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A COMPUTATIONAL METHOD FOR OPTIMAL CONTROL OF MULTI-RESERVOIR SYSTEM BASED ON DIFFERENTIAL DYNAMIC PROGRAMMING

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SYNOPSIS

A computational method is proposed for the optimal control problem of the multi-reservoir system having a convex cost that function consists mainly of drought damage and secondary of a special penalty function. The method is based on differential dynamic programming (DDP) under the condition that inflows to reservoirs and water demands can be assumed to be deterministic. If the cost function of multi-reservoir operations is expressed as a function of water demands and supplies only, finding the optimal control generally causes singular control problems which are very difficult to solve. In order to cope with the singular control problems, our method is based on the modification of two methods, i.e., the K.Ohno's new DDP algorithm with Newton's method for discrete time systems [15] and the Bell-Jacobson's ε-algorithm for singular control problems [2].

The simulation study shows that the proposed method can accurately solve the optimal control problem of the multi-reservoir system on the relatively long time interval with the moderate computational load.

INTRODUCTION

Efficient operations of the reservoir systems are important for the effective use of water resources. As for the techniques on reservoir operations, many studies have been presented . In this paper, we have referred to Takasao, et al. [19], Takeuchi [20], Kikkawa, et al. [21], Murray and Jacowitz [12], Yeh [23], Ikebuchi [7] and so forth.

There are two types of models handling the operations of reservoir systems, namely deterministic and stochastic models. Stochastic models are very difficult to compute, if state constraints are considered. Practically there are many cases where the optimal operations of a multi-reservoir system can be treated deterministically. Standard or drought duration curves of inflows to reservoirs can be treated as deterministic by the results of the suitable statistical analysis on the long actual duration curves, for example. Further-more the water demands of a region and its sub-regions can be estimated deterministically by a suitable regional planning.

In this paper, we assume that the state transition equation of reservoirs can be expressed by a discrete-time dynamic system. And from the above-mentioned we treat of the deterministic case on the optimal control problem of a multi-reservoir system. In formulating the practical optimal control problem of a multi-reservoir system, the selection of the cost function is very important. For urban water supply problems other than electric power use, a useful cost function is what can be evaluated by economical drought damage. Therefore we adopted the drought damage function as the main cost function of control problems.

In our model, there are several water demand sites assigned in the sub-areas of a region. These are regarded as the sink nodal points of a network which connects reservoirs (main part of sources) and demand sites with water channels or rivers. The defined main cost function is incorporated with suitable drought damage functions assigned to each demand site.

With the above model configuration, the optimal control problem of the multi-reservoir system is realistically and generally formulated. However, it will usually cause singular control problems to adopt the drought damage function only as the cost function. For example, the solution (i.e., the discharge from each reservoir) for a parallel two-reservoir system with one demand site can not be decided uniquely by minimizing the drought damage function. In order to treat these singularities we introduced a special penalty function. Few papers directly consider the possible singularities for optimal control problems of multi-reservoir systems.

In this paper, we have firstly adopted the DDP algorithm with Newton's method which was proposed by Ohno [15] to solve discrete optimal control problems with state and control constraints. In our case we have used the damped Newton's method for the stability of numerical computation. As a result, the convergent domain of starting variables is extended remarkably. We refer to the method as DN-DDP (DDP with damped Newton's method).

Secondly, we have coped with singular control problems by combing the sequential regularization process, similar to the ε-algorithm proposed by Bell and Jacobson [2], with DN-DDP under a newly introduced penalty function. This regularization process is referred to as SR-process. As a whole, a new general computational method for the optimal control problem of the multi-reservoir system is proposed here, and it is named as SDN-DDP (Sequential regularized DDP with damped Newton's method).

SDN-DDP was applied to the model shown as Fig. 1, i.e., the multi-reservoir system composed of three reservoirs, the five component control variable, two demand sites and the drought damage function on 216 time periods as the main cost function. The solution of DN-DDP converged sufficiently on the criterion deduced from Khun-Tucker optimality conditions, and the value of the penalty function decreased monotonously by the SR-process. We obtained satisfactory computation results described later.

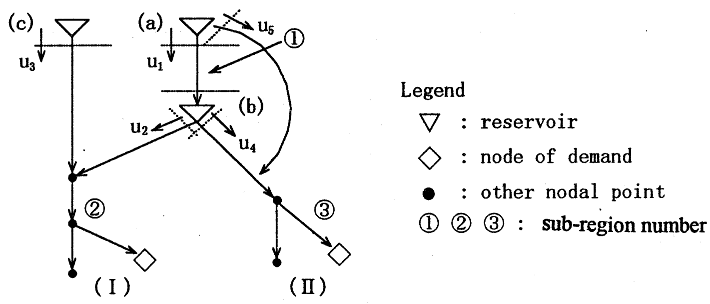


Fig. 1 Configuration of the example computation model

STATE EQUATION OF MULTI-RESERVOIR AND COST FUNCTION

(1) State Equation of Multi-reservoir System and Its Optimal Control Problem

Similar to ordinary reservoir operation models, we adopt storages of reservoirs as the state variable and discharges of reservoirs as the control variable. The discrete state transition equation of the multi-reservoir is expressed by the following linear difference equation:

$$x_{n+1} = x_n + Bu_n\Delta t + q_n\Delta t$$
$$= f_n(x_n, u_n, q_n, \Delta t) = f_n(x_n, u_n), \quad n = 1,2,\cdots,N-1 \tag{1}$$

Further, initial and terminal boundary conditions can be expressed as:

$$x_1 = a_0 \tag{2}$$

$$x_N = a_N \tag{3}$$

where

$x_n \underline{\underline{\Delta}} (x_{n1}, \cdots, x_{ni}, \cdots, x_{nm})^T$ = state variable at the n -th stage , i.e., storages of reservoirs at the n -th stage, and m is the maximum number of reservoirs;

$u_n \underline{\underline{\Delta}} (u_{n1}, \cdots, u_{nj}, \cdots, u_{nr})^T$ = control variable during the n -th period, i.e., discharges and/or intakes of reservoirs during the n -th period, and r is the maximum number of control components and each reservoir must have at least one control component . $m \leqq r$;

$q_n \underline{\underline{\Delta}} (q_{n1}, \cdots, q_{ni}, \cdots, q_{nm})^T$ = inflows to reservoirs during the n -th period;

Δt = time interval , i.e., length of the period, and maximum period number is $(N-1)$;

stage = the time at the starting and/or end point of each period , and maximum stage number is N ;

n = stage number or period number in the finite time horizon, $n = 1,2,\cdots,N$, and $t_n = (n-1)\Delta t$;

N = maximum value of n ;

B = matrix indicating the relation between x_{n+1} and u_n , $(m \times r\ matrix)$;

a_0 = initial state (vector), i.e., initial storages of reservoirs;

a_N = designated final state (vector), i.e., designated final storages of reservoirs at the N -th stage ;

As for the constraints on x_n and u_n, only the linear functions regarding to x_n and u_n are considered and it is supposed that each state constraint always contains the components of u_n corresponding to that of x_n. For example,

$x_{(n+1)i} \leq M_{ni}$, $u_{nj} \geqq 0$, etc. are expressed as follows:

$$g_{ns}(x_n,u_n) = f_{ni}(x_n,u_n) - M_{ni} \leqq 0, \quad g_{n(s+1)}(x_n,u_n) = -u_{nj} \leqq 0.$$

here, i = constrained component number of the state variable; j = constrained component number of the control variable;

s = constraint number.

The equality constraints can be expressed similarly. By putting in order, these are written as follows:

$$g_n(x_n,u_n) \leqq 0, \ \ h_n(x_n,u_n) = 0, \ \ n = 1,2,\cdots,N-1 \tag{4}$$

here g_n and h_n are the m_n-th and the r_n-th dimensional vector respectively. As for the terminal constraint, we treat here the fixed end constraint only.

In this paper, the optimal control problem finding $\left\{x_n^*, u_n^*\right\}^\ell$ which satisfies Eq. 1, Eq. 2, Eqs. 3 and 4 and minimizes the cost function $J(\ell)$ that is defined in the next section under given ℓ and involves the penalty function, is referred to as problem (A). Where ℓ is the sequential regularization process number defined later, and upper subscript * denotes the optimal value.

A part of the final parameter of the penalty function, namely the final pilot trajectory X^* mentioned later, can not be estimated accurately beforehand, since it involves the equivalent of $\left\{x_n^*\right\}$. Then, it is necessary, for obtaining the final optimal solution $\left\{u^*\right\}$, to adopt the sequential process that solves the problem (A) relating to each ℓ, such as ℓ =0,1,2,···, for the improvement of the pilot trajectory X^ℓ and the convergence of $J(\ell)^*$. We shall refer to the finding of the final optimal solution $\left\{x_n^*, u_n^*\right\}$ and X^* that satisfy $J(\bullet)^*|_{\{x_n,u_n,X_n\}^*} = \min\left\{J(\ell)^* | \ell = 0,1,2,\cdots\right\}$, as problem (T).

(2) Cost Function and Stabilization Function

It is a very difficult problem to express simply the cost function for the effective control of the multi-reservoir system in practice. In this paper, referring to [9], [14] and so forth, we adopt the drought damage function as the main component of the cost function for the optimal control problem of the multi-reservoir system and a special penalty function as remainders. We assume referring to the above cited literatures that the drought damage function of a region where the multi-reservoir is assigned can be expressed by the sum of $L_{0n}, (n = 1,2,\cdots,N-1)$ defined below.

Remark 1: Here, for convenience, a region implies the water supply district connected by a water-channel network containing reservoirs, and the multi-reservoir implies the group of reservoirs connected by one water-channel network. And a sub-region implies the suitable part of a region defined later.

L_{0n} is composed of the minimum of the quadratic function of the supply cut ratio under given control

u_n on the n-th period in the region, and the provided supply cut ratio is defined as shortage of water supply divided by demand.

In order to show concretely how to obtain L_{0n}, we will consider the multi-reservoir system drawn in Fig. 1. The positions of reservoirs, sources of the rivers and water channels, their junctions, demand sites and non-effective discharge points of the region, etc. can be expressed as nodes. Suitable sections of rivers and water channels having a flow direction are expressed as arcs . We shall compose the directed graph from these nodes and arcs, and express the configuration of the demand-supply system of the region including the multi-reservoir. Subsequently, if the connections between the arcs expressing channels, etc are cut at the nodal points correspond to reservoirs, sub-graphs can be obtained. We shall call them sub-regions for convenience.

In the model depicted in Fig. 1, the configuration of the demand-supply system is composed of three sub-regions. The j-th component u_{nj} of u_n and the demand D_{np} at the demand site p are allotted to nodes of the corresponding network of sub-regions.

It is assumed that the channel flowing into at each demand node is limited to one channel. If it is not satisfied, an assumed channel is added to the demand node and transfer the demand node to the opposite node, so that the assumption will be satisfied. It is evident that the component of L_{0n} corresponding to a sub-region is independent of the others under the given control u_n of the multi-reservoir.

In our problem, even if the control u_n are given, there are some cases, in which L_{0n} cannot be easily obtained, unless we solve the flow problems of the sub-region networks. Sometimes the flows in the network problem for finding the component of L_{0n} can not be solved uniquely even L_{0n} exists. In these cases, we can avoid this singularity by giving small quadratic transport costs to some arcs. Based on the above, the objective function (cost function) L_{1n} on the n-th period is defined under given u_n as follows:

$$
\begin{aligned}
L_{1n}(u_n) &= \min\left\{\sum_R\left(L_{0n}+\sum_\ell C_\ell Q_\ell^2\right)_R\right\}_n \\
&= \min\left\{\sum_R\left(\sum_p C_p D_{np}\left(\Delta Q_{np}/D_{np}\right)^2+\sum_\ell C_\ell Q_\ell^2\right)_R\right\}_n \\
&= \sum_R\left\{\min\left(\sum_p C_p D_{np}\left(\Delta Q_{np}/D_{np}\right)^2+\sum_\ell C_\ell Q_\ell^2\right)_R\right\}_n \qquad (5)
\end{aligned}
$$

where

R = sub-region number in the region;

v = node number in the sub-region R;

ℓ = water-channel number in the sub-region R;

p = demand node number in the sub-region R;

L_{1n} = main cost function of the system on the n-th period, i.e., the drought damage function of the region on the n-th period;

C_p = coefficient of the drought damage function at the demand node p;

D_{np} = water demand at the demand node p in the sub-region R on the n-th period;

Q_ℓ = flow (or discharge) of the channel ℓ in the sub-region R on the n-th period;

C_ℓ = coefficient to Q_ℓ^2 in the case where the cost of the water flow is necessary, and it is usually set at 0 ;

$\Delta Q_{np} = (Q_{\ell_p})_n - D_{np}$ = shortage of the flow of the demand channel ℓ_p absorbed into the demand node p on the n-th period;

Furthermore, in order to treat with excess water which appears when flooding, it is frequently necessary to assign to the configuration of each sub-region one non-effective discharge channel which generally has no transport cost. The assignment of non-effective channels to the configuration of the region should be done under suitable observations. L_{1n} is the main component of the cost function for the optimal control problem of the multi-reservoir system on the n-th period.

Some remarks for modeling the configuration of the multi-reservoir system are as defined follows:

① The configuration of the demand and supply system of the region should be simplified by the equivalent transformation, if possible. The control components which are dependent on others should be eliminated beforehand, so as to reduce the number of unknowns and constraints.

② The number of control component u_{nj} from one reservoir into the same sub-region should be set up to one. The rivers flowing into reservoir i are not treated as inflow arcs to i, but these flows are expressed by q_{ni} as the inhomogeneous term of the state equation.

③ The component u_{nj} from an upstream reservoir to the downstream reservoir i, should be treated independently of q_{ni} and u_{ne} from another, if there is no divergence into the demand nodes.

④ If there are more than one path from one node to another, the excessive paths should be assigned their transport costs .

⑤ If the problem finding $\{Q_\ell\}_R$, etc. which minimize $(L_{1n})_R$ under the given $(u_n)_R$ becomes singular, the configuration of the sub-region should be reformulated to proper one.

The iterative process of the Newton's method finding u_n that minimizes the cost function, requires the approximations of $\nabla_u L_{1n}$ and $\nabla_u^2 L_{1n}$. In this paper , for scalar L_n and vector f_n following symbols are used.

$$\nabla_u L_n = \left(\partial L_n / \partial u_{n1}, \partial L_n / \partial u_{n2}, \cdots, \partial L_n / \partial u_{nr}\right), \quad \nabla_u^2 L_n = \left(\partial^2 L_n / \partial u_{ni} \partial u_{nj}\right),$$

$$\nabla_u f_n = (\partial f_{ni} / \partial u_{nj}).$$

Since there is no case in which one component u_{nj} of u_n is supplied over more than one sub-region, symbol 'min' and 'Σ_R' of Eq. 5 are exchangeable. Accordingly L_{1n}, its derivative $\nabla_u L_{1n}$ and $\nabla_u^2 L_{1n}$ at u_n can be obtained by searching for these components in each sub-region. Notice that there are also sub-regions which have no contribution to L_{1n} directly.

In the sub-region R of a very simple configuration, $(L_{1n})_R$ and its first and second derivatives with respect to $(u_n)_R$ are directly obtained. When the configuration of the sub-region R is somewhat complicated, these are obtained by adopting quadratic programming. In the later case, $\{Q_\ell\}_R$ of the sub-region is adopted as the unknown variable to formulate the problem. Then, continuity equations of flows at sources and intermediate nodes become constraints on the quadratic programming. For example, constraints can be expressed as follows:

$$\left.\begin{array}{l}\text{for the source node } v \text{ supplied } u_{nj};\ Q_\ell|_v = u_{nj} \\ \text{for the intermediate node } v;\ \ \Sigma Q_\ell|_v = 0\end{array}\right\} \qquad (6)$$

And for the demand node p the new variable ΔQ_{np} is introduced as Eq. 5, and the other sink nodes except the special cases have no constraints.

In this paper, as the solution method of quadratic programming, the Wolfe's method (short-form) is adopted, because it is frequently used and $\partial L_{1n}/(\partial u_n)_R$ and $\partial^2 L_{1n}/(\partial u_n)_R^2$ are obtained easily. In the solution of this method, the values of Lagrangian multipliers to constraints Eq. 6 are contained , and these consist of the negative derivatives of $(L_{1n})_R$ by right-hand side inhomogeneous terms of constraints containing $\{u_{nj}\}_R$. Therefore, because of the linearity of the Wolfe's tableau, we place the perturbation (matrix) terms correspond to each $\{u_{nj}\}_R$ at the right-hand side of inhomogeneous terms of the constraint equations (6). By solving it through expanding the range of sweeping, we can obtain $\partial^2 (L_{1n})_R / (\partial u_n^2)_R$ together with the original solution. From these , we can construct $\partial L_{1n}/\partial u_n$ and $\partial^2 L_{1n}/\partial u_n^2$ in the n-th period.

In order that the problem can be solved by the Wolfe's method, the network of the sub-region must be constituted so that the solution $\{Q_\ell^*\}_R$ is uniquely determinable. But the judgment of solvability of the problem seems to be done easily by the observation of the sub-region configuration

Remark 2 : The technique called as depth-first search on a graph was useful in our experience to decompose the graph of the region to sub-graphs and to formulate the first tableau of Wolfe's method in the computer memories

When reservoirs are placed in parallel or there is flooding and in some other cases, the inverse matrix of $\partial^2 L_{1n}/\partial u_n^2$ does not exist. Accordingly, if we construct the cost function of problem (A) by $L_{1n}(u_n)$ alone, there is the large possibility that the control becomes generally singular, namely, the optimal control can not be determined uniquely even if it exists. In such cases, the procedure to find the desirable control becomes very complicated.

As mentioned above Bell and Jacobson proposed an iterative computational method called as ε-algorithm for solving the singular optimal control problems. Stating with regard to our problem, it is adding the penalty function such as $\varepsilon_\ell \frac{1}{2} u_n^T u_n$ to L_{1n} and converts the singular control problem to nonsingular one. Next, minimizing the normalized functional by the suitable method, ε_ℓ is progressively reduced toward zero and the solution close to original one is obtained. Note that the numerical ill-condition may happen as ε_ℓ approaches to zero. Bell and Jacobson suggested that the penalty function works even if it is the function of $\{x_n, u_n\}$. By referring that, we provide the water storage sequence $X^\ell = \{X_n(pt)\}^\ell$ of reservoirs, called the pilot trajectory in this paper, which is temporarily assumed to be optimal. The penalty function proportional to the square of the difference between $\{X_n(pt)\}^\ell$ and computed storages $\{x_n\}^\ell$ is introduced. Furthermore instead of reducing ε_ℓ, we improve the pilot trajectory progressively solving the problem (A) and finally we can obtain a suitable solution in practice which is very close to one of original singular optimal solutions.

If there are more than one components $u_{n(ik)}$ of u_n supplying sub-regions from one reservoir i, we introduce ε-penalty function of $u_{n(ik)}$ components except one passing through the same channel as the discharge from the spillway.

Putting in order the above, we define our penalty function of the n-th period as follows:

$$L_{2n}^\ell = \sum_i \left\{ \varepsilon_{ni} (f_{n,i}(x_n, u_n) - X_{n+1,i}(pt)|_\ell)^2 \right\}, \qquad (i = 1,2,\cdots,m) \tag{7}$$

$$L_{3n} = \sum_i \left\{ \sum_k \tfrac{1}{2} \varepsilon_{ik} \left(u_{n(ik)} \right)^2 \right\} \tag{8}$$

where, ε_{ni} = coefficient of the penalty function correspond to the reservoir i, and ordinary constant;

ℓ = process number of solving problem (A) and progressing $\{X(pt)_n\}^\ell$, $\ell = 0,1,2,\cdots$.

And k = above mentioned component number of $u_{n(ik)}$ of u_n in the reservoir i, $ik \Leftrightarrow j$ which indicate a component of u_n; ε_{ik} = small constant coefficient of the penalty function correspond $u_{n(ik)}$, if need, usually $\varepsilon_{ik} = 0$.

We refer to L_{2n}^ℓ and its functional as the stabilization function or regularization function and, L_{3n} and its functional as ε-penalty function.

By adding L_{2n}^ℓ and L_{3n} to L_{1n}, the cost function on the n-th period is set as:

$$L_n^\ell = L_{1n} + L_{2n}^\ell + L_{3n} \tag{9}$$

Based on this, the cost function $J(\ell)$ of problem (A) is defined as:

$$J(\ell) = \sum_{n=1}^{N-1} L_n^\ell \tag{10}$$

Here, $L_N^\ell = 0$ is assumed. It is evident that $J(\ell)$ is a strictly convex function of $\{x_n\}$ and $\{u_n\}$, and separable on n. We shall refer to $J(\ell)$ as the modified drought damage cost function. For simplicity, ℓ is expressed by the subscript letter or is omitted occasionally.

Using SDN-DDP that is always accompanied by $J(\ell)$, we can obtain the accurate approximation of the optimal control $\{u_n\}^*$ which satisfies following relation;

$$J(\bullet)^*|_{\{x_n, u_n, X_n\}^*} = \min \left\{ J(\ell)^* | \ell = 0,1,2,\cdots \right\}.$$

The starting pilot trajectory $\{X_n(pt)\}^0$ should be set at high level as possible as being explained in the chapter of Numerical Computation Method for Optimal Control, so that the numerical optimal solution $\{x_n^*, u_n^*\}$ can be obtained at high water level. However, getting the numerical optimal solution by introducing the stabilization function generally implies that one solution supposed to be useful is selected from innumerable singular optimal solutions. The characteristics of the numerical solution and matters concerning to it will be also mentioned in the above cited chapter. In the case of $\varepsilon_{ik} \neq 0$, although the numerical optimal solution may be slightly strained, we shall permit to adopt small ε_{ik} for stability of the numerical analysis and ε_{ik} is kept constant.

We define J_1, J_1^ℓ and J_2^ℓ correspond to ℓ as:

$$J_1 = J_1^\ell = \sum_{n=1}^{N-1}(L_{1n} + L_{3n})|_\ell \quad (11)$$

$$J_2^\ell = \sum_{n=1}^{N-1}\tilde{L}_{2n}^\ell \quad (12)$$

Remark 3: By a process similar to that of $\tilde{L}_{2n}^\ell$, the function L_{3n} may be improved, but such process is not adopted for the sake of simplicity here.

In the numerical computation, the original $\nabla_u L_{2n}$ and $\nabla_u^2 L_{2n}$ are modified like Eq. 13 and 14 by expressing with same symbol. In this case, $\nabla_u^2 L_{2n}$ is regarded as a consistent approximation [17].

$$\nabla_u L_{2n} \underline{\Delta} (original \nabla_u L_{2n}) \times s \quad (13)$$

$\nabla_u^2 L_{2n} \underline{\Delta}$ (the diagonal elements of original $\nabla_u^2 L_{2n}$ are kept unchanging, and the other elements of it are multiplied by $0.5) \times s^2$ (14)

here, s = the correcting factor which will be explained in the later chapter.

In this paper the computational model mentioned above is referred as the water demand and supply model.

PRELIMINARIES TO DDP

(1) Assumption of the Existence of Feasible Solutions

If the conditions and the constraints of the computational model of a reservoir system are suitably defined, we can expect that problem (A) of the model has feasible solutions, because the control $u(t)$ always exists in the real reservoir system. In this paper, the existence of feasible solutions of problem (A) is assumed.

(2) Properties of the Solution for Optimal Control Problem (A)

Problem (A) can be converted to the problem of mathematical programming equivalent to it. Subsequently, based on some theorems of convex programming, the assumption of existence of feasible solutions and strict convexity of the cost function, the existence of the unique global solution can be easily proved [4], [6]. Accordingly, if the numerical solution for problem (A) converges locally, it is the global solution.

(3) Dynamic Programming (DP) and Pay-Off Function

Based on the property of the state equation and the cost function, DP can be applied to problem (A) for each ℓ. If the pay-off function $V_n(x_n)$ is defined concerning to $n = 1,2,\cdots,N$ as follows:

$$V_n(x_n) = \min\left\{\sum_{j=n}^{N-1} L_j + L_N(x_N) \mid x_{j+1} = f_j(x_j,u_j), g_j \leqq 0, h_j = 0, j = n,\cdots,N-1\right\} \quad (15)$$

Eq. 16 can be obtained from the principle of optimality.

$$V_n(x_n) = \min\left\{L_n(x_n,u_n) + V_{n+1}(f_n(x_n,u_n)) \mid g_n \leqq 0, h_n = 0\right\}, n = 1,2,\cdots,N-1 \quad (16)$$

In the case of problem (A), terminal condition Eq. 3 can be transformed into Eq. 17. (refer to Eq. 42)

$$V_N(x_n) = L_N(x_N) = 0 \tag{17}$$

The minimum value of the cost function $J(\ell)$ for each ℓ can be written by the definition of Eq. 15 as follows:

$$J^*(\ell) = V_1(x_1) = V_1(a_0) \tag{18}$$

In usual DP, the optimal control $\{u_n^*\}$ can be obtained by solving Eq. 16 recursively starting from the terminal condition Eq. 3. On the other hand, in the case of DDP, in order to alleviate the dimensional difficulty, $\{u_n^*\}$ is obtained by solving equations derived from Eqs. 16 and 17 iterativelly under optimality conditions.

(4) Necessary Conditions for Optimality and so forth

In this paper, we adapt many materials from Ohno [15] concerning DDP. In the analysis of optimal control problems, it is usual to set some assumptions in regard to the functions. These will be briefly described.

Assumption 1 : All the functions f_n, g_n, h_n, L_n, $n=1,2,\cdots N-1$ and L_N are twice differentiable and all their second derivatives are piecewise continuous.

Assumption 2 : For the activated g_n^* and h_n^*, $\nabla_u g_{ni}^*$ and $\nabla_u h_{nj}^*$ are linearly independent.

Assumption 3 : As for g_n^*, strict complementarity holds, that is , for all activated g_{ni}^*, $\lambda_{ni}^* > 0$.

It is clear that these assumptions do not add new constraints to problem (A).

In order to obtain the formulation for the minimization problem of $V_n(x_n)$, i.e., Eq. 16, we shall introduce the Lagrangian function $F_n, (n=1,2,\cdots N-1)$ as follows:

$$F_n(x_n,u_n,\lambda_n,\mu_n) = L_n(x_n,u_n)+V_{n+1}(f_n(x_n,u_n))+\lambda_n^T g_n(x_n,u_n)+\mu_n^T h_n(x_n,u_n) \tag{19}$$

where λ_n and μ_n are Lagrange multipliers.

The following Kuhn-Tucker conditions, Eq. 20, Eq. 21, Eqs. 22 and 23 hold as necessary conditions that u_n^* be an optimal solution of Eq. 16;

$$\nabla_u F_n^* = \nabla_n L_n^* + \nabla V_{n+1}^* \nabla_u f_n^* + (\lambda_n^*)^T \nabla_u g_n^* + (\mu_n^*)^T \nabla_u h_n^* = 0 \tag{20}$$

$$diag(\lambda_n^*) g_n^* = 0, h_n^* = 0 \tag{21}$$

$$g_n^* \leqq 0, \lambda_n^* \geq 0 \tag{22}$$

and for the vector Z which satisfies $\nabla_u g_{ni}^* Z = 0$ for all activated g_{ni}^* and $\nabla_u h_{nj}^* Z = 0$, .

$$Z^T \nabla_u^2 F_n^* Z \geq 0 \tag{23}$$

where

$$\nabla_u^2 F_n^* = \nabla_u^2 L_n^* + \nabla V_{n+1}^* \nabla_u^2 f_n^* + (\lambda_n^*)^T \nabla_u^2 g_n^* + (\mu_n^*)^T \nabla_u^2 h_n^* + (\nabla_u f_n^*)^T \nabla^2 V_{n+1}^* \nabla_u f_n^* \tag{24}$$

Furthermore from the strict convexity of L_n, the following holds for above vector Z.

$$Z^T \nabla_u^2 F_n^* Z > 0 \tag{25}$$

Eqs. 20 ~ 23 and Eq. 25 are the necessary and sufficient conditions for u_n^* to be an optimal solution of Eq.16, and obviously $V_1(x_1)$ equals $J^*(\ell)$ of the problem (A). Now, we define y_n and $T_n(x_n, y_n)$ as follows, for $n = 1,2,\cdots,N-1$:

$$y_n = (u_n^T, \lambda_n^T, \mu_n^T)^T \tag{26}$$

$$T(x_n, y_n) = (\nabla_u F_n, g_n^T diag(\lambda_n), h_n^T)^T \tag{27}$$

The Eqs. 20 and 21 can be rewritten as:

$$T(x_n^*, y_n^*) = 0, (n = 1,2,\cdots,N-1) \tag{28}$$

For fixed x_n, $T_n(x_n, y_n) = 0$ is a system of $(r + m_n^* + r_n)$ equations for the same number of unknown y_n. It is evident that $\{x_n^*, y_n^*\}^\ell$ satisfying the initial and terminal conditions and Eq. 28 are the optimal solution of problem (A), since these satisfy the necessary and sufficient conditions for optimality and $J(\ell)$ is strictly convex.

Regularity of the Jacobian matrix $J_n(x_n^*, y_n)$ of $T_n(x^*, y)$ with respect to y_n is evident from Eq. 25, assumption-3, etc., then y_n^* can be obtained from $T_n(x_n^*, y_n) = 0$. And $J_n(x_n, y_n)$ is expressed as follows:

$$J_n = \begin{bmatrix} \nabla_u^2 F_n & \nabla_u g_n^T & \nabla_u h_n^T \\ diag(\lambda_n)\nabla_u g_n & diag(g_n) & 0 \\ \nabla_u h_n & 0 & 0 \end{bmatrix} \tag{29}$$

Although the same letter J is adopted to denote the cost function, this will not cause any confusion since the subscript is different.

Now, $K_n(x_n, y_n)$, i.e., the Jacobian matrix of $T_n(x_n, y_n)$ regarding x_n is expressed as:

$$K_n(x_n, y_n) = (\nabla_{ux}^2 F_n^T, \nabla_x g_n^T diag(\lambda_n), \nabla_x h_n^T)^T \tag{30}$$

here

$$\nabla_{ux}{}^2 F_n = \nabla_{ux}{}^2 L_n + \nabla_u f_n^T \nabla^2 V_{n+1} \nabla_x f_n + \nabla V_{n+1} \nabla_{ux}^2 f_n + \lambda_n^T \nabla_{ux}^2 g_n + \mu_n^T \nabla_{ux}^2 h_n \tag{31}$$

From implicit function theorem [17] the following holds:

$$\nabla_x y_n^*(x_n) = (\nabla_x u_n^*(x_n)^T, \nabla_x \lambda_n^*(x_n)^T, \nabla_x \mu_n^*(x_n)^T)^T = -(J_n(x_n, y_n^*(x_n)))^{-1} K_n(x_n, y_n^*(x_n)) \tag{32}$$

The strict complementarity condition for g_n leads to

$$g_n^T(x_n, u_n^*) \nabla \lambda_n^*(x_n) = 0 \tag{33}$$

Accordingly when considering $V_N(x_N) = L_N$, the following can be proved for $n = 1,2,\cdots,N-1$:

$$V_n(x_n) = F_n(x_n, y_n^*(x_n)) \tag{34}$$

$$\begin{aligned} \nabla V_n(x_n) &= \nabla_x F_n(x_n, y_n^*(x_n)) \\ &= \nabla_x L_n + \nabla V_{n+1} \nabla_x f_n + \lambda_n^*(x_n)^T \nabla_x g_n + \mu_n^*(x_n)^T \nabla_x h_n \end{aligned} \tag{35}$$

$$\nabla^2 V_n(x_n) = \nabla_x^2 L_n + \nabla_x f_n^T \nabla^2 V_{n+1} \nabla_x f_n + \nabla V_{n+1} \nabla_x^2 f_n + \lambda_n^*(x_n)^T \nabla_x^2 g_n + \mu_n^*(x_n)^T \nabla_x^2 h_n$$
$$+ \left(\nabla_{xu}^2 L_n + \nabla_x f_n^T \nabla^2 V_{n+1} \nabla_u f_n + \nabla V_{n+1} \nabla_{xu}^2 f_n + \lambda_n^*(x_n)^T \nabla_{xu}^2 g_n + \mu_n^*(x_n)^T \nabla_{xu}^2 h_n\right) \cdot \nabla_x u_n^*(x_n)$$
$$+ \nabla_x g_n^T \nabla_x \lambda_n^*(x_n) + \nabla_x h_n^T \nabla_x \mu_n^*(x_n) \qquad (36)$$

Here, all functions are assumed values at $(x_n, u_n^*(x_n))$. Since, functions f_n, g_n, h_n of problem (A) are linear concerning x_n, their second derivatives become zero. Such terms will be omitted in the later algorithm.

NUMERICAL COMPUTATION METHOD FOR OPTIMAL CONTROL

The numerical computation method used in this paper for the optimal control of the multi-reservoir system is composed of the double iterative computational processes. The first process is similar to Bell-Jacobson's ε-algorithm, and corresponds to the outer loop of the flowchart of computational processes. In the process, pilot trajectory $\{X_n(pt)\}^\ell$ of stabilization function which has been introduced to cope with the singular control, is improved and the cost function $J(\ell)$ is reformed. We shall refer to this process as the sequential regularization process (SRP).

The second process is the iterative computational process performed by DDP with damped Newton's method for problem (A) of index ℓ. The process is referred to as DN-DDP. The process putting the ℓ-th SRP and the ℓ-th DN-DDP together is referred to as the ℓ-th sequential process of SDN-DDP.

As mentioned previously, Ohno [15] proposed DDP with Newton's method as an algorithm of DDP to solve optimal control problems of the discrete-time dynamic systems with state and control constraints. He proved the convergence of the method and showed the rate of convergence. In his method, it was assumed that the starting values of variables are close enough to the optimal values. In our paper, considerable improvement of his original algorithm has been done in practice as DN-DDP by adopting the line search in Newton's method and Lagrange multiplier method for the terminal constraint. With the strictly convex cost function $J(\ell)$, DN-DDP has obtained the robust stability of the numerical computation and has expanded the convergence area of the starting variables. As the result, DN-DDP has been made to be an effective computational method for problem (A).

In the following, we describe the algorithm combining SRP and DN-DDP, i.e., SDN-DDP which is assumed to be always accompanied by the $J(\ell)$ type cost function. Subsequently, the convergence of the algorithm and some characteristics of the numerical optimal solution are briefly described.

(1) Supplementary Notes on DN-DDP

The necessary and sufficient conditions which the optimal solution of problem (A) must satisfy are Eq. 28 and the initial and terminal boundary conditions. Eq. 22 will be satisfied automatically, if the choice of active inequality constraints are suitable.

In solving optimal control problems by the iterative procedure in DDP, the following amount of correction δy_n^k is commonly used;

$$\delta y_n^k = \delta y_{no} + \partial y_n^k / \partial x_n \cdot \delta x_n \tag{37}$$

δy_{no}^k is the amount of correction obtained by Newton's method , and is sought for the backward time direction and δx_n^k and δy_n^k are sought for the forward time direction. Here k is the iteration number.

In this paper, by introducing the step-length α into standard Newton's method, δy_{no}^k is obtained in the following form,

$$\delta y_{no}^k = -\alpha \cdot (J_n(x_n^k, y_n^k))^{-1} T_n(x_n^k, y_n^k)$$

here, T_n, J_n, etc. contain unknown values $\nabla V_{n+1}(x_{n+1}^k)$ and $\nabla^2 V_{n+1}(x_{n+1}^k)$. These approximate values can be obtained by Eqs. 35 and 36. Denote these approximate values of $\nabla V(x^k)$ and $\nabla^2 V(x_n^k)$ by the $\nabla \overline{V}_n^k$ and $\nabla^2 \overline{V}_n^k$ respectively. In the following, the symbol $\overline{\bullet}_n$ denotes variable $\bullet_n$ with ∇V_{n+1} substituted by $\nabla \overline{V}_{n+1}$, and $\left[\left(\overline{J}_n\right)^{-1}\overline{K}_n\right]_\theta$ denotes the sub-matrix of $(\overline{J}_n)^{-1}\overline{K}_n$ corresponding to variable θ. We adopt damped Newton's method as follows,

$$\overline{y}_n^{k+1} = \overline{U}_n(x_n^k, y_n^k) = y_n^k - \alpha \cdot (\overline{J}_n(x_n^k, y_n^k))^{-1} \overline{T}(x_n^k, y_n^k) \tag{38}$$

The approximate value of $\partial y_n^k / \partial x_n^k$ is obtained from Eq. 32. The succeeding computational procedure will be shown in section (6) .

As for the criterion of the line search and the convergence of iterations of DN-DDP, we adopt the following error estimating functions for the n -th period:

$$E_n^k(\alpha) = \left\| \nabla_u \overline{F}(x_n^{k+1}, y_n^{k+1}, (\nabla \overline{V}_{n+1}^{k+1}\Big|_{\alpha=0} + \nabla \overline{V}_{n+1}^k\Big|_\alpha) / 2, \alpha) \right\|$$

and for the total time horizon:

$$E^k(\alpha) = \Sigma_{N-1} E_n^k(\alpha) \tag{39}$$

$$E'^k(\alpha) = \sqrt{\Sigma_{N-1} E_n^k(\alpha)^2} \tag{40}$$

(2) Treatment of Inequality Constraints and Approximation of J_n

In this paper, for avoiding Lagrangian multipliers λ_n appearing in J_n directly like Eq. 29, we adopt a technique of mathematical programming which can treat active inequality constraints in the same way as equality constraints. The validity of the technique is shown in reference [3]. For the sake of conciseness of descriptions, we shall omit symbol h_n according to circumstances and shall represent it by g_n including active inequality constraints.

Furthermore in SDN-DDP of this paper, in place of the original J_n, the following consistent approximation J_{Mn} is used, and expressed as J_n instead of original one.

$$J_{Mn} = \begin{bmatrix} \nabla_u^2 F_n & \nabla_u g_n^T \\ \nabla_u g_n & diag(-C_{Mn})^{-1} \end{bmatrix} \tag{41}$$

where, $C_{Mn} = (C_{Mj_1}, C_{Mj_2}, \cdots)$

This is effective when the assumption 2 is temporarily broken on the way of numerical computation. Although the computational speed a little bit decreases by adopting J_{Mn}, the stability of computation increases sufficiently. The element of C_{Mn} is usually taken larger than that of C_{Ni} of the terminal boundary condition which will be explained in the next section.

(3) Treatment of Terminal State Constraint

The terminal boundary condition, Eq. 3 at the stage $n = N$, can be expressed as:

$$\phi(x_N) = x_N - a_N = 0 \tag{42}$$

In SDN-DDP, we apply the Lagrangian multiplier method[3] to this boundary condition. The validity of the method is given in [18]. The augmented Lagrangian function J_c^ℓ containing $\phi(x_N)$ is introduced as an extension of the cost function $J(\ell)$ to be minimized. J_c^ℓ is defined as:

$$J_c^\ell = J^\ell + b^k \phi(x_N) + \sum_i C_{Ni}^k \phi_i (x_N)^2 \tag{43}$$

First, set up $b^0 = \left(diag\left(C_N^0\right)\phi\left(x_N^0\right)\right)^T$, and for $k = 0,1,2,\cdots$, $\nabla \overline{V}_N^k\left(x_N^k\right)$ and b^{k+1} can be obtained as:

$$\nabla \overline{V}_N^k(x_N) = b^k \phi\left(x_N^k\right) \tag{44}$$

$$b^{k+1} = b^k + \left(diag\left(C_N^k\right)\phi\left(x_N^k\right)\right)^T \tag{45}$$

where b^k is a row vector. In addition, $\nabla^2 \overline{V}_N^k = diag\left(C_N^k\right)$.

In the numerical example presented in this paper, the initial value of the component C_{Ni}^0 of C_N^k is set as about $0.1/x_{ni\max}$, thereafter it is increased gradually at a rate of around 1% per step with the upper limit to be an order of $5C_{Ni}^0$.

(4) Pilot Trajectory and its Improvement

The relation between the pilot trajectory $\{X_n(pt)\}^\ell$ and the stabilization function L_{2n}^ℓ is shown as Eq. 7. Here ε_{ni} is settled by considering coefficients of L_{1n} and the stability of computation. For example, it is placed like $\varepsilon_{ni} = C_i/(x_{ni})_{\max}$. Here, i is the reservoir number, and $(x_{ni})_{\max}$ is the capacity limit. C_i is the adequate positive number. The starting pilot trajectory at $\ell = 0$ is set at the water level nearly as high as possible. For example, it is set at a water level that is several centimeters lower than the upper limit water level in almost every stages, and the boundary of feasible domain of $\{x_n\}$ should be avoided. In the neighboring n of terminal N, the trajectory should be pointed to the designated terminal water level.

By the optimal solution $\{x_n^*\}^\ell$ obtained through the ℓ-th sequential process of SDN-DDP and $\{X_n(pt)\}^0$, the improvement of the trajectory $X^{\ell+1} = \{X_n(pt)\}^{\ell+1}$ is performed using either next two ways.

$$X_n^{\ell+1}(pt) = \left(x_n^*\right)^\ell, \quad n = 1,2,\ldots,N-1 \tag{46}$$

or $X_n^{\ell+1}(pt) = 0.5^{\ell+1} X_n^0(pt) + (1-0.5^{\ell+1}) \cdot (x_n^*)^\ell$, $n = 1,2,\cdots,N-1$ (47)

here, ℓ is the sequential regularization process number, ℓ = $0,1,2,3,\cdots$

(5) Others

The correction factor s mentioned in the chapter State Equation of Multireservoir··· should be selected so that $\nabla_u^2 L_n$ becomes considerably large compared with $\nabla_x^2 L_n$, for example something around s ≒1～5. s would depend on the length of the unit period Δt. As for $\nabla_x L_n$, when the small penalty function $1/2\, c g_{nd}^2(x_n)$ concerning the activated g_{ns} is added to L_n, there are some cases in which the computational efficiency is somewhat improved.

(6) Algorithm of SDN-DDP

The outline of computational procedure of SDN-DDP accompanied always by $J(\ell)$ is as follows.

Step(0): ① Select various constants required for the computation. (refer to section (8) of this chapter and the examples of the chapter Numerical Simulation)

② Assume the starting pilot trajectory $\{X_n(pt)\}^0$ and set $\ell = 0$.

③ Select the nominal control variable $\{u_n^0\}^0$ and state variable $\{x_n^0\}^0$ satisfying the Eq. 1, Eq. 2, Eqs. 3 and 4. Set up $\{\lambda_n^0\}$ and $\{\mu_n^0\}$ at 0, and calculate b^0, and set $k = 0$.

Step(1): Find $\nabla \overline{V}_N^k(x_N^k)$ and $\nabla^2 \overline{V}_N^k(x_N^k)$ from Eq. 44 and section (3) of this chapter. In addition update b^{k+1} by Eq. 45.

Step(2): ① From step (3)～(6), obtaining $\{y_n^{k+1}\}_m$, $\{x_n^{k+1}\}_m$ and $E'(\alpha_m)^k$ for each interpolation step-length α_m in the line $[\alpha_{\min}, \alpha_{\max}]$, find α_{opt}^k minimizing $E'(\alpha_{opt})^k$. Where m is the interpolation number of the step-length and $0 < \alpha_{\min} \le \alpha_{opt}^k \le \alpha_{\max} \le 1$ is hold. $\alpha_{\min}$ and $\alpha_{\max}$ are adequately given.

② From step (3)～(6), calculate $E(\alpha_{opt})^k$, $E'(\alpha_{opt})^k$, $\{y_n^{k+1}\}_{opt}^\ell$, and $\{x_n^{k+1}\}_{opt}^\ell$ for α_{opt}^k.

③ If constraints are satisfied within the prescribed accuracy and if $E(\alpha_{opt})^k \le \varepsilon_E$ or $k \ge k_{\max}$, go to step (7). Where ε_E and $k_{\max}$ is the prescribed value.

④ Check the active inequality constraints, and update them if necessary. Set $k \leftarrow k+1$ and go to step (1).

Step(3): For $n = N-1, \ldots, 2, 1$; $\alpha = \alpha_m$; carry out the following computations. Here subscript m of α_m shows the interpolation number of the step-length or optimal one mentioned at step (2).

$$\overline{y}_n^{k+1} = \overline{U}_n(x_n^k, y_n^k) = y_n^k - \alpha \cdot (\overline{J}_n(x_n^k, y_n^k))^{-1} \overline{T}(x_n^k, y_n^k) \quad (48)$$

$$\nabla\overline{V}_n^k = \nabla_x L_n + \nabla\overline{V}_{n+1}^k \nabla_x f_n + \beta\left(\overline{u}_n^{k+1} - u_n^k\right)^T \nabla_u f_n\left(x_n^k, u_n^k\right)^T \nabla^2\overline{V}_{n+1}^k \nabla_x f_n$$

$$+\left(\overline{\lambda}_n^{k+1}\right)^T \nabla_x g_n + \left(\overline{\mu}_n^{k+1}\right)^T \nabla_x h_n \tag{49}$$

$$\nabla^2\overline{V}_n^k = \nabla_x^2 L_n + \nabla_x f_n^T \nabla^2\overline{V}_{n+1}^k \nabla_x f_n - \left(\nabla_{xu}^2 L_n + \nabla_x f_n^T \nabla^2\overline{V}_{n+1}^k \nabla_u f_n\right)\left[r\overline{J}_n^{-1}\overline{K}_n\right]_u$$

$$-\nabla_x g_n^T\left[r\overline{J}_n^{-1}\overline{K}_n\right]_\lambda - \nabla_x h_n^T\left[r\overline{J}_n^{-1}\overline{K}_n\right]_\mu + R(0) \tag{50}$$

here, $\beta \doteqdot 0.2 \sim 1.0$; $r \doteqdot Min(1, \max(0.5, 2\alpha))$; R(0) = the term which becomes 0 for the problem (A).

Step(4): For $n=1$, $\alpha=\alpha_m$,

$$x_1^{k+1} = a_0, y_1^{k+1} = \overline{U}_1\left(x_1^{k+1}, y_1^k\right) \tag{51}$$

Step(5); For $n = 2, \cdots, N-1$, $\alpha = \alpha_m$, compute the followings.

$$x_n^{k+1} = f_{n-1}\left(x_{n-1}^{k+1}, u_{n-1}^{k+1}\right) \tag{52}$$

$$y_n^{k+1} = \overline{y}_n^{k+1} - \left[\left(\overline{J}_n\left(x_n^k, y_n^k\right)\right)^{-1} \overline{K}_n\left(x_n^k, y_n^k\right)\right]\left(x_n^{k+1} - x_n^k\right) \tag{53}$$

In addition, at $n = N$, compute.

$$x_N^{k+1} = f_{N-1}\left(x_{N-1}^{k+1}, u_{N-1}^{k+1}\right) \tag{54}$$

Step(6): For given $\alpha = \alpha_m$, by getting $E(\alpha)^k$ and $E'(\alpha)^k$ via $\{x_n^{k+1}\}, \{y_n^{k+1}\}$, $\{\nabla\overline{V}_n^k\}$, etc. obtained in the above and with Eqs. 39-40, return to the prescribed place of step (2).

Step(7): ① By setting $\{x_n^*\}^\ell = \{x_n^{k+1}\}_{opt}^\ell$, calculate $\{X_n(pt)\}^{\ell+1}$ through Eq. 46.

② When the variation value of the trajectory X^ℓ improvement falls within the range prescribed or $\ell \geqq \ell_{\max}$ holds, go to step (8). Here, $\ell_{\max}$ is the prescribed integer.

③ Substitute $\{u_n^*\}^\ell$ and $\{x_n^*\}^\ell$ to the nominal starting variables $\{u_n^0\}^{\ell+1}$ and $\{x_n^0\}^{\ell+1}$, update $\{\lambda_n^0\}$ and $\{\mu_n^0\}$ (for example 0), calculate b_0, set $\ell \leftarrow \ell+1$ and set $k=0$, and go to the step (1).

Step(8): Stop the computation, and output the necessary results. [End of algorithm]

Remark 4 : $\overline{y}_n^{k+1}$ is an improved estimate to the solution of $T_n\left(x_n^k, y_n\right) = 0$, and not to the solution of $T_n\left(x_n^{k+1}, y_n\right) = 0$. Eq. 53 adapts $\overline{y}_n^{k+1}$ for the old state x_n^k to y_n^{k+1} for the new state x_n^{k+1}. Eq. 49 is obtained by approximating Eq. 35 at $\overline{y}_n^{k+1}$.

Remark 5 : Nominal starting variables $\{u_n^0\}^0$ and $\{x_n^0\}$ are estimated so that $\{x_n^0\}^0$ belong to the neighborhood of $\{X_n(pt)\}^0$ by suitable conventional methods. This request can be fairly loosen, but the variables must satisfy the state transition Eq. 1.

(7) Convergence of the Algorithm

a) Convergence of DN-DDP

Since the convergence proof of DN-DDP would be done modifying that of DDP with the standard Newton's method [15], the only outline of the convergence proof of DN-DDP will be explained adapting from standard one.

The assertion of [15] for DDP with Newton's method is roughly as follow:

"Put $\delta_n^k = \left\| y_n^k - y_n^*\left(x_n^k\right) \right\|_1$ and $\delta^k = \left(\delta_1^k, \delta_2^k, \cdots, \delta_{N-1}^k\right)^T$.

Arranging the computational process of DDP with Newton's method, the following inequality can be proved for δ^k at the appropriate neighborhood of the origin of δ^k.

$$\delta^{k+1} \le Q^k\left(\delta^k\right)\delta^k$$

where, $Q^k\left(\delta^k\right)$ is a $(N-1)\times(N-1)$ matrix obtained in [15], And $\left\|Q^k\left(\delta^k\right)\right\| < 1$ hold for above δ^k [15]. Thus, δ^k converges to 0."

Based on the same logic, we can express the result of the k-th iterative process of DN-DDP as

$$\delta^{k+1} \le Q^k\left(\alpha, \delta^k\right)\delta^k$$

where, $Q^k\left(\alpha, \delta^k\right)$ is a $(N-1)\times(N-1)$ matrix correspond to above $Q^k\left(\delta^k\right)$.

If we choose $\alpha \le 1$, $\left\|Q^k\left(\alpha, \delta^k\right)\Big|_{\alpha\le1}\right\| \le \left\|Q^k\left(\delta^k\right)\right\|$ will hold for same δ^k. Accordingly the iterative process of DN-DDP for $J(\ell)$ will converge for remarkably wider domain of starting variables than standard one under. This can be confirmed by the numerical computation. Namely the problem of the numerical example of this paper does not converge under $\alpha = 1$, but it converges under suitable $\left\{\alpha^k \le 1\right\}$. The distinctive feature of problem (A) under the modified drought damage cost function is that, if the various constants are appropriately chosen, $\nabla^2 V_n^k$ obtained by the difference equation of Riccati type Eq. 50 is inferred to be symmetric, positive definite and bounded. This fact would serve the stability of DN-DDP process of problem (A). However, it would be obvious that DN-DDP process converges within the suitable limit of the number m of reservoirs and N of stages, etc. Moreover, the rate of convergence of DN-DDP is presumed to be R-linear and the process is uniformly asymptotically stable in the neighborhood of the origin of δ^k.

b) Convergence of Sequential Regularization Process

The cost function $J(\ell)$ can be divided into two functions J_1 and J_2^ℓ. J_1 consists of the drought damage fnnction defined by Eq. 11. J_2^ℓ is composed of the stabilization function defined by Eq.12. When the pilot trajectory $\left\{X_n(pt)\right\}^\ell$ is updated by Eq. 46, $J_2\left(\left\{X_n\right\}^{\ell+1}, \left\{x_n^*\right\}^\ell\right)$ becomes smaller than $J_2\left(\left\{X_n\right\}^\ell, \left\{x_n^*\right\}^\ell\right)$. In addition, $J(\ell+1)$ is minimized under the given pilot trajectory $\left\{X_n(pt)\right\}^{\ell+1}$. As the result, the following relation holds.

$$\left.\begin{aligned}&\left[J_1\left(\{x_n^*\}^{\ell+1},\{u_n^*\}^{\ell+1}\right)+J_2\left(\{X_n\}^{\ell+1},\{x_n^*\}^{\ell+1}\right)\right]\\&\leqq\left[J_1\left(\{x_n^*\}^{\ell},\{u_n^*\}^{\ell}\right)+J_2\left(\{X_n\}^{\ell+1},\{x_n^*\}^{\ell}\right)\right]\\&\leqq\left[J_1\left(\{x_n^*\}^{\ell},\{u_n^*\}^{\ell}\right)+J_2\left(\{X_n\}^{\ell},\{x_n^*\}^{\ell}\right)\right]\end{aligned}\right\}\tag{55}$$

From the above inequality, $J^{*\ell}$ decreases monotonously, and $J^{*\ell}\to J^*$ and $J_2^{*\ell}\to 0$ (or ε_{E2}) as ℓ becomes large, resulting in the convergence of the pilot trajectory $\{X_n(pt)\}^{\ell}$ into $\{X_n(pt)\}^{*}$. On the above, the optimal solution $\{x_n^*, u_n^*\}$ of problem (T) can be obtained by SDN-DDP accompanied always by the $J(\ell)$ type cost function.

(8) Characteristics of the Numerical Optimal Solution

As previously described, we have introduced the stabilization function to eliminate the singularity of control and have solved the problem of the multi-reservoir system by SDN-DDP. This means to select out a precise approximate solution among innumerable singular optimal controls which consist of drought damage only as their cost function. The selected approximate solution must be of practical use. In order to satisfy this request, the selected solution should be characterized to have the high water level transition curve for each reservoir, if possible. We will explain roughly how the numerical optimal solution having such property can be obtained by SDN-DDP.

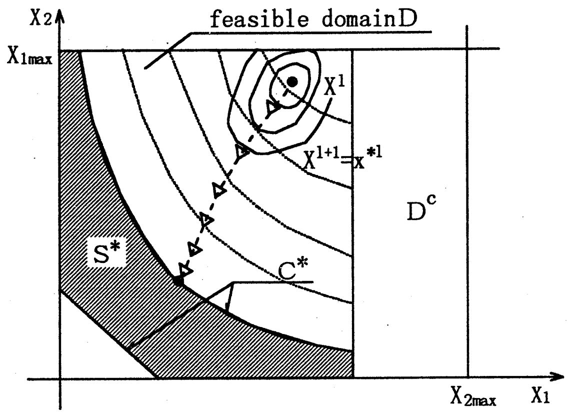


Fig. 2 Illustration of the convergence process of the optimal control in SDN-DDP

Let M equals to $m\times N$. Here m and N are the maximum numbers of reservoirs and stages respectively. Next, consider E^M which basis is provided with the unit storage of each reservoir $i,(i=1,2,\cdots,m)$, for all $n,(n=1,2,\cdots,N)$. Accordingly, arbitrary $\{x_n\}^{\ell}$ and $\{X_n(pt)\}^{\ell}$ are expressed by x^{ℓ} and X^{ℓ} as points of E^M respectively, and $\{u_n\}^{\ell}$ is uniquely determined for given x^{ℓ} and X^{ℓ} from the strict convexity of $J(\ell)$. As the result, J_1 and J_2 are defined on E^M, and the feasible domain of x^ℓ is expressed as domain D on E^M for the given parameters of the problem (A). Fig. 2 shows a conceptual drawing of the convergence process of the optimal control.

Let C^* be the envelope of the set of original (singular) optimal solutions S^* of the multi-reservoir control problem on E^M. Suppose that X^0 is set up at the neighborhood of the upper right corner of D. $x^{*\ell}$ and $u^{*\ell}(x^{*\ell})$ are obtained through the ℓ-th iteration process of SDN-DDP. The point $x^{*\ell}$ on the drawing shows this position. In this stage, $x^{*\ell}$ is the point minimizing $J(\ell)$. The contour z^ℓ of $J_2^*(\ell)$ passing through $x^{*\ell}$ on E^M is an (hyper-)ellipsoid, and its shape is determined by the coefficient $\{\varepsilon_{ni}\}$ of $J_2(\ell)$. $x^{*\ell}$ is also the point minimizing J_1^ℓ on the common part of z^ℓ and D. $X^{\ell+1}$ is obtained from $x^{*\ell}$ by Eq.46. X^{l+1} is the point such that descend form X^ℓ as far as the minimum point of $J(\ell)$ on D, namely, the minimum point of $J_1(\ell)$ on $z^\ell \cap D$.

In general the starting pilot rajectory X^0 is the outside of S^*, then $x^{*\ell}$ and $X^{*\ell+1}$ converge to a point on C^* without reaching the inside of S^*.

The convergent point is determined if all parameters and coefficients of the problem (T) including X_0 are given. SRP can be interpreted as a kind of steepest descent method under a suitable norm.

It could be qualitatively said that the position of x^* is located on the point not relatively far away from X^0 on C^*. x^* is prescribed by parameters X_0 and $\{\varepsilon_{ni}\}$, etc. through the cost function J.

If the solution obtained by SDN-DPP under above mentioned X_0 is somewhat unsatisfactory for practical use, it can be improved mainly by adjusting $\{\varepsilon_{ni}\}$.

In the next simulation, ε_{ni} is adopted from the suitable range of $C_i/x_{ni\text{-max}}$, so we can shift x^* on C^* by adjusting the input data C_i. When making some C_i to be larger under fixed X^0, the water level transition curve of the reservoir i at x^* becomes relatively higher, and if making C_i smaller, it becomes relatively lower. In this paper, we selected C_i by the inspection of computation results in several trials.

NUMERICAL SIMULATION

In order to demonstrate the effectiveness of SDN-DDP that is always accompanied by the $J(\ell)$ type cost function, a numerical simulation was carried out for the multi-reservoir system modeled as Fig. 1.

(1) Coefficients and Parameters of the Simulation

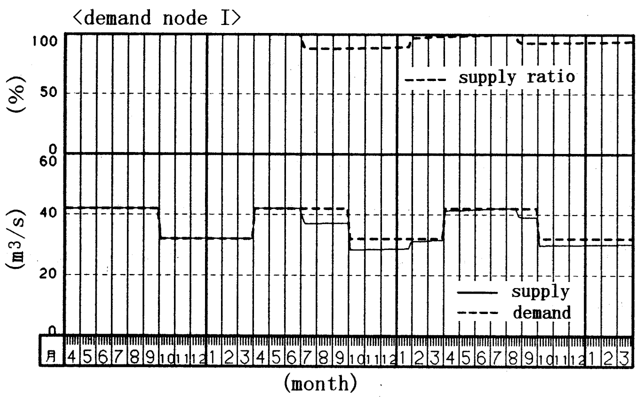
The common coefficient of drought damage function: $C_p = 10.0(m^3/s)^{-1}$; $\Delta t = 5day$; $s = 2.0$; $\beta = 0.5$; maximum number of periods $N-1 = 216$ (correspond to 3 years); coefficients of the penalty of activated g_n, $C_g = 0.4/x_{i\text{-max}}$; $a_0 = a_N =$ $(304.0, 214.1, 163.4)^T\ m^3/s \cdot day$; lower limit of $u_{nj} = 0$ or $0.5\,m^3/s$; for starting value of $\{u_n^0\}^0$ and $\{x_n^0\}^0$ refer to Remark 5. Table.1 shows coefficients and parameters used in this simulation. Other data; $\{x_{ni.\max}\}$, $\{q_{ni}\}$ and $\{D_{np}\}$ are given in the diagrams of results.

Table 1 Coefficients, etc. concerning each Reservoir

	x_{max}(m^3/s·day)	Coef. of L_{2n}: ε_{ni}	terminal C_{Ni}^0	C_{Mni} of J_{Mn}	upper limit of x_n
Reservoir a	605	$0.16/x_{na,max}$	$0.1/x_{a,max}$	$20.0/x_{a,max}$	exist
Reservoir b	545	$0.16/x_{nb,max}$	$0.1/x_{b,max}$	$20.0/x_{b,max}$	exist
Reservoir c	680	$0.14/x_{nc,max}$	$0.1/x_{c,max}$	$20.0/x_{c,max}$	no exist

notice : coefficient of inequality g_n consisted with only u_{nj}; C_{Mnj}=5.0;

(2) Results



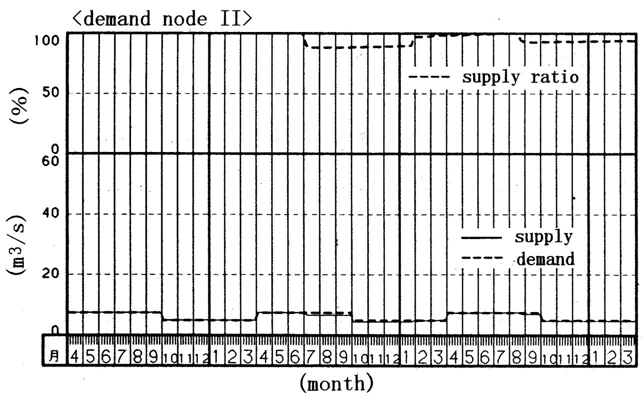


Fig.3 Duration curves of supply ratio and demand

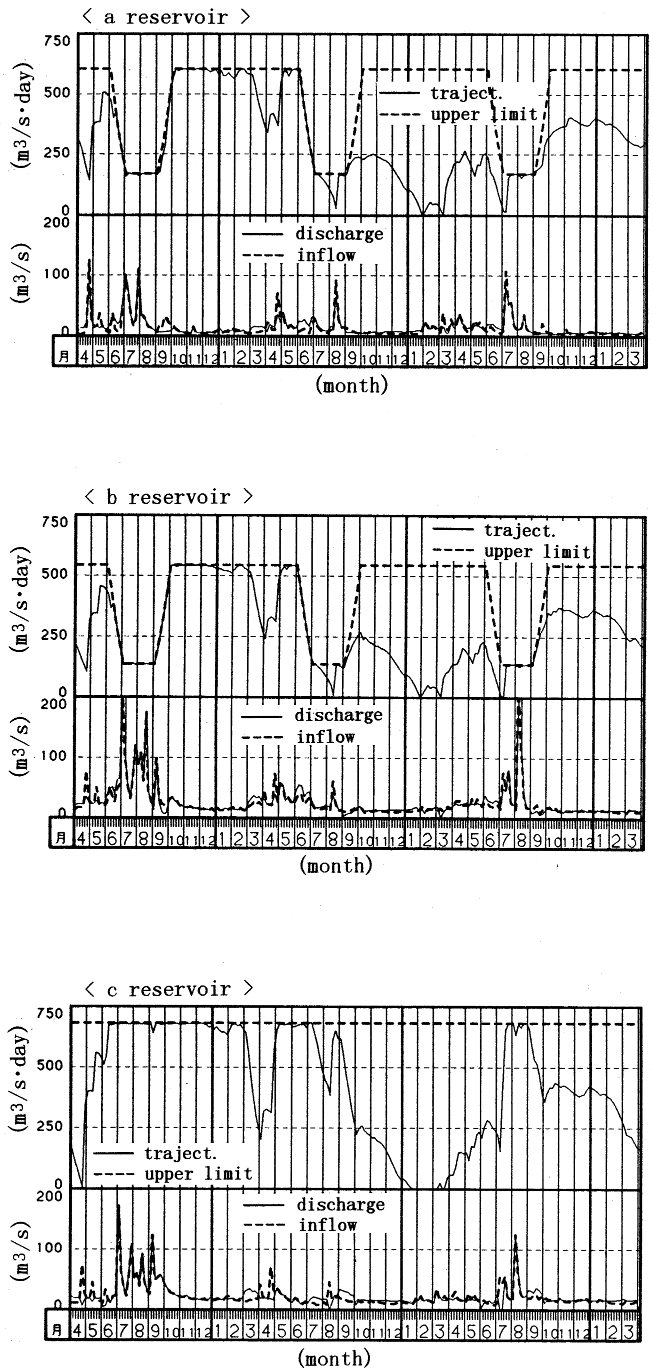


Fig.4 Calculated optimal trajectories, upper bound of Xn, inflow, etc

Each optimal storage trajectory of reservoirs obtained by SDN-DDP is shown in Fig. 4. The duration curve of the ratio of supply Q_{ℓ_d} to D_{np} at each demand node is shown in Fig. 3. Where, we have defined supply ratio as: 100% minus supply-cut ratio. We can conclude that the numerical solution obtained from this simulation converges very close to the optimal solution ,which is judged by the above mentioned criterion and obtained diagrams, etc.

As for E^k of the Eq. 39, at first $\left(E^k\right)^{\ell}_{k=0,\ell=0}$ was about 2300, and it becomes below 0.80 in every ℓ as the increase of k. Where, ℓ is the iteration number of SR-process and k is the iteration number of DN-DDP. In this simulation, the maximum k is 139 and the maximum value of $\left|X^{\ell+1}_{ni}(pt) - X^{\ell}_{ni}(pt)\right| / x_{i\max}$ for each ℓ was at first 0.298, but it became 0.016 at $\ell = 13$. The average of maximum k for each ℓ was about 105. Execution time of the simulation was about 30 minutes using the engineering work station : DEC-500 au.

(3) Comparison of Efficiency with Other Methods

According to several references, e.g. [1], [12], CDDP and the method based on the maximum principle [18] are seemed to be effective methods among many computational methods for finding the optimal control of the state constrained problem on a finite long time horizon. They are originally formulated to treat normal control problems. The optimal control problem in our case, however, is a bit complex; so that both methods are modified slightly to be suitable to our problem. Here, the adjusted CDDP is referred as Method A and the adjusted method based on the maximum principle as Method B. For the simplicity, the comparison of these methods with SDN-DDP was carried out within the limit of DN-DDP omitting SR process.

(a) Models for Comparison Computations and its Results

The model shown in Fig. 5 firstly used for the comparisons of computation efficiency. The model is a simplified version of that shown in Fig. 1.

As the various inputs and constants of the model of Fig. 5, we assumed that the capacity of reservoirs, inflows, demands and the cost function and so forth were same as those of the example model of Fig. 1. But some parameters were selected more suitable for each method. The results in table 2 were obtained by the comparison computations of the model shown in Fig. 5.

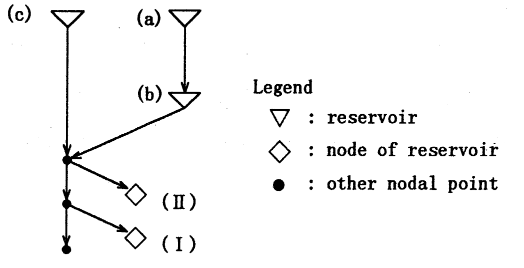


Fig. 5 Configuration of the computation model

Next, the test computations were also done for the model of Fig. 1. In these case, the computational periods were

216 steps (3years). But the results obtained by Method A and Method B were not so good. On the other hand, SDN-DDP converged within $E^k = 0.8$ at $k = 139$ on the same model. For the detail, refer to [13].

(b) Comparison of Results

Table 2 Comparison of computed results (Δ t=5days)

		SDN-DDP	Method-A	Method-B
72 periods (1 year)	Iteration N_D	29	26	43
	Truncation E^k	0.30	0.30	0.30
144 periods (2 years)	Iteration N_D	83	130	190
	Truncation E^k	0.50	0.50	0.50
288 periods (4 years)	Iteration N_D	146	361	508
	Truncation E^k	0.70	0.70	0.70

notice: results are for the model shown in Fig.3

From the above result, SDN-DDP is more robust and more rapid on its convergence for the medium to traditional scale models, so that it is judged that SDN-DDP has the wider applicability than Method A and Method B for the simulation of the multi-reservoir system.

CONCLUDING REMARKS

In this paper, we proposed a new computational method SDN-DDP solving the deterministic discrete optimal control problem of the multi-reservoir system which has a general configuration of the demand-supply system and the modified drought damage cost function. From the result of numerical simulations, it is shown that SDN-DDP is sufficiently applicable to practical problems under the above cost function. In addition, SDN-DDP is superior to standard computational methods of optimal control in robustness and rapidity of the convergence for the medium to ordinary scale multi-reservoir models.

SDN-DDP can cope with the singularity of controls of the multi-reservoir system having the practical drought damage cost function by virtue of introducing the stabilization function as a penalty function. However, introducing the stabilization function implies that one solution is chosen from the original singular optimal solutions having innumerable degrees of freedom. It is advisable from the viewpoint of practical use to balance the coefficients of the stabilization function for reservoirs so as to obtain the desirable optimal solution.

When we apply SDN-DDP to actual multi-reservoir operations, some problems may occasionally occur in Japan. On one hand, these are suitable estimation problems of future inflow duration curves of reservoirs. However, there are several excellent papers on this subject, for example [21], and the subject is beyond the scope of our paper, so we do not mention it more. On the other hand, if there are priorities in water rights and/or target discharges of rivers depending on seasons, i.e., variable new constraints. The quadratic programming part described in this paper can be slightly modified to correspond to the new constraints. This can be essentially done by changing the concerned demand nodes to the absorbed flow specified sinks (or enlarging the coefficients C_p) and/or adding the flow specified imaginary channels (equality constraints) or changing ordinary unconstraint channels to the constraints in the configuration of the sub-region. Wolfe's method has the criterion phase of compatibility for the problem. By the use of this criterion phase, it is straight-forward to successively formulate the adaptable Wolfe's method to above-mentioned variable constraints. As a rule, SDN-DDP is mostly applicable to actual control problems of the multi-reservoir system, if the control problems have the suitable modified drought damage cost function.

Although the controllable time horizon and the number of reservoirs of the system have their limits, it is safely said that SDN-DDP is applicable effectively to a wider range of optimal control problems of the multi-reservoir system than the conventional methods from the results of comparison simulations.

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APPENDIX - NOTATION

The following symbols are used in this paper. Although few symbols have been used reduplicatively according to each precedent, but we suppose that this will not cause any confusion.

a_0 = initial state, initial storages of reservoirs;

a_N = designated final state of reservoirs at $t = (N-1)\Delta t$;

b^k = Lagrangian multiplier regarding the terminal constraint;

B = coefficient of u_n in Eq. 1;

C_ℓ = coefficient of Q_ℓ^2 as the penalty in the cost function;

C_p = coefficient of drought damage function at the demand node p;

C_g =coefficient of the activated penalty function g_n;

D_{np} = water demand at the demand node p on the n-th period;

$E_n^k(\alpha)$	= error estimating function of DN-DDP as defined by Eq. 39;
$E_n'^k(\alpha)$	= error estimating function of DN-DDP as defined by Eq. 40;
$f_n(x_n,u_n)$	= state transition function from x_n to x_{n+1} as defined by Eq. 1;
F_n	= Lagrangian function as defined by Eq. 19;
g_n	= inequality constraint to (x_n,u_n);
h_n	= equality constraint to (x_n,u_n);
$J(\ell)$, J^ℓ	= cost function at the process ℓ, $(\ell = 0,1,\ldots.)$, as defined by Eq. 10;
J^*	= minimum value of the cost function $J^\ell, (\ell \to \infty)$;
J_1^ℓ	= drought damage function component of the cost function J^ℓ, as defined by Eq. 11;
J_2^ℓ	= penalty component of the cost function J^ℓ, as defined by Eq. 12;
J_c^ℓ	= Augmented Lagrangian function of J^ℓ, as defined by Eq. 43;
$J_n(x_n,y_n), J_n$	= Jacobian matrix of T_n originally, and slight modified matrix of original one lately, as defined by Eqs. 29 and 41;
J_{Mn}	= consistent approximation of J_n, lately refer to this as J_n, see Eq. 41;
k	= iteration number of DN-DDP process;
K_n	= Jacobian Matrix of T_n with respect to x_n;
L_{1n}	= the main component of the cost function on the n-th period, see Eq. 5;
L_{2n}^ℓ	= the component of the cost function on the n-th period consisted of penalty function as defined by Eq. 7;
L_n^ℓ	= cost function on the n-th period as defined by Eq. 9;
ℓ	= sequential number in SRP or SDN-DDP;
m	= max. number of reservoirs or max. component number of x_n;
m_n	= max. number of active inequality constraints at the n-th period;
n	= stage number or period number;
N	= max. value of n, integer indicating the final time $t_N = (N-1)\Delta t$;
p	= demand node number in the sub-region

q_n	= Inflows to reservoirs on the n -th period;
Q_ℓ	= flow of channel ℓ in the subregion R on the n -th period;
$Q^k(\alpha,\delta^k)$	= $(N-1)\times(N-1)$ matrix which transform δ^k to δ^{k+1} at the k -th process of DN-DDP under parameter α ;
ΔQ_{np}	= $Q_{\ell_p} - D_{np}$, shortage flow of demand channel ℓ_p on the n -th period;
γ	= max. component number of u_n ;
γ_n	= max. number of equality constraints at n -th period, refer to Eq. 4;
R	= sub-region number in the region;
s	= correcting factor of consistent approximation of $\nabla_u L_{2n}$ and $\nabla_u^2 L_{2n}$;
T_n	= $\left(\nabla_u F_n, g_n^T diag(\lambda_n), h_n^T\right)^T$: refer to Eq. 27;
u_n	= $(u_{n1}, u_{n2}, \ldots\ldots, u_{nr})^T$: discharges and/or intakes of reservoirs on the n -th period;
v	= node number in sub-region R
$V_n(x)$	= minimum value of cost function starting from x at the beginning of n -th period;
x_n	= $(x_{n1}, x_{n2}, \ldots\ldots, x_{nm})^T$: storages of reservoirs at the n -th stage;
X^ℓ	= $\{X_n(pt)\}^\ell$;
$\{X_n(pt)\}^\ell$	= ℓ -th pilot trajectory for $\{x_n\}^\ell$;
y_n	= $\left(u_n^T, \lambda_n^T, \mu_n^T\right)^T$;
Z	= any vector satisfying $\left(\nabla g^*\right)^T Z = 0$ and $\nabla_u h^T Z = 0$;
ε_{ni}	= parameter of stabilization function of reservoir i on the n -th period;
ε_{ik}	= parameter of penalty function of $u_{n(ik)}$, refer to Eq. 8;
α^k	= damping factor in Newton's method at the k-th iteration, step length;
λ_n	= Lagrange multiplier for activated g_n ;
μ_n	= Lagrange multiplier for h_n ;
$\lVert\bullet\rVert_p$	= ℓ_p norm of $\bullet$, but if $p=2$, p is omitted for convenience sake;
$\{\bullet_n\}$	= the ordered set of variable $\bullet_n$;

Notice for the subscripts;

$\bullet^*$	= subscript * denotes the optimal value of variable $\bullet$;
$\bar{\bullet}$	= subscript - denotes that variable $\bullet$ is substituted by approximate values $\nabla\overline{V}^k_n$ and/or $\nabla^2\overline{V}^k_{n+1}$;
$(\bullet)_\theta$ or $[\bullet]_\theta$	= these symbols express the component of $(\bullet)$ or $[\bullet]$ correspond to some variable θ :
i	= i -th component number of state variable or reservoir number of the multi-reservoir;
j	= j -th component number of control variable u_n and y_n ;
k	= k -th iteration number of DN-DDP in the ℓ -th sequential process of SDN-DDP;
(ik)	= a component number in control $(u_n)_i$ of reservoir i , if $(u_n)_i$ are plural. $(ik) \leftrightarrow j$;
ℓ	= ℓ -th sequential process number of SDN-DDP. secondary, water channel number in sub-region R ;
n	= n indicate the stage number or period number;
p	= demand node number in the sub-region;
R	= sub-region number in the region.

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