0 Introduction : resonances in hadron physics

Hadrons

- Hadrons: particles interacting through the strong force
- Observed hadrons [1]
  - Baryons $(p, n, \Lambda, \cdots)$: about 150 species
  - Mesons $(\pi, K, \eta, \cdots)$: about 210 species
- All states emerge from QCD, color SU(3) gauge theory
  $$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} + \bar{q}(i\not{D} - m_q)q$$
  color triplet quarks : $\bar{q}, q$
  color octet gluons in $G_{\mu\nu}, D_\mu$
- Quarks have six kinds of flavor (u, d, s, c, b, t)

Regularity of observed states

- Hadrons belong to color singlet
  No rule in QCD to forbid the appearance of colored states
  $\rightarrow$ problem of color confinement
- Flavor quantum numbers can be described by qqq or $\bar{q}q$
  No rule in QCD to forbid the appearance of $\bar{q}\bar{q}qq$, $\bar{q}qqqq$, $\cdots$
  $\rightarrow$ problem of exotic hadrons (genuine exotics)
- Experimental fact, nothing to do with quark models
- J^{PC} exotic mesons have been observed : $\pi_1(1400)$ and $\pi_1(1600)$ have $J^{PC} = 1^{-+}$

Exotic structure candidates (heavy quark system : c, b)

- Pentaquarks $P_c(4312), P_c(4440), P_c(4457)$ [2, 3] (Fig. 1, left)
  $$P_c \rightarrow J/\psi(\bar{c}c) + p(uud)$$
- Tetraquarks $Z_b(10610), Z_b(10650)$ [4] (Fig. 1, right)
  $$Z_b^{\pm} \rightarrow \Upsilon(\bar{b}b) + \pi^{\pm}(\bar{d}u/\bar{u}d)$$
- Only ~ 8 candidates out of 360 hadrons
- Internal structure (multiquarks, hadronic molecules, $\cdots$) has not been determined yet
- Flavor quantum numbers can be described by $P_c \sim uud$, $Z_b^+ \sim \bar{d}u$
  But it is unnatural to explain their mass without heavy quark pairs

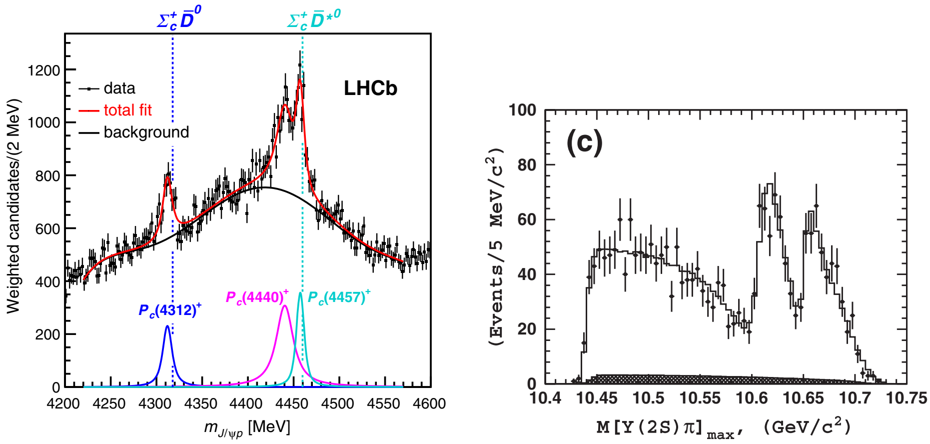

Figure 1: Examples of spectra of exotic hadrons. Left: spectrum of pentaquark P_c, adopted from R. Aaij *et al.*, (LHCb collaboration), Phys. Rev. Lett. **122**, 222001 (2019). Right: spectrum of tetraquark Z_b, adopted from A. Bondar *et al.*, (Belle Collaboration) Phys. Rev. Lett. **108**, 122001 (2012).

Exotic structure candidates (light quark system : u, d, s)

- $\Lambda(1405)$: $S = -1$, $I = 0$, $J^P = 1/2^-$ [5]
  - Difficult to describe in quark models
    uds with orbital angular momentum $\ell = 1 \to \sim 1.6$-1.7 GeV
    Experiments : ~ 1.4 GeV
  - $\bar{K}N$ molecular state? Pentaquarks?
- Scalar mesons: $\sigma, \kappa, f_0(980), a_0(980)$, $J^P = 0^+$ [6]
  - Difficult to describe in quark models
    Masses of $\bar{q}q$ states : $\bar{n}n(I = 0, I = 1) < \bar{n}s, \bar{s}n < \bar{s}s(I = 0)$
    Experiments : $\sigma(I = 0) < \kappa < f_0, a_0(I = 0, I = 1)$
  - Meson-meson molecular state? Tetraquarks?
- Flavor quantum numbers can be described by $\Lambda(1405) \sim uds$, Scalar mesons $\sim \bar{q}q$

Excitation mechanism of hadrons

- Excitation in constituent quark models : internal excitation (Fig. 2)
- Excitation with $\bar{q}q$ pair creation is possible in QCD $\to$ multiquarks and hadronic molecules
- States with same quantum numbers can mix with each other

$$|\,\Lambda(1405)\,\rangle = C_{3q}|\,uds\,\rangle + C_{5q}|\,uds\bar{q}q\,\rangle + C_{MB}|\,MB\,\rangle + \cdots .$$

How can we determine the weight C_i? Well-defined decomposition?

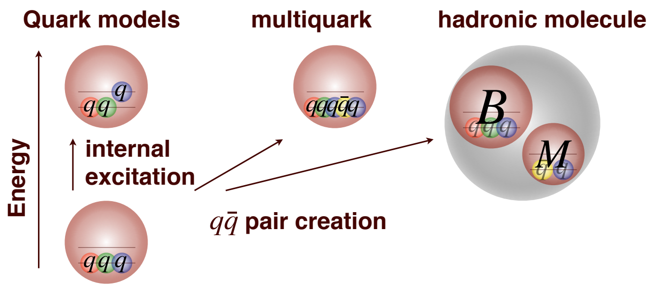

Figure 2: Schematic illustration of the excitation mechanisms of baryons.

Decays via strong interaction

- Hadrons of interest are unstable against strong decay

$$
\begin{aligned}
P_c &\to J/\psi + p \\
Z_b &\to \Upsilon + \pi^{\pm} \\
\Lambda(1405) &\to \pi + \Sigma \\
\sigma &\to \pi + \pi, \quad \kappa \to \pi + K, \quad \cdots
\end{aligned}
$$

  In fact, stable hadrons are $\sim 20/360$

- Unstable states should be treated as resonances in hadron scattering

Goal and plan of this lecture

- Structure of exotic hadrons
  How can we define the "structure" of unstable resonances?

- What are resonance "states"?

  - §1 Resonances in quantum mechanics (5 Oct.)
    → eigenstates with complex energy
  - §2 Scattering theory primer (12 Oct.)
    → definition of scattering amplitude
  - §3 Resonances in scattering theory (12 Oct.)
    → poles of scattering amplitude
  - §4 Theory of Feshbach resonances (19 Oct.)
    → bound state embedded in continuum
  - §5 Nonrelativistic effective field theory (19 Oct.)
    → description of low-energy scattering
  - §6 Compositeness and weak-binding relation (26 Oct.)
    → application to hadron systems

Relation to other fields

- Nuclear physics
  Cluster structure of near-threshold excited states : ^{8}Be $\sim \alpha\alpha$, Hoyle state of ^{12}C $\sim \alpha\alpha\alpha$, etc.

- Particle physics
  Higgs particle : $H \to \gamma\gamma$, $H \to ZZ$
  Observed through the decays into known particles $\to$ a resonance
  Higgs in the standard model? Composite of new particles?

- Atomic physics
  Feshbach resonance by cold atoms [7]
  $\to$ controlling scattering length (interaction strength) via external magnetic field
  Broad/narrow Feshbach resonance : entrance channel fraction $\approx$ compositeness

References

- Resonances in quantum mechanics
  Textbook : A. Bohm [8], Kukulin-Krasnopol'sky-Horacek [9], N. Moiseyev [10]
  Review article : Ashida-Gong-Ueda [11]

- Scattering theory
  Textbook : J.R. Taylor [12], R.G. Newton [13]
  Review article : Hyodo-Niiyama [14]

- Feshbach resonances
  Review articles : Köhler-Góral-Julienne [15], Chin-Grimm-Julienne-Tiesinga [16]

- Effective field theory
  Review article : Braaten-Kusunoki-Zhang [17]

- Compositeness and weak-binding relation
  Original papers : S. Weinberg [18], Kamiya-Hyodo [19, 20]
  Review article : T. Hyodo [21]
  JPS journal (in Japanese) : T. Hyodo [22]

- This lecture is partly based on the intensive lectures by Naomichi Hatano (YITP, Kyoto Univ., Feb. 2017) and by Yusuke Nishida (YITP, Kyoto Univ., Feb. 2014)

- For Japanese students : 同じ内容の都立大講義の講義ノート（日本語版）が
  `http://www.comp.tmu.ac.jp/hyodo/2020Tokuron.html`
  にあります。

1 Resonances in quantum mechanics

1.1 Overview of resonance states

Resonances and scattering states

- Resonance : quantum mechanically formed quasi-stable "state" which decays as time goes by
- Schrödinger equation is time reversal invariant
  $\leftrightarrow$ Resonances only decay, namely, not invariant under time reversal
  Solution breaks symmetry of theory (equation) : spontaneous breaking?
- Decay products are scattering states (continuum) $\rightarrow$ need for scattering theory
  Example) $\Lambda(1405) \rightarrow \pi\Sigma$: $\Lambda(1405)$ is a resonance in the $\pi\Sigma$ scattering
- Inelastic scattering and scattering channels
  - Elastic scattering : initial state = final state ($\pi\Sigma \rightarrow \pi\Sigma$)
  - Inelastic scattering : transition to different final states ($\pi\Sigma \rightarrow \bar{K}N$, $\pi\Sigma \rightarrow \pi\pi\Sigma$, etc.)
  - Channels : states connected through inelastic scatterings ($\pi\Sigma$, $\bar{K}N$, $\pi\pi\Sigma$, etc.)
- Threshold : lowest energy of scattering states
  Example) Threshold is $E = 0$ if potential vanishes at $r \rightarrow \infty$

Exercise 1
1) Let Θ be the time-reversal operator. Considering the classical time-reversal operation for the coordinate $\boldsymbol{r}$ and the momentum $\boldsymbol{p}$, derive $\Theta\boldsymbol{r}\Theta^{-1}$ and $\Theta\boldsymbol{p}\Theta^{-1}$.
2) Calculate the commutation relation $[L_i, p_j]$ with the angular momentum $L_i = \epsilon_{ijk} r_j p_k$ ($\hbar = 1$).
3) To satisfy the same commutation relation $[L_i, p_j]$ after the time reversal, it turns out that Θ must be an antilinear operator ($\Theta\alpha\psi = \alpha^*\Theta\psi$ for $\alpha \in \mathbb{C}, \psi \in \mathcal{H}$ where $\mathcal{H}$ is the Hilbert space). When the Hamiltonian is time-reversal invariant ($\Theta H \Theta^{-1} = H$), show the time-reversal invariance of the Schrödinger equation

$$i\frac{\partial\Psi(\boldsymbol{r}, t)}{\partial t} = H\Psi(\boldsymbol{r}, t),$$

namely, show that $\Theta\Psi(\boldsymbol{r}, t)$ follows the same equation.

Characterization of resonances

- Various definitions : how are they related?
  - Peak in spectra/cross sections : Fig. 3(a)
  - $\pi/2$ crossing of phase shift $\delta(E)$: Fig. 3(b)
  - Pole of scattering amplitude in complex energy plane : Fig. 3(c)
  - Eigenstate of Hamiltonian (with complex energy)

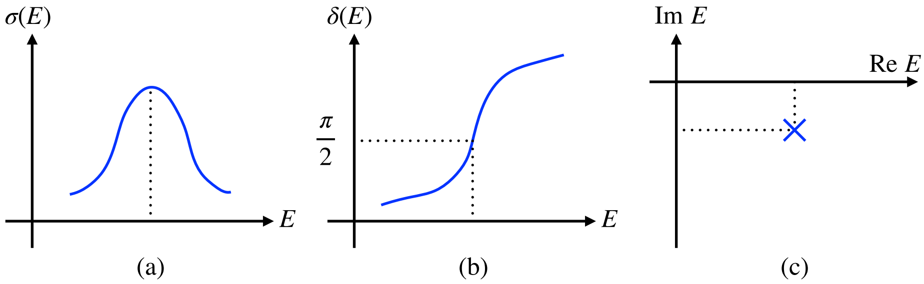

Figure 3: Schematic illustration of characterization of resonances. (a): peak in total cross section $\sigma(E)$, (b) : $\pi/2$ crossing of phase shift $\delta(E)$, (c) : pole of scattering amplitude.

Shape resonance and Feshbach resonance

- Resonances can be classified into two classes
- Shape resonance, potential resonance : Fig. 4(b)
  - Single-channel scattering
    Typical potential : short range attraction + repulsive barrier
  - Energy $E > 0$
  - Unstable via tunneling effect
- Feshbach resonance : Fig. 4(c)
  - Coupled-channel scattering
    P : entrance channel, Q : closed channel
  - Threshold of Q at $E = \Delta > 0$ with threshold of P being $E = 0$
  - A bound state of channel Q at $0 < E < \Delta$
  - Unstable via $Q \to P$ transition
- They are different in origin
  Method to distinguish $\to$ compositeness

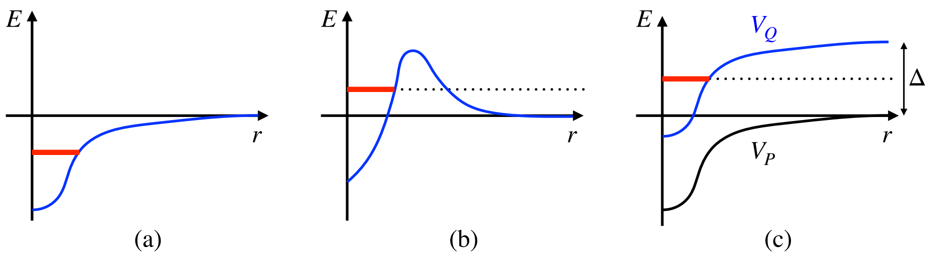

Figure 4: Illustration of resonances. (a) : bound state, (b) : shape resonance, (c) : Feshbach resonance.

1.2 Resonances as eigenstates of Hamiltonian

- Ref. [23] : To describe α decay of atomic nuclei, imaginary part of eigenenergy is introduced by hand (opposite sign from current convention Im $E < 0$)

$$E = E_0 + i\frac{h\lambda}{4\pi} = E_0 + i\frac{\hbar\lambda}{2}$$

λ : decay constant, related to the decay width through $\Gamma = \hbar\lambda$

- Time evolution of wave function (opposite sign from current convention e^{-iEt})

$$\Psi(t) \propto \exp\{+iEt/\hbar\} = \exp\{+iE_0 t/\hbar\} \exp\{-\lambda t/2\}$$

Probability decreases exponentially $|\Psi(t)|^2 \propto \exp\{-\lambda t\}$

- Inconsistent with "expectation value of hermitian operator is real"?
$\leftarrow$ Space on which the operator acts (domain $D(H)$) needs to be specified
c.f. definition of hermitian conjugate : $\langle H^\dagger \Psi, \Phi\rangle = \langle \Psi, H\Phi\rangle$, $\Psi, \Phi \in D(H)$

  – Eigenvalues are real if $D(H)$ is Hilbert space ($\sim$ square integrable function space $L^2(\mathbb{R}^d)$)

$$\int |\Psi(x)|^2 dx < \infty$$

  If domain is extended, H can have complex eigenvalues

  – Example of non-square-integrable wave function : plane wave $\Psi(x) \sim e^{\pm ipx}$

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = \int_{-\infty}^{\infty} 1\ dx \to \infty$$

  Resonances couple with scattering states through decay : non-square-integrable wave function

1.3 Square well potential

Definitions and scattering states

- Schrödinger equation ($\hbar = 1, m = 1$)

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + V(r)\right)\chi(r) = E\chi(r), \quad 0 \leq r \leq \infty \tag{1}$$

  – In this unit system, physical quantities are counted by dimension of length

$$(\text{energy}) = (\text{length})^{-2}, \quad (\text{momentum}) = (\text{length})^{-1}$$

  – $\chi(r) \sim$ radial wave function of spherical 3d potential : $\psi_{\ell,m}(\boldsymbol{r}) = \dfrac{\chi_\ell(r)}{r} Y_\ell^m(\hat{\boldsymbol{r}})$

- Attractive square well potential ($V_0 > 0$, Fig. 5(a))

$$V(r) = \begin{cases} -V_0 & 0 \leq r \leq b \\ 0 & b < r \end{cases} \tag{2}$$

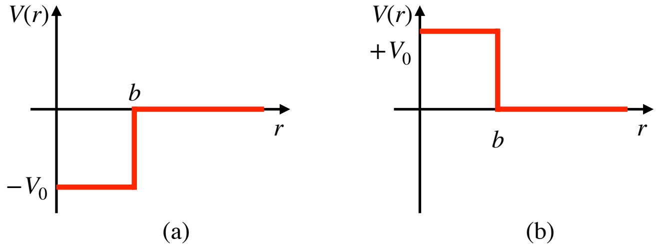

Figure 5: Width b rectangular potentials. (a) : attraction with depth V_0, (b) : repulsion with height V_0.

- General solutions (no boundary condition)

$$\chi(r) \propto \begin{cases} e^{\pm ikr} & 0 \le r \le b, \quad k = \sqrt{2(E+V_0)} \\ e^{\pm ipr} & b < r, \quad p = \sqrt{2E} \end{cases}$$

- Scattering solutions (boundary condition $\chi(r \to 0) = 0$)

$$\chi(r) = \begin{cases} C\sin(kr) & 0 \le r \le b \\ A^-(p)e^{-ipr} + A^+(p)e^{+ipr} & b < r \end{cases} \tag{3}$$
$$A^\pm(p) = \frac{C}{2}\left[\sin(kb) \mp i\frac{k}{p}\cos(kb)\right]e^{\mp ipb}$$

  - Scattering solutions are not normalizable (non-vanishing at $r \to \infty$)
    $\to$ Overall normalization C is arbitrary
  - $A^\pm(p)$ is determined by continuity of χ and $d\chi/dr$ at $r = b$
  - Scattering phase shift is determined by the wave function at $r \to \infty$ (see §2)

- Scattering solutions satisfy Schrödinger equation (1) for any $E > 0$: continuous spectrum

- Wave $e^{\pm ipr}$ propagates in $\pm r$ direction : A^+ (A^-) is the amplitude of outgoing (incoming) wave

Discrete eigenstates and boundary conditions

- Discrete eigenstates are obtained by imposing boundary conditions both at $r \to 0$ and $r \to \infty$

- Bound state solution : eigenenrgy $E < 0 \Leftrightarrow$ pure imaginary eigenmomentum $p = \sqrt{2E}$

$$p = i\kappa, \quad \kappa > 0$$

  Wave function at $r \to \infty$ behaves as

$$\chi(r) = A^-(i\kappa)e^{+\kappa r} + A^+(i\kappa)e^{-\kappa r} \quad (r \to \infty)$$

Table 1: Numerical solutions of Eq. (4) (discrete eigenstates of attractive square well) with $V_0 = 10b^{-2}$.

	p $[b^{-1}]$	$E = p^2/2$ $[b^{-2}]$
Bound state B	$+3.68i$	-6.78
1st resonance R_1	$1.06 - 1.02i$	$0.05 - 1.08i$
2nd resonance R_2	$6.29 - 1.41i$	$18.8 - 8.86i$
3rd resonance R_3	$9.90 - 1.69i$	$47.6 - 16.8i$
$\vdots$		

- Boundary condition : $\chi(r)$ is square integrable $\to$ eliminate diverging component $e^{+\kappa r}$

$$A^-(i\kappa) = 0$$

For $p = i\kappa$, incoming wave (e^{-ipr}) vanishes, leaving outgoing wave (e^{+ipr}) only

- $A^-(p) = 0$: **outgoing boundary condition**

$$\tan(\sqrt{p^2 + 2V_0}\ b) = -i\frac{\sqrt{p^2 + 2V_0}}{p} \tag{4}$$

Substituting $p = i\kappa$, we obtain bound state condition for square well potential $\kappa = -k\cot(kb)$

Resonance solutions

- Bound states : solution of Eq. (4) with pure imaginary p
  $\leftrightarrow$ physical scattering occurs for real and positive p
  $\Rightarrow$ bound state solution is obtained by analytic continuation of (4)

- Resonance states : solution of Eq. (4) with **complex** p

- Attractive square well potential have infinitely many resonance solutions [24, 10]
  Table 1 : numerical solutions of Eq. (4) with $V_0 = 10b^{-2}$
  Poles of $1/|A^-(p)|$ in complex p plane (Fig. 6)

- Imaginary part of eigenmomentum is negative

$$p = p_R - ip_I, \quad p_R, p_I > 0$$

behavior of wave function

$$\chi(r) \to A^+(p)e^{ipr} \propto \underbrace{e^{ip_R r}}_{\text{oscillation}} \underbrace{e^{+p_I r}}_{\text{increasing}}$$

$\chi(r)$ diverges with oscillation for $r \to \infty$, not square integrable (Fig. 6, right)

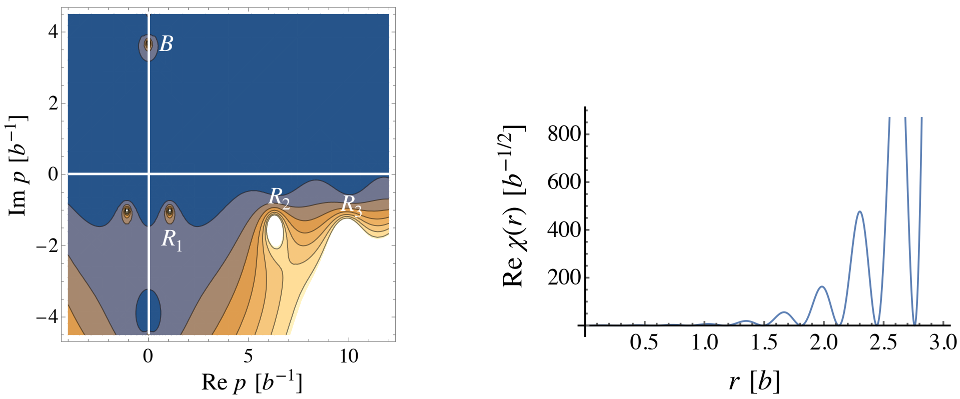

Figure 6: Left: contour plot of $1/|A^-(p)|$ of square well potential (2) with $V_0 = 10b^{-2}$. Right : real part of wave function of the third resonance R_3.

1.4 Localization of resonance wave function

Resonance phenomena at real energies

- Only real energies are experimentally accessible
- Repulsive square barrier potential ($V_0 > 0$, Fig. 5(b)) [10, 25]

$$V(r) = \begin{cases} +V_0 & 0 \le r \le b \\ 0 & b < r \end{cases}$$

  (Solutions of attractive potential are special examples, not suitable to see localization.)

- Condition for solution : replace $V_0 \to -V_0$ in the solution of attractive case

$$\tan(\sqrt{p^2 - 2V_0}\; b) = -i\frac{\sqrt{p^2 - 2V_0}}{p} \tag{5}$$

- No bound state solution, but infinitely many resonances mainly in the region $E > V_0$ (Table 2) (shifting origin of energy to $E = +V_0$, there is attraction for $r > b$)
- Behavior of scattering wave functions at real energies (Fig. 7)
  - Wave function localizes in $r < b$ (interaction region) near resonance energies
  - Away from resonances, approximately plane wave

Quantification of localization

- Ratio of amplitudes of interaction region and outer region

$$\chi(r) = \begin{cases} C\sin(kr) & 0 \le r \le b \\ C^{\mathrm{out}}\sin(pr + \delta) & b < r, \quad \delta : \text{phase shift} \end{cases}$$

Table 2: Numerical solutions of Eq. (5) (discrete eigenstates of repulsive barrier) with $V_0 = 10b^{-2}$.

	p $[b^{-1}]$	$E = p^2/2$ $[b^{-2}]$
1st resonance R_1	$5.37 - 0.36i$	$14.4 - 1.9i$
2nd resonance R_2	$7.56 - 0.92i$	$28.2 - 6.9i$
$\vdots$		

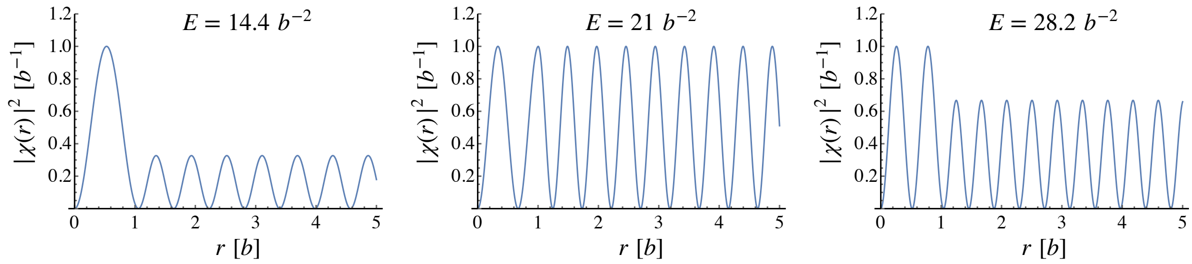

Figure 7: Localization of wave function. Wave functions at near-resonance energies $E = 14.4b^{-2}$, $E = 28.2b^{-2}$ and at far from resonance $E = 21b^{-2}$ by repulsive square barrier potential with $V_0 = 10b^{-2}$.

- Localization rate R : from continuity at $r = b$,

$$R = \left|\frac{C}{C^{\text{out}}}\right|^2 = \left(1 + \frac{k^2 - p^2}{p^2}\cos^2(kb)\right)^{-1}$$

From $p = \sqrt{2E}$, $k = \sqrt{2(E - V_0)}$ and $V_0 > 0$, we have $k < p$, so $R \geq 1$

- Numerical calculation : Resonance with small imaginary part (narrow width) localizes strongly

$$R = \begin{cases} 3.05 & (E = 14.4b^{-2},\ \text{first resonance}) \\ 1.00 & (E = 21b^{-2}) \\ 1.49 & (E = 28.2b^{-2},\ \text{second resonance}) \end{cases}$$

1.5 Summary of §1

- Discrete eigenstates $\leftarrow$ outgoing boundary condition
- Resonances : eigenstates of Hamiltonian with complex eigenenergy
  (Same with bound states, analytic continuation of eigenmomentum)
- Resonance wave function
  - diverges at $r \to \infty$ (complex p)
  - localises in interaction region (real p)