EFFECT OF DIFFERENT NUMERICAL SOLUTION METHODS ON USF MODEL

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1. INTRODUCTION

The flood flow in urban areas constitutes a serious hazard to both the population and infrastructure. Because most of the population is concentrated near the flood plain and a lot of buildings are along the rivers. The extreme events of urban flood cause damages, resulting in the loss of human life, property, crop and other losses (Sahoo and Saritha, 2015). Also, the presence of sewer system, impervious surfaces, etc. convert a high percentage of rainfall into runoff more quickly and the floods become flashier. For this reason, implementation of appropriate flood control activities and mitigation strategies are important to keep flood damage to a less level. Therefore, the advance prediction of flood characteristics such as peak, volume, etc. is essential in order to control its harmful effects.

Different models are available for the rainfall-runoff modelling in which the lumped model has been widely used. The storage function model, one of the lumped model, have gained popularity in different parts of the world, especially in Japan, due to the ease of expressing the nonlinear relationship of rainfall-runoff events with simple equations and its ability to provide easy computation (Kawamura et al., 2004). The first storage function model with three parameters was originally proposed by Kimura (1961) and it can express the nonlinearity in the outflow process with a relatively simple structural formula. It still has been widely used in Japan. Later, Prasad (1967) directly expressed the bivalence of the storage and discharge using the nonlinear storage equation with three parameters without using the lag time introduced in the Kimura’s storage function model. Furthermore, Hoshi and Yamaoka (1982) modified Prasad’s storage function model by adding another parameter and improved the strength of the model.

All the above models are using effective rainfall as their input for the estimation of direct runoff, which involves errors and parameter uncertainty. In order to overcome this, Baba et al. (1999) developed a storage function model with loss mechanism that directly uses observed rainfall and observed discharge and is applied to the mountainous river basin in Hokkaido, Japan. However, the outflow characteristics of the urban watershed largely differ from those in the mountainous river basin due to the presence of impervious surface and sewer system in the urban area. Hence, it is difficult to reproduce the hydrograph of urban river discharge using Baba's storage function model. Therefore, Takasaki et al. (2009) proposed Urban Storage Function (USF) model which uses the observed rainfall and runoff directly without effective rainfall estimation and base flow separation for the flood prediction. It considers all the possible inflow and outflow components of the urban basin with combined sewer system such as storm drainage from the basin through the combined sewer system as one of the outflow, to improve the accuracy of the storage function model for flood prediction.

The USF model solution can be achieved by using different numerical solution methods. In this study, we are comparing the effect of two numerical solution methods of (1) Runge-Kutta-Gill (RKG) method and (2) Difference method (DM) for solving the USF model. The root mean squared error (RMSE) and elapsed time taken for the program execution for different time increments were considered as the effect evaluation criteria of solution methods. For this comparison, we selected one flood event of the Kanda River basin, a small to medium-sized urban watershed in Tokyo, Japan where the flood occurs every year.

2. METHODOLOGY

2.1 USF model

The USF model proposed by Takasaki et al. (2009) is a lumped conceptual model. **Fig. 1** shows the schematic diagram of all possible inflow and outflow components of an urban watershed. The inflow components in **Fig.1** comprise rainfall *R* (mm/min) and urban specific and ground water inflows from other basins *I* (mm/min). The urban specific inflows include leakage from water distribution pipes, irrigational flow, etc. The outflow components consist of river discharge *Q* (mm/min), evapotranspiration *E* (mm/min), storm drainage from the basin through the combined sewer system q_R (mm/min), water intake from the basin *O* (mm/min) and ground water related loss q_l (mm/min). In addition, the domestic sewage q_w and total discharge from combined sewer system q_s $(q_R + q_w)$ are also depicted in **Fig.1** even though they do not directly contribute to the watershed storage *s* (mm).

The bivalent relationship between the outflow from the basin $(Q + q_R)$ and storage s can be expressed by the following equation:

$$s = k_1(Q + q_R)^{p_1} + k_2 \frac{d}{dt}\{(Q + q_R)^{p_2}\} \qquad (1)$$

where *t*: time (min), k_1, k_2, p_1, p_2: model parameters.

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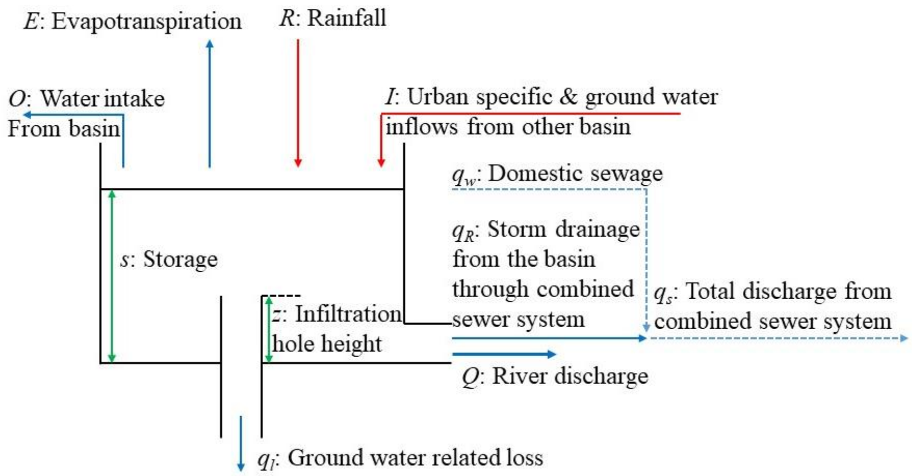

Fig. 1 Schematic diagram of all inflow and outflow components of an urban watershed.

Combining the above expression of storage with the following continuity equation yields the nonlinear expression of USF model.

$$\frac{ds}{dt} = R + I - E - O - (Q + q_R) - q_l \qquad (2)$$

Further, the ground water related loss (q_l) was defined by considering the infiltration hole height (z) and is given by:

$$q_l = \begin{cases} k_3(s-z) & (s \geq z) \\ 0 & (s < z) \end{cases} \qquad (3)$$

where k_3, z: model parameters.

The expression for storm drainage q_R from the combined sewer system discharged out of the basin is developed by assuming a linear relationship between total discharge $Q + q_R$ and the storm drainage q_R immediately after the rainfall. The q_R is defined as:

$$q_R = \begin{cases} \alpha(Q + q_R - Q_0), & \alpha(Q + q_R - Q_0) < q_{R_{max}} \\ q_{R_{max}} & , \alpha(Q + q_R - Q_0) \geq q_{R_{max}} \end{cases} \qquad (4)$$

where α: the slope of the linear relationship between total discharge $Q + q_R$ and drainage q_R, Q_0: the initial river discharge just before the rain starts.

The maximum volume of q_R cannot exceed the sewer maximum carrying capacity $q_{R_{max}}$. Substituting Eq. (1) into Eq. (2) will lead to a second order Ordinary Differential Equation (ODE) as follows:

$$k_2 \frac{d^2}{dt^2}(Q + q_R)^{p_2} = -k_1 \frac{d}{dt}(Q + q_R)^{p_1} + R + I - E - O - (Q + q_R) - q_l \qquad (5)$$

In order to solve the ODE, change of variables is performed as follows:

$$x_1 = (Q + q_R)^{p_2} \qquad (6)$$

$$x_2 = \frac{d}{dt}\{(Q + q_R)^{p_2}\} \qquad (7)$$

By substituting Eq. (3) into Eq. (5) and performing change of variables will lead to the emergence of two first order ODE's concerning two conditions shown in Eq. (3).

When $s \geq z$, the first order ODE for is given as:

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{k_1}{k_2}\frac{p_1}{p_2} x_1^{(p_1/p_2 - 1)} x_2 - \frac{1}{k_2} x_1^{(1/p_2)} - \frac{k_1 k_3}{k_2} x_1^{(p_1/p_2)} \\ \qquad -k_3 x_2 + \frac{1}{k_2}(R + I - E - O + k_3 z) \end{cases} \qquad (8a)$$

In the case of $s < z$, the first order ODE is given as:

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{k_1}{k_2}\frac{p_1}{p_2} x_1^{(p_1/p_2 - 1)} x_2 - \frac{1}{k_2} x_1^{(1/p_2)} \\ \qquad + \frac{1}{k_2}(R + I - E - O) \end{cases} \qquad (8b)$$

Eq. (8) can be expressed in an expanded way as follows, (Morinaga et al., 2001)

$$\frac{dX(t)}{dt} = \begin{bmatrix} f_1(X) \\ f_2(X) \end{bmatrix} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = F(X) \qquad (9)$$

By solving the two simultaneous non-linear ODE's of Eq. (8) numerically, we will obtain the total discharge $Q + q_R$. In order to solve this two first order simultaneous ODE's, we used the RKG method and difference method. The river discharge Q will obtain as the solution after subtracting the q_R, which is calculated using Eq. (4), from the total discharge.

2.2 Runge-Kutta-Gill method

In order to solve the Eq. (9), the formula for RKG numerical solution is as follows (Kojima and Machida, 1982):

$$X_{n+1} = X_n + \frac{1}{6}\left[u_1 + (2-\sqrt{2})u_2 + (2+\sqrt{2})u_3 + u_4\right] \quad (10)$$

where

$$u_1 = \Delta t F(t_n, X_n) \quad (11)$$

$$u_2 = \Delta t F\left(t_n + \frac{1}{2}\Delta t, X_n + \frac{1}{2}u_1\right) \quad (12)$$

$$u_3 = \Delta t F\left(t_n + \frac{1}{2}\Delta t, X_n + \frac{\sqrt{2}-1}{2}u_1 + \frac{2-\sqrt{2}}{2}u_2\right) \quad (13)$$

$$u_4 = \Delta t F\left(t_n + \Delta t, X_n - \frac{\sqrt{2}}{2}u_2 + \frac{2+\sqrt{2}}{2}u_3\right) \quad (14)$$

where Δt: time increment.

2.3 Difference method

The nonlinear vector function in Eq. (9) is calculated by using the Taylor's theorem using known $X = X^*$ in advance and then the vector function $F(X)$ is transformed into a linear equation expressed by Eq. (15) by omitting terms after the quadratic term.

$$\frac{dX(t)}{dt} = J(X^*)X + B(X^*) \quad (15)$$

$J(X^*)$ is the Jacobian of $F(X)$ and it is expressed by the following equation,

$$[J(X^*)]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]_{X=X^*} = \begin{bmatrix} 0 & 1 \\ j_1 & j_2 \end{bmatrix} \quad (16)$$

Here j_1 and j_2 are given by,

$$j_1 = -\frac{k_1}{k_2}\frac{p_1}{p_2}\left(\frac{p_1}{p_2} - 1\right)x_1^{(p_1/p_2-2)}x_2 - \frac{1}{k_2}\frac{1}{p_2}x_1^{(1/p_2-1)} - \frac{k_1 k_3}{k_2}\frac{p_1}{p_2}x_1^{(p_1/p_2-1)}$$

$$j_2 = -\frac{k_1}{k_2}\frac{p_1}{p_2}\left(\frac{p_1}{p_2} - 1\right)x_1^{(p_1/p_2-2)}x_2$$

The matrix $B(X^*)$ can be expressed as,

$$B(X^*) = F(X) - J(X^*)X^* \quad (17)$$

In addition, we transformed the continuous linear Eq. (15) into a discretized format in order to make the calculations easier and is given as,

$$X(k+1) = \Psi(k)X(k) + \Lambda(k)B(k) \quad (18)$$

Here, k is the discretized computation time point and the matrices $\Psi(k)$ and $\Lambda(k)$ are calculated using the following equation,

$$\Psi(k) = exp(J\Delta t) \equiv I + J\Delta t + \frac{(J\Delta t)^2}{2!} + \frac{(J\Delta t)^3}{3!} + \frac{(J\Delta t)^4}{4!} + \cdots \quad (19)$$

$$\Lambda(k) = [exp(J\Delta t) - I]J^{-1} = \Delta t\left[I + \frac{J\Delta t}{2!} + \frac{(J\Delta t)^2}{3!} + \frac{(J\Delta t)^3}{4!}\cdots\right] \quad (20)$$

3. RESULTS AND DISCUSSION

The inflow component I was fixed at 0.0012 mm/min based on the business annual report of the TMG. The water intake O and evapotranspiration E were set at 0 since there is no water intake from the target basin and evapotranspiration during heavy rainfall is insignificant. The maximum drainage, $q_{R_{max}}$ was estimated at 0.033 mm/min using the Manning's equation.

We synthetically generated the discharge values using the inflow and outflow components along with fixed parameters with RKG as the solution method and 0.01 as the time increment. The fixed parameters were set as k_1=50, k_2=600, k_3=0.05, p_1=0.3, p_2=0.4, z=30, α=0.5 based on the average values described by Takasaki et al. (2009). This synthetically generated discharge forms the base value for further calculations.

Fig. 2 shows the RMSE values generated by the USF model for different solution methods with different time increments. We considered a total of 5 time increments of 0.01, 0.1, 0.2, 0.5, and 1.0. From **Fig. 2**, we can see that both the methods generate same RMSE values and they are overlapping for a time increment of 0.01. When the time increment has increased by 10 times ($\Delta t = 0.1$), there was no significant difference in the RMSE values by both the methods. However, it was found that there was a notable change in the RMSE value estimated by the difference method as compared with the RKG method at a time increment of 0.2. The difference method produced a higher RMSE value even though it is of very small magnitude (0.81×10^{-5}). When Δt equals to 0.5 and 1.0, the RMSE values estimated using RKG method were quite consistent. However, there was an abrupt change in the values produced by the difference method. The results indicate that the RKG method is

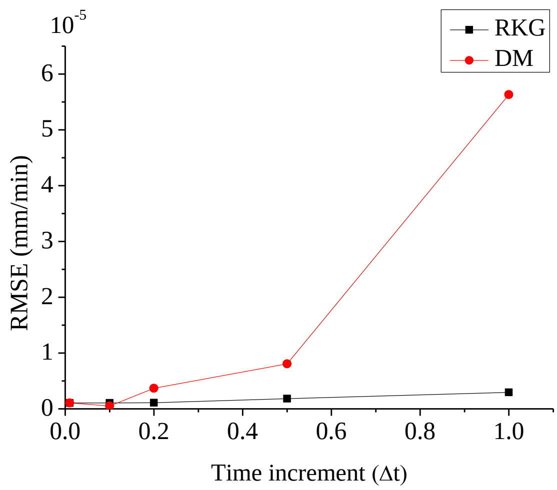

Fig. 2 Effect of time increment on RMSE values generated by USF model for RKG and Difference solution methods.

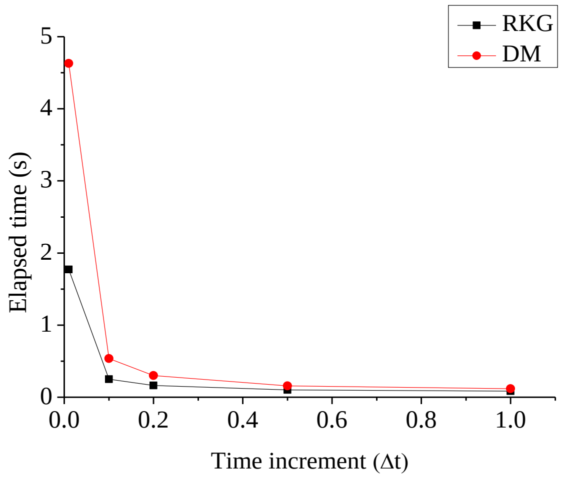

Fig. 3 Effect of time increment on elapsed time taken for the program execution of USF model by both the methods.

more stable compared with difference method for all the considered time increments. The difference method generated comparable results with RKG method only at very small time increments of 0.01 and 0.1.

Fig. 3 shows the effect of time increment on the elapsed time taken by the USF model for the program execution using both the solution methods. From **Fig. 3**, we can see that the elapsed time is higher for the difference method compared with RKG method at the time increment of 0.01. When the time increment starts increasing, the elapsed time taken by difference method has gradually reduced and become equal with that of taken by RKG method at Δt equals to 0.5. The RKG method also shows a higher elapsed time at a time increment of 0.01 compared with other time increments. The results reveal that the RKG method is faster compared to the difference method in program execution with very small increments. Even though the differences in the elapsed time between two methods are very small, it will considerably affect long data sets for years.

From both the figures, we can envisage that the RKG method is giving better performance over the difference method in all aspects. Even though the difference method giving almost same RMSE as that of RKG method at lower time increments, the elapsed time taken by the difference method is high. In a similar way, the elapsed time taken by both the methods are same for time increments of 0.2, 0.5, and 1. However, the RMSE generated by the difference method at these time increments is far higher than that produced by RKG method.

In **Fig. 2**, If we are comparing the RMSE of RKG method when Δt equals to 0.5 (0.18×10^{-5}) with that of difference method when Δt equals to 0.2 (0.37×10^{-5}), the RMSE value of RKG method is 2 times smaller than that of the difference method. Therefore, we can conclude that the method is dominating over the time increment.

CONCLUSION

In this study, we compared the effect of different time increment used in two different numerical solution methods of RKG method and difference method on USF model with fixed parameters in terms of RMSE and elapsed time taken for the program execution.

The results indicated that the RKG method exhibits better performance compared with difference method for solving USF model for all the considered time increments in terms of RMSE. The RMSE values were stable for the RKG method, while the same estimated by difference method was highly fluctuating with increasing time increments. Furthermore, the RKG method showed faster program execution in terms of elapsed time at smaller time increments compared with difference method.

As a conclusion, the RKG method was found to be superior to the difference method in terms of both RMSE and elapsed time. Even though the difference method is not demonstrating an equal performance with RKG method, it is needed for the real time flood prediction using Kalman filter.

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