Q ($E=\Delta$) is

$$B.E. = \Delta - \epsilon_0$$

  $B.E.$ is fixed by $\hat{H}_{QQ}$ $\Rightarrow$ If Δ is proportional to magnetic field, ϵ_0 can be controlled

4.4 Scattering amplitude and resonance

Lippmann-Schwinger equation

- $\hat{H}^{\rm eff}$ is a single-channel Hamiltonian for P $\Rightarrow$ apply scattering theory in §2

$$\hat{H}^{\rm eff} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2\mu_P}, \quad \hat{H}_0|\,\boldsymbol{p}\,\rangle = \frac{\boldsymbol{p}^2}{2\mu_P}|\,\boldsymbol{p}\,\rangle$$

- Schrödinger equation ($|\,P\,\rangle$ is eigenstate of $\hat{H}^{\rm eff}$)

$$\begin{aligned}
\hat{H}^{\rm eff}|\,P\,\rangle &= E|\,P\,\rangle \\
(\hat{H}_0 + \hat{V})|\,P\,\rangle &= E|\,P\,\rangle \\
\hat{V}|\,P\,\rangle &= (E - \hat{H}_0)|\,P\,\rangle
\end{aligned}$$

Add $(E-\hat{H}_0)|\,\boldsymbol{p}\,\rangle = 0$ in right hand side (for scattering state $|\,P\,\rangle \to |\,\boldsymbol{p}\,\rangle$ at $\hat{V}\to 0$)

$$\begin{aligned}\hat{V}|\,P\,\rangle &= (E-\hat{H}_0)(|\,P\,\rangle - |\,\boldsymbol{p}\,\rangle)\\ (E-\hat{H}_0)^{-1}\hat{V}|\,P\,\rangle &= |\,P\,\rangle - |\,\boldsymbol{p}\,\rangle\\ |\,P\,\rangle &= |\,\boldsymbol{p}\,\rangle + (E-\hat{H}_0)^{-1}\hat{V}|\,P\,\rangle\end{aligned}$$

- Green's operator (resolvent)

$$\hat{G}(E) = (E-\hat{H}_0)^{-1}$$

with this,

$$|\,P\,\rangle = |\,\boldsymbol{p}\,\rangle + \hat{G}\hat{V}|\,P\,\rangle \tag{34}$$

- T operator : relating eigenstate of $\hat{H}_0$ ($|\,\boldsymbol{p}\,\rangle$) and that of $\hat{H}^{\text{eff}}$ ($|\,P\,\rangle$)

$$\begin{aligned}\text{(definition)}\quad \hat{T}|\,\boldsymbol{p}\,\rangle &= \hat{V}|\,P\,\rangle\\ &= \hat{V}|\,\boldsymbol{p}\,\rangle + \hat{V}\hat{G}\hat{V}|\,P\,\rangle \quad \leftarrow \text{(Eq. (34))}\\ &= \hat{V}|\,\boldsymbol{p}\,\rangle + \hat{V}\hat{G}\hat{T}|\,\boldsymbol{p}\,\rangle \quad \leftarrow \text{(definition)}\end{aligned}$$

This leads to Lippmann-Schwinger equation for T operator

$$\begin{aligned}\hat{T} &= \hat{V} + \hat{V}\hat{G}\hat{T}\\ &= \hat{V} + \hat{V}\hat{G}(\hat{V} + \hat{V}\hat{G}\hat{T}) \quad \text{(iterative substitution)}\\ &= \hat{V} + \hat{V}\hat{G}\hat{V} + \hat{V}\hat{G}\hat{V}\hat{G}\hat{V} + \cdots\end{aligned}$$

$\hat{T}$ depends on energy E because $\hat{G}$ does (even if $\hat{V}$ does not)

- Relation with (on-shell) T matrix

$$\langle\,\boldsymbol{p}'\,|\hat{T}(E+i0^+)|\,\boldsymbol{p}\,\rangle = t(\boldsymbol{p}'\leftarrow\boldsymbol{p}) = -\frac{1}{(2\pi)^2\mu_P}f(E,\theta)$$

Poles of $t(\boldsymbol{p}'\leftarrow\boldsymbol{p})$ are poles of scattering amplitude, representing discrete eigenstates

- Lippmann-Schwinger equation for T matrix

$$\begin{aligned}t(\boldsymbol{p}'\leftarrow\boldsymbol{p}) &= \langle\,\boldsymbol{p}'\,|\hat{V}|\,\boldsymbol{p}\,\rangle + \langle\,\boldsymbol{p}'\,|\hat{V}\hat{G}\hat{T}|\,\boldsymbol{p}\,\rangle\\ t(\boldsymbol{p}'\leftarrow\boldsymbol{p}) &= \langle\,\boldsymbol{p}'\,|\hat{V}|\,\boldsymbol{p}\,\rangle + \int d\boldsymbol{q}\langle\,\boldsymbol{p}'\,|\hat{V}|\,\boldsymbol{q}\,\rangle\langle\,\boldsymbol{q}\,|\hat{G}\hat{T}|\,\boldsymbol{p}\,\rangle \quad \leftarrow \hat{I} = \int d\boldsymbol{q}|\,\boldsymbol{q}\,\rangle\langle\,\boldsymbol{q}\,|\\ &= \langle\,\boldsymbol{p}'\,|\hat{V}|\,\boldsymbol{p}\,\rangle + \int d\boldsymbol{q}\langle\,\boldsymbol{p}'\,|\hat{V}|\,\boldsymbol{q}\,\rangle\frac{1}{E-\boldsymbol{q}^2/(2\mu_P)+i0^+}t(\boldsymbol{q}\leftarrow\boldsymbol{p}) \quad \leftarrow \hat{H}_0|\,\boldsymbol{q}\,\rangle = \frac{\boldsymbol{q}^2}{2\mu_P}|\,\boldsymbol{q}\,\rangle\end{aligned}$$

Integral equation for $t(\boldsymbol{p}'\leftarrow\boldsymbol{p})$

Separable interaction

- Separable interaction (product of functions of $\boldsymbol{p}$ and $\boldsymbol{p}'$)

$$\langle \boldsymbol{p}' | \hat{V} | \boldsymbol{p} \rangle = \lambda F(\boldsymbol{p}')F(\boldsymbol{p})$$

In this case, T matrix is ($G(E,\boldsymbol{q}) = [E - \boldsymbol{q}^2/(2\mu_P) + i0^+]^{-1}$)

$$\begin{aligned}
t(\boldsymbol{p}' \leftarrow \boldsymbol{p}) &= \langle \boldsymbol{p}' | \left[\hat{V} + \hat{V}\hat{G}\hat{V} + \hat{V}\hat{G}\hat{V}\hat{G}\hat{V} + \cdots\right] | \boldsymbol{p} \rangle \\
&= \lambda F(\boldsymbol{p}')F(\boldsymbol{p}) + \int d\boldsymbol{q} \lambda F(\boldsymbol{p}')F(\boldsymbol{q})G(E,\boldsymbol{q})\lambda F(\boldsymbol{q})F(\boldsymbol{p}) \\
&\quad + \int d\boldsymbol{q} \int d\boldsymbol{q}' \lambda F(\boldsymbol{p}')F(\boldsymbol{q})G(E,\boldsymbol{q})\lambda F(\boldsymbol{q})F(\boldsymbol{q}')G(E,\boldsymbol{q}')\lambda F(\boldsymbol{q}')F(\boldsymbol{p}) + \cdots \\
&= \lambda F(\boldsymbol{p}')F(\boldsymbol{p}) + \lambda F(\boldsymbol{p}') \left[\lambda \int d\boldsymbol{q} F(\boldsymbol{q})G(E,\boldsymbol{q})F(\boldsymbol{q})\right] F(\boldsymbol{p}) \\
&\quad + \lambda F(\boldsymbol{p}') \left[\lambda \int d\boldsymbol{q} F(\boldsymbol{q})G(E,\boldsymbol{q})F(\boldsymbol{q})\right] \left[\lambda \int d\boldsymbol{q}' F(\boldsymbol{q}')G(E,\boldsymbol{q}')F(\boldsymbol{q}')\right] F(\boldsymbol{p}) + \cdots \\
&= \lambda F(\boldsymbol{p}')F(\boldsymbol{p}) \left[1 + \mathcal{G}(E) + [\mathcal{G}(E)]^2 + \cdots\right] \quad \leftarrow \mathcal{G}(E) = \lambda \int d\boldsymbol{q} F(\boldsymbol{q})G(E,\boldsymbol{q})F(\boldsymbol{q}) \\
&= \lambda F(\boldsymbol{p}')F(\boldsymbol{p}) \left[1 - \mathcal{G}(E)\right]^{-1} \\
&= \frac{\lambda F(\boldsymbol{p}')F(\boldsymbol{p})}{1 - \lambda \displaystyle\int d\boldsymbol{q} F(\boldsymbol{q})G(E,\boldsymbol{q})F(\boldsymbol{q})} \\
&= \frac{F(\boldsymbol{p}')F(\boldsymbol{p})}{\dfrac{1}{\lambda} - \displaystyle\int d\boldsymbol{q} \frac{F(\boldsymbol{q})F(\boldsymbol{q})}{E - \boldsymbol{q}^2/(2\mu_P) + i0^+}}
\end{aligned}$$

Scattering amplitude

$$f(E,\theta) = -(2\pi)^2\mu \frac{F(\boldsymbol{p}')F(\boldsymbol{p})}{\dfrac{1}{\lambda} - \displaystyle\int d\boldsymbol{q} \frac{F(\boldsymbol{q})F(\boldsymbol{q})}{E - \boldsymbol{q}^2/(2\mu_P) + i0^+}}$$

- Potential $\hat{V}$ corresponds to the Hamiltonian in Eq. (33)

$$\hat{V}(E) = \hat{V}_P + \frac{\hat{V}_t | \phi_0 \rangle \langle \phi_0 | \hat{V}_t}{E - \epsilon_0}$$

If ϵ_0 is sufficiently small, the second term is dominant at low energy :
(A special case for $\hat{V}_P \neq 0$ will be discussed in §6)

$$\hat{V}(E) \approx \frac{\hat{V}_t | \phi_0 \rangle \langle \phi_0 | \hat{V}_t}{E - \epsilon_0}$$

This is a separable potential

$$\langle \boldsymbol{p}' | \hat{V}(E) | \boldsymbol{p} \rangle = \frac{\langle \boldsymbol{p}' | \hat{V}_t | \phi_0 \rangle \langle \phi_0 | \hat{V}_t | \boldsymbol{p} \rangle}{E - \epsilon_0}$$

$$\Rightarrow \quad \lambda = \frac{1}{E - \epsilon_0}, \quad F(\boldsymbol{p}) = \langle \phi_0 | \hat{V}_t | \boldsymbol{p} \rangle \quad \text{(form factor)}$$

- Scattering amplitude in P channel

$$f(E,\theta) = -\frac{N(E,\theta)}{E-\epsilon_0-\Sigma(E)}, \quad N(E,\theta) = (2\pi)^2\mu\langle \boldsymbol{p'}\,|\hat{V}_t|\,\phi_0\,\rangle\langle\,\phi_0\,|\hat{V}_t|\,\boldsymbol{p}\,\rangle$$

$$\Sigma(E) = \int d\boldsymbol{q}\frac{\langle\,\phi_0\,|\hat{V}_t|\,\boldsymbol{q}\,\rangle\langle\,\boldsymbol{q}\,|\hat{V}_t|\,\phi_0\,\rangle}{E-\boldsymbol{q}^2/(2\mu_P)+i0^+} \quad \text{(self energy)}$$

- When $\hat{V}_t = 0$, $\Sigma(E) = 0$, so the pole position is

$$E = \epsilon_0 \in \mathbb{R}$$

corresponds to the bound state by $\hat{H}_{QQ}$

- When $\hat{V}_t \neq 0$, in general the solution of $E - \epsilon_0 - \Sigma(E) = 0$, but for weak $\hat{V}_t$, we have

$$E \approx \epsilon_0 + \Sigma(\epsilon_0) \quad \text{(perturbative approximation)}$$

When $\epsilon_0 > 0$, $\Sigma(\epsilon_0)$ has an imaginary part (dq integration starts from $q = 0$)
$\Rightarrow$ **resonance with complex eigenenergy**

- Physically, bound state by $\hat{H}_{QQ}$ acquires decay width through transition to the continuum of P

4.5 Controlling scattering length by magnetic field

- For s wave ($\ell = 0$) scattering, $N(E,\theta)$ had no θ dependence, and scattering length a_0 is (see §3)

$$a_0 = -f(E=0) = \frac{N(0)}{-\epsilon_0 - \Sigma(0)}$$

where $N(0) > 0$ and

$$\Sigma(0) = \int d\boldsymbol{q}\frac{\langle\,\phi_0\,|\hat{V}_t|\,\boldsymbol{q}\,\rangle\langle\,\boldsymbol{q}\,|\hat{V}_t|\,\phi_0\,\rangle}{-\boldsymbol{q}^2/(2\mu_P)} = -\int d\boldsymbol{q}\frac{2\mu_P|\langle\,\boldsymbol{q}\,|\hat{V}_t|\,\phi_0\,\rangle|^2}{\boldsymbol{q}^2} < 0$$

- When Δ (namely ϵ_0) is proportional to the external magnetic field B

$$\epsilon_0 = CB + \epsilon_0^{(0)}$$

$C > 0$ because the splitting increases with the magnetic field
$\epsilon_0^{(0)}$ is the energy of the bound state with $B = 0$ where $E_{\text{th}}(Q) = 0$, so $\epsilon_0^{(0)} < 0$
Scattering length depends on B as

$$a_0(B) = -\frac{N(0)}{C(B-B_0)}, \quad B_0 = \frac{-\Sigma(0)-\epsilon_0^{(0)}}{C} > 0 \tag{35}$$

Scattering length diverges at $B = B_0$: unitary limit

- With $\hat{V}_P \neq 0$, we obtain (Fig. 12 right)

$$a_0(B) = a_{\text{BG}}\left[1 - \frac{\Delta B}{B-B_0}\right] \tag{36}$$

a_{BG} is the scattering length only by $\hat{V}_P$

Exercise 4

1) When the S matrix $s(p)$ (suppressing ℓ) has a pole at $p = p_R \in \mathbb{C}$, it is written as $s(p) = A(p)(p-p_R)^{-1}$ with a function $A(p) \in \mathbb{C}$. From the unitarity condition (14), show that in general we can write $A(p) = C(p)(p - p_R^*)$ with a complex function with unit magnitude $C(p)$.

2) From this, the S matrix is given by ($\delta_{\mathrm{BG}}(p) \in \mathbb{R}$)

$$s(p) = s_{\mathrm{BG}}(p)s_{\mathrm{BW}}(p), \quad s_{\mathrm{BG}}(p) = e^{2i\delta_{\mathrm{BG}}(p)}, \quad s_{\mathrm{BW}}(p) = C_{\mathrm{BW}}\frac{p - p_R^*}{p - p_R}$$

From Eq. (13), show that $s(p = 0) = 1$ for $|f(p = 0)| < \infty$, and determine C_{BW} when $s_{\mathrm{BW}}(p)$ follows this condition.

3) Let the scattering lengths of $s(p)$, $s_{\mathrm{BG}}(p)$, $s_{\mathrm{BW}}(p)$ be a_0, a_{BG}, a_{BW}, respectively. Express a_0 by using a_{BG} and a_{BW}.

4) Let the scattering length with $\hat{V}_P = 0$ be a_{BW}, and that of only $\hat{V}_P \neq 0$ be a_{BG}. Show that B dependence of the total scattering length a_0 is given in the form of Eq. (36), and determine ΔB.

4.6 Summary of §4

- Coupled-channel Hamiltonian of P and Q
- Eliminating channel Q to obtain effective Hamiltonian in P
- Bound state $|\,\phi_0\,\rangle$ in Q couples with P to generate complex energy state

5 Nonrelativistic effective field theory

5.1 Effective field theories

- Microscopic quantum field theory $\mathcal{L}_{\text{micro}}$
- Λ : ultraviolet cutoff scale (see Fig. 14)
- Ω : low-energy/long-wavelength phenomena below Λ
- Effective Field Theory, EFT $\mathcal{L}_{\text{EFT}}$
  - $\mathcal{L}_{\text{EFT}}$ describes the same Ω with $\mathcal{L}_{\text{micro}}$ does
  - can be elaborated systematically
  - Λ : applicability bound of EFT

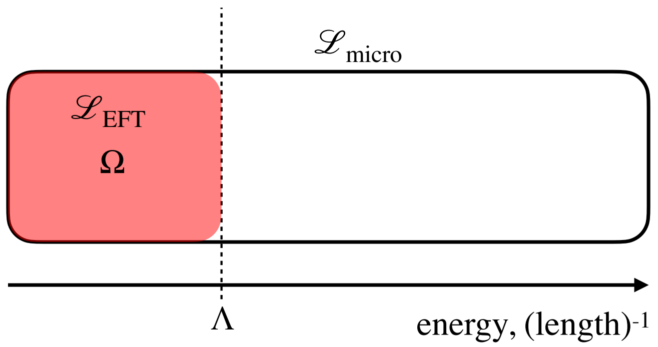


Figure 14: Schematic figure of effective field theory.

- Example 1 : Electromagnetic interaction
  - Microscopic theory : QED

$$\mathcal{L}_{\text{micro}} = \mathcal{L}_{\text{QED}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{kinetic term of photons}} + \underbrace{\bar{e}(i\not{D} - m_e)e}_{\text{kinetic, mass, interaction terms of photons}}$$

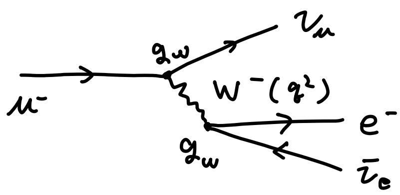
massless photons and electrons with mass m_e

  - EFT : Euler-Heisenberg theory [33]

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underbrace{c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}_{\text{interaction terms of photons}} + \cdots$$

Electrons are "heavy", only photons ($\Lambda \sim m_e$)

Coefficients are calculable from QED : $c_1 = \dfrac{\alpha^2}{90m_e^4}$, $c_2 = \dfrac{7\alpha^2}{360m_e^4}$



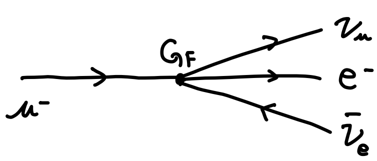


Figure 15: Schematic figure of μ^- decay. Left : Weinberg-Salam theory, Right : Fermi theory

- Example 2 : Weak interaction

  - Microscopic theory : Weinberg-Salam theory

$$\mathcal{L}_{\text{micro}} = \mathcal{L}_{\text{WS}}(\text{leptons, neutrinos, } W^{\pm}, Z, \ldots)$$

    Interaction is mediated by exchange of $W^{\pm}, Z$ (Fig. 15, left)

$$\text{interaction} \propto \frac{g_w^2}{q^2 - m_W^2} = -\frac{g_w^2}{m_W^2}\left(1 + \mathcal{O}\left(\frac{q^2}{M_W^2}\right)\right)$$

  - EFT : Fermi theory

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{F}}(\text{leptons, neutrinos, } \ldots)$$

    four-Fermi (contact) interaction (Fig. 15, right)
    $W^{\pm}, Z$ are "heavy", only fermions ($\Lambda \sim m_{W^{\pm}}, m_Z$)

$$\text{interaction} \propto G_F \left(\propto -\frac{g_w^2}{m_W^2}\right)$$

- Example 3 : strong interaction

  - Microscopic theory : QCD

$$\mathcal{L}_{\text{micro}} = \mathcal{L}_{\text{QCD}}(\text{quarks, gluons})$$

    not calculable at low energy, hadrons are degrees of freedom (color confinement)

  - Weinberg's "theorem" [34]
    The most general $\mathcal{L}_{\text{EFT}}$, consistent with the symmetries of $\mathcal{L}_{\text{micro}}$, effectively describes Ω
  - EFT : chiral perturbation theory (having chiral symmetry as QCD) [35]

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{ChPT}}(\text{hadrons})$$

    The most general Lagrangian contains infinitely many terms → sorted out by importance

$$\mathcal{L}_{\text{ChPT}} = \mathcal{L}^{(\text{LO})} + \mathcal{L}^{(\text{NLO})} + \cdots$$

5.2 Zero-range model

- Description of nonrelativistic two-body scattering in EFT
- $R_{\rm typ}$: typical length scale of interaction
  - Square well potential : $R_{\rm typ} = b$ (well width)
  - Yukawa potential $V(r) = g\dfrac{e^{-\kappa r}}{r}$: $R_{\rm typ} = 1/\kappa$
  - van der Waals potential $V(r) = -\dfrac{C_6}{r^6}$: $R_{\rm typ} \sim \ell_{\rm vdW} = (mC_6/\hbar^2)^{1/4}$
- Zero-range model $\mathcal{L}_{\rm ZR}$: s-wave scattering with larger $|a_0|$ than $R_{\rm typ}$ ($\ell = 0$ abbreviated) [17]

$$f(p) = \frac{1}{-\dfrac{1}{a_0} - ip} \tag{37}$$

  - Nucleons : long-range tail is of Yukawa form by π exchange

$$a_0(^1S_0) \simeq 20 \text{ fm}, \quad a_0(^3S_1) \simeq -4 \text{ fm}, \qquad |a_0| \gg R_{\rm typ} \sim \frac{1}{m_\pi} \sim 1 \text{ fm}$$

(Nucleons : fermions with spin and isospin degrees of freedom)

$$\mathcal{L}_{\rm micro} = \mathcal{L}_{NN} \quad (\text{or } \mathcal{L}_{\rm QCD})$$

  - ^{4}He atoms : long-range tail is of van der Waals form by polarization

$$a_0 \simeq 200 \text{ [Bohr radius]} \qquad |a_0| \gg R_{\rm typ} \sim \ell_{\rm vdW} \sim 10 \text{ [Bohr radius]}$$

$$\mathcal{L}_{\rm micro} = \mathcal{L}_{\rm atom}$$

- At low-energy $p \ll 1/R_{\rm typ}$, both can be described by the same $\mathcal{L}_{\rm ZR}$

Lagrangian

- Lagrangian density of zero-range model

$$\mathcal{L}_{\rm ZR} = \underbrace{\psi^\dagger \left(i\partial_t + \frac{\boldsymbol{\nabla}^2}{2m}\right)\psi}_{\text{kinetic term}} - \underbrace{\frac{\lambda_0}{4}(\psi^\dagger\psi)^2}_{\text{interaction term}} \tag{38}$$

$\psi(t, \boldsymbol{x})$: boson field

m : boson mass

λ_0 : (bare) coupling constant

(two-fermions with antisymmtric spin w.f. is essentially same with two-bosons)

- Quantization : equal-time commutation relation

$$[\psi(t, \boldsymbol{x}), \psi(t, \boldsymbol{x}')] = 0$$

$$[\psi(t, \boldsymbol{x}), \psi^\dagger(t, \boldsymbol{x}')] = \delta^3(\boldsymbol{x} - \boldsymbol{x}')$$

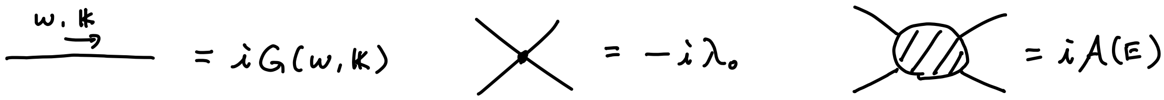


Figure 16: Feynman rules of zero-range model (38). Left : boson propagator iG, Middle : vertex $-i\lambda_0$, Right : four-point function $i\mathcal{A}$

- Interaction term : four-point contact interaction $\sim$ 3d δ function potential

$$-\mathcal{L}_{\text{int}} = \frac{\lambda_0}{4}(\psi^\dagger\psi)^2 \sim \mathcal{H}_{\text{int}} \sim (\text{energy})$$

$$\begin{cases} \lambda_0 > 0 & \text{increase energy} \Rightarrow \text{repulsion} \\ \lambda_0 < 0 & \text{decrease energy} \Rightarrow \text{attraction} \end{cases}$$

- Symmetries : space-time translation, rotation, parity, Galilean boost
  phase symmetry

$$\psi(t, \boldsymbol{x}) \to e^{i\theta}\psi(t, \boldsymbol{x})$$

  corresponding conserved charge

$$N = \int d\boldsymbol{x}\ \psi^\dagger\psi \quad (\text{particle number})$$

  $\Rightarrow \mathcal{L}_{\text{int}}$ does not change the particle number (two-body is always two-body)

Feynman rules

- Calculation of physical quantities in quantum field theory

  1. Derive Feynman rules (peaces of Feynman diagrams)
  2. Sum up all possible Feynman diagrams (two-body sector of $\mathcal{L}_{\text{ZR}}$ is possible)

  2'. Perform perturbation theory (when 2. is not doable)

- Propagator : propagation of particle (Fig. 16, left)

$$iG(\omega, \boldsymbol{k}) = \frac{1}{\omega - \boldsymbol{k}^2/(2m) + i0^+}$$

  only positive energy component : only forward going in time

- vertex : interaction (Fig. 16, middle)

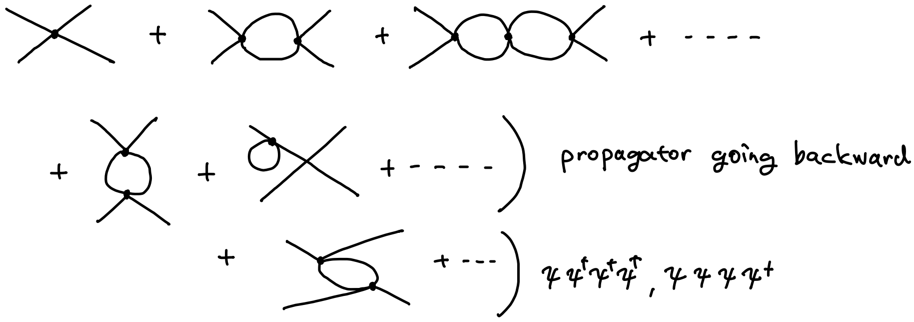


Figure 17: Candidates of Feynman diagrams.

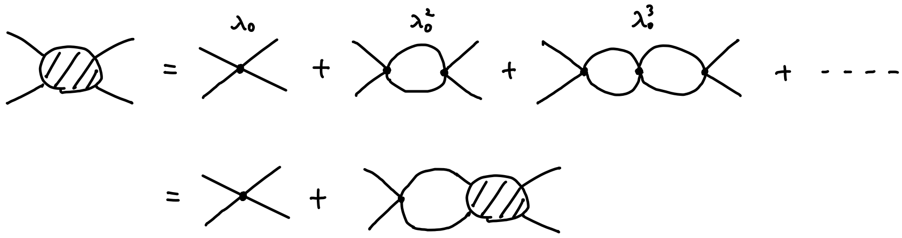


Figure 18: Possible Feynman diagrams.

5.3 Two-boson scattering

- two-body scattering amplitude $\leftarrow$ four-point function $i\mathcal{A}(E)$ (2 in, 2 out, Fig. 16, right)
- Write down all diagrams from Feynman rules with keeping initial and final sates (Fig. 17)
- Eventually, same structure with Lippmann-Schwinger equation remains (Fig. 18)
- Different $(\lambda_0)^n$ terms are summed to all orders : nonperturbative scattering amplitude
- Perturbative expansion with small λ_0 leads the first term : $i\mathcal{A}(E) = -i\lambda_0$

Calculation of scattering amplitude

- Two-body scattering amplitude $\mathcal{A}(E)$

$$i\mathcal{A}(E) = -i\lambda_0 - i\lambda_0 \frac{1}{2} \int \frac{d\omega d\boldsymbol{q}}{(2\pi)^4} iG(\omega, \boldsymbol{q}) iG(E-\omega, -\boldsymbol{q}) i\mathcal{A}(E) \tag{39}$$

  - 1/2 is the symmetry factor
  - Here completeness relation is $1 = \int \frac{d\boldsymbol{q}}{(2\pi)^3} |\, \boldsymbol{q} \,\rangle\langle\, \boldsymbol{q} \,|$
  - dq integration diverges : introduce cutoff Λ (integral range $0 \leq q \leq \Lambda$)
  - $i\mathcal{A}(E)$ in right hand side in not in the integration : same with separable interaction

- $\mathcal{A}(E)$ can be determined algebraically

$$\mathcal{A}(E) = \left[-\frac{1}{\lambda_0} - \frac{m}{4\pi^2} \left(\Lambda - \sqrt{-mE - i0^+} \arctan \frac{\Lambda}{\sqrt{-mE - i0^+}} \right) \right]^{-1} \tag{40}$$

- Energy E and momentum p

$$E = \frac{p^2}{2\mu} = \frac{p^2}{m} \quad \leftarrow \quad \mu = \frac{mm}{m+m} = \frac{m}{2}$$

For physical scattering $E > 0$, $p > 0$,

$$\sqrt{-mE - i0^+} = -i\sqrt{m|E|} = -i\sqrt{p^2} = -ip$$

- For a small momentum $p \ll \Lambda$ than the cutoff Λ,

$$\arctan \left(\frac{\Lambda}{-ip} \right) = \frac{\pi}{2} + \mathcal{O}\left(\frac{p}{\Lambda} \right)$$

then, Eq. (40) is

$$\mathcal{A}(p) = \left[-\frac{1}{\lambda_0} - \frac{m}{4\pi^2} \left(\Lambda + ip\frac{\pi}{2} \right) \right]^{-1} = \left[-\frac{1}{\lambda_0} - \frac{m}{4\pi^2}\Lambda - ip\frac{m}{8\pi} \right]^{-1}$$

- Scattering amplitude

$$f(p) = \frac{m}{8\pi}\mathcal{A}(p) = \frac{1}{-\frac{8\pi}{m}\left(\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\Lambda \right) - ip}$$

Comparing with Eq. (37), scattering length is

$$a_0 = \frac{m}{8\pi} \left(\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\Lambda \right)^{-1} \tag{41}$$

Unitarity

- Scattering amplitude is nonperturbative

$$f_{\mathrm{NP}}(p) = \frac{1}{-1/a_0 - ip}$$

- Scattering amplitude with $\mathcal{O}(\lambda_0^1)$ perturbation theory

$$f_{\mathrm{P}}(p) = -\frac{m}{8\pi}\lambda_0 = C \quad \text{(constant)}$$

Fourier transformation of the interaction term, namely, Born approximation

- From Eq. (13), S matrix is given by $s(p) = 2ipf(p) + 1$, so

$$s_{\mathrm{NP}}(p) = \frac{2ip}{-1/a_0 - ip} + 1 = \frac{2ip - 1/a_0 - ip}{-1/a_0 - ip} = \frac{-1/a_0 + ip}{-1/a_0 - ip}$$

$$s_{\mathrm{P}}(p) = 2ipC + 1$$

- Unitarity condition (14)

$$s^*_{\rm NP}(p)s_{\rm NP}(p) = \frac{-1/a_0 - ip}{-1/a_0 + ip}\frac{-1/a_0 + ip}{-1/a_0 - ip} = 1$$

$$s^*_{\rm P}(p)s_{\rm P}(p) = (2ipC+1)(-2ipC+1) = 1 + 4C^2p^2 \neq 1$$

- Nonperturbative scattering amplitude $f_{\rm NP}(p)$ satisfies unitarity, but $f_{\rm P}(p)$ dose not

- Perturbative calculation violates unitarity (If f is constant, so is $\sigma \sim f^2$, violating unitarity bound)

5.4 Renormalization

- Scattering length a_0 is observable and independent of cutoff Λ

- Λ dependent coupling constant λ_0

$$\lambda_0(\Lambda) = \left(1 - \frac{2a_0}{\pi}\Lambda\right)^{-1}\frac{8\pi}{m}a_0 \tag{42}$$

  coupling constant λ_0 for a given Λ to give fixed scattering length a_0

- Under Eq. (42), the limit $\Lambda \to \infty$ can be taken

- Renormalization group equation : behavior of coupling constant with respect to cutoff Λ

$$\frac{d}{d(\ln\Lambda)}\hat{\lambda}(\Lambda) = \hat{\lambda}(\Lambda)\left[1 + \hat{\lambda}(\Lambda)\right] \tag{43}$$

  dimensionless coupling constant

$$\hat{\lambda}(\Lambda) = \frac{m}{4\pi^2}\Lambda\lambda_0(\Lambda)$$

- Fixed point $\hat{\lambda}^*$: the value at which $\hat{\lambda}$ is Λ independent, RHS of Eq. (43)= 0, so that $\hat{\lambda}^* = 0$ or -1
  - $\hat{\lambda}^* = 0$: $a_0 = 0$, noninteracting, trivial
  - $\hat{\lambda}^* = -1$: $a_0 = \pm\infty$, unitary limit, nontrivial

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Exercise 5

1) Show that a_0 is cutoff independent if the coupling constant λ_0 has Λ dependence as in Eq. (42).
2) Show that the renormalization group equation for $\hat{\lambda}(\Lambda)$ is Eq. (43).
3) Show that $a_0 = 0$ ($a_0 = \pm\infty$) when $\hat{\lambda}^* = 0$ ($\hat{\lambda}^* = -1$).

5.5 Summary of §5

- EFT : description of low-energy physics

- Zero-range model : nonperturbative (unitary) scattering amplitude

$$f(p) = \frac{1}{-1/a_0 - ip}$$