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Application of the Extended Kalman Filter for Reconstructing Systems from Chaotic Numerical Time Series

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Many hydro-meteorological time series possess chaotic characteristics. In order to study models which can represent this chaotic behavior and their predictability, chaotic time series were generated by the Rössler equations and the parameters of given system models were evaluated by using the Extended Kalman Filter. First, in order to verify the proposed approach, the original structure of the Rössler equations was used as system model. Second, the possibility of reconstructing the system was studied. The results indicate that when a proper system model is given, the Extended Kalman Filter can efficiently evaluate the system parameters from a numerical time series, even when the time series is chaotic.

Keywords: chaos, Extended Kalman Filter, Rössler equations,
reconstructing system, time series

1. INTRODUCTION

Research in the field of hydro-meteorology relies mainly on observed time series of parameters such as sunspots, temperature, and rainfall. Although resulting from continuous physical processes, these phenomena are very difficult to model, chiefly due to the problem of correctly describing their considerable variability on basis of only short observed parts of the theoretically infinite time series. Traditionally, a stochastic approach has been regarded as the only way to deal with this problem, but recent findings indicate that by using the theory of chaos, more efficient ways to characterize these natural phenomena are opened (e.g., Kurths and Herzel, 1987; Rodriguez-Iturbe et al., 1989; Tsonis and Elsner, 1989).

The theory of chaos originates from the work of Rössler and Lorenz, where it was found that certain mathematical models without any explicit random component are able to produce a behavior characterized by some highly specific features, e.g., the appearance is very irregular, the process never exactly repeats itself, the future development is highly sensitive to the initial conditions appearance - a chaotic behavior. These models can be described by a system of purely deterministic but nonlinear equations. When investigating hydro-meteorological processes these equations are beforehand unknown, but by analyzing a time series of a certain observed variable (which is the outcome of all interacting variables in fact), information about the system as a whole may be obtained. By studying the so-called phase space attractor, i.e., a trajectory in space representing the time-evolution of the variables, the requisite number of equations may be estimated. Much work has been devoted to this subject (e.g., Grassberger and Procaccia, 1983), but the following step of identifying these equations is still very little investigated.

The aim of this preliminary study was to test the applicability of the Extended Kalman Filter for identifying the governing equations from a chaotic numerical time series, i.e., reconstructing the system.

2. APPLICATION OF THE EXTENDED KALMAN FILTER FOR CHAOTIC ANALYSIS

2.1 Extended Kalman Filter

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The Kalman Filter algorithm is expressed as differential equations in its original form. By using a difference method, the system equation and its observation equation can be expressed in discrete form (Ueda et al., 1984):

$$X(k+1)=\Phi(k)X(k)+\alpha(k)+u(k) \quad (1)$$
$$Y(k+1)=H(k+1)X(k+1)+\beta(k+1)+w(k+1) \quad (2)$$

where X: system vector to be estimated; Φ: known state transition matrix; α: known constant vector; u: white system noise vector; Y: observation vector; H: known observation matrix; β: known constant vector; w: white observation noise vector; k: time step.

2.2 The Rössler equations

To study the applicability of the Extended Kalman Filter to analysis of chaotic numerical time series, the Rössler equations, a set of differential equations, is used:

$$\begin{cases} \dot{x}=-y-z \\ \dot{y}=x+ay \\ \dot{z}=b+z(x-c) \end{cases} \quad (3)$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ are the parametric derivatives of x, y and z with respect to t which is regarded as time. The parameters a, b and c are constant. The behavior of the equation system is very sensitive to the parameter values. The system obtained when a=0.398, b=2.0, and c=4.0, is often used as an example of a chaotic phenomenon (e.g., Gouesbet, 1991). If the time interval $\Delta t=0.1$, the initial values of x, y, and z are 2.55, -2.4, and 0.7 respectively, and the equations are solved by the Runge-Kutta method, the state space portrait of the system exhibits the appearance shown in Figure 1. This is a typical example of a chaotic (or strange) attractor acting on the system.

2.3 Original system

By using a numerical calculation method to solve the Rössler equations, three time series may be obtained, x(t), y(t), and z(t), which all contain information about the chaotic system. However, when studying hydro-meteorological processes, usually only one time series is available. Therefore, emphasis is put on the possibility to use the Extended Kalman Filter to evaluate system parameters based on only one of the three time series.

When the time series x(t) is used to evaluate parameters, using the original structure of the Rössler equations as model, the number of system variables in the system vector X is six. The components $x_1, x_2, \ldots\ldots, x_6$ of X correspond to x, y, z, a, b and c in equations (3). The system equation for the vector X is expressed as a vector function $\mathbf{f}$ as follows:

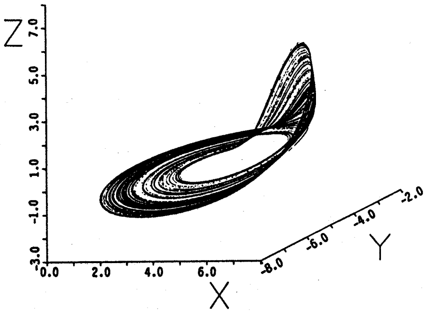

Figure 1. The state space portrait of the Rössler equations using the parameters: a=0.398, b=2.0, c=4.0.

$$\dot{X}=\mathbf{f}(X) \quad (4)$$

In developed form:

$$\begin{cases} \dot{x}_1=f_1(X)=-x_2-x_3 \\ \dot{x}_2=f_2(X)=x_1+x_2x_4 \\ \dot{x}_3=f_3(X)=x_5+x_3(x_1-x_6) \\ \dot{x}_4=f_4(X)=0 \\ \dot{x}_5=f_5(X)=0 \\ \dot{x}_6=f_6(X)=0 \end{cases} \quad (5)$$

The system state transition matrix Φ(k) and constant vector α(k) in the system equation (1) are:

$$\Phi(k)=J_f[X^*(k)]\Delta t+I \quad \text{(I: Unit Matrix)} \quad (6)$$
$$\alpha(k)=\{\mathbf{f}[X^*(k)]-J_f[X^*(k)]X(k)\}\Delta t \quad (7)$$
$$\{J_f[X^*(k)]\}_{ij}=\partial f_i[X(k)]/\partial x_j|_{X(k)=X^*(k)} \quad (J_f\text{: Jacobian Matrix}) \quad (8)$$

The vector $X^*(k)$ is replaced by X(k|k-1), which is the estimation of the state vector X at time step k based on the observation at time step k-1. In this case the observation equation (2) is a function of X as follows:

$$Y=\mathbf{g}(X) \quad (9)$$

If only one of the three time series is observed, the observation vector Y in the equation (2) becomes scalar, and its element y_1 is expressed as:

$$y_1 = g_1(X) = x_1 \tag{10}$$

The observation matrix H(k+1) and constant vector β(k+1) in the observation equation (2) are:

$$H(k) = J_g[X^*(k)] \tag{11}$$
$$\beta(k) = g[X^*(k)] - J_g[X^*(k)]X^*(k) \tag{12}$$
$$\{J_g[X^*(k)]\}_{ij} = \partial g_i[X(k)]/\partial x_j \,|_{X(k)=X^*(k)} \quad (J_g\text{: Jacobian Matrix}) \tag{13}$$

The term $X^*(k)$ is replaced by $X(k|k\text{-}1)$.

During the calculation, predictions of $x_1(t)$, $x_2(t)$,..., $x_6(t)$, corresponding to the time series x(t), y(t), z(t) and the constants a, b, c in the original equations, are received. From the results shown in Figure 2, it is evident that by using one time series and the Extended Kalman Filter, the exact original system can be identified because of the chaotic behavior of the system. If y(t) or z(t) are used as observation time series, the results are similar. However, when z(t) is used as observation time series, the convergence is somewhat slower than for x(t) and y(t). This is due to the smaller variation of z(t), as compared to x(t) and y(t). If two or all of the time series are used to evaluate the parameters, the results are better than in case when only one is used.

2.4 Reconstructing system

According to the results of 2.3, when the structure of the original Rössler equation system is used as model, the parameters can be efficiently evaluated using the Extended Kalman Filter. In practice, however, the exact mathematical description of the system is unknown. In this case a general model framework must be used as a starting-point. The first step is to calculate the number of equations (or variables) required to describe the system, the second is to determine the form of the equation system, and the third is to evaluate the model parameters - all three steps based on observed time series. To complete the first step, a fractal analysis of the time series attractor may be used (e.g., Hense, 1987; Mundt et al., 1991), but for performing steps two and three, no established methods exist at present.

As previously mentioned, when studying hydro-meteorological processes, usually only one time series is available. Besides, we do not know which other variables are included in the system. All the available information is contained in the time series at hand, for example x(t). In this case, for three dimensional systems, the following equation system can be a suitable starting-point:

$$\begin{cases} \dot{x} = Y \\ \dot{Y} = Z \\ \dot{Z} = f(x,Y,Z) \end{cases} \tag{14}$$

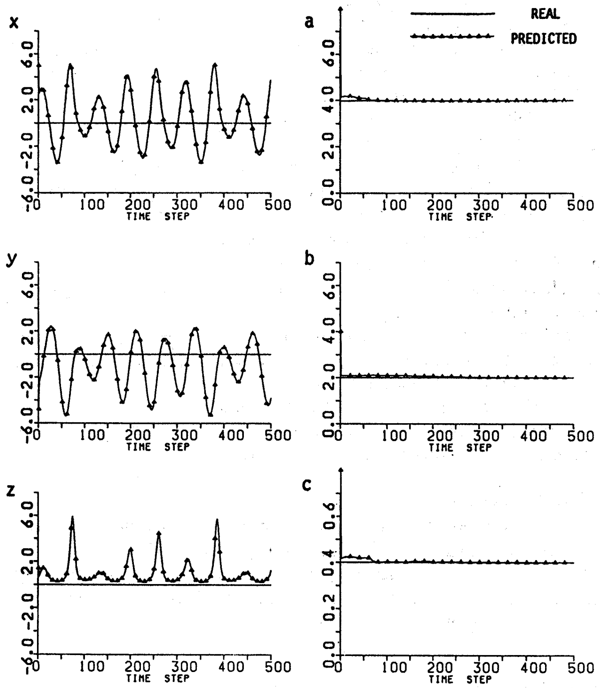

Figure 2. Series (left) and parameters (right) of original system evaluated from the time series x(t) by use of Extended Kalman Filter

where Y and Z may be real variables of system or complex variables based on real variables. Therefore the equations (14) is a reconstructing system. When the function f(x,Y,Z) has a proper form and the parameters are correct, equations (14) can be equivalent to the original ones. For both the original system and the reconstructing system, it is important that the variable x in fact corresponds to the observed time series.

When the time series x(t) of the Rössler equations is considered, one form of the function f(x,Y,Z) is derived from the original equations (3) as follows (Gouesbet, 1991):

$$f(x,Y,Z) = ab - cx + x^2 - axY + xZ + (ac-1)Y + (a-c)Z - Y(x+b-aY+Z)/(a+c-x) \tag{15}$$

From equations (14) and (15) and the time series x(t) obtained from the original Rössler equations, we can use the Extended Kalman Filter to receive the time series x(t), Y(t), Z(t), and calculate the parameters a, b, and c. As shown in Figure 3, the results indicate that although the convergence towards the true parameter values is slower than for the previous case, the proposed technique can efficiently be used for reconstructing systems from chaotic numerical time series, on condition that a proper system structure is given.

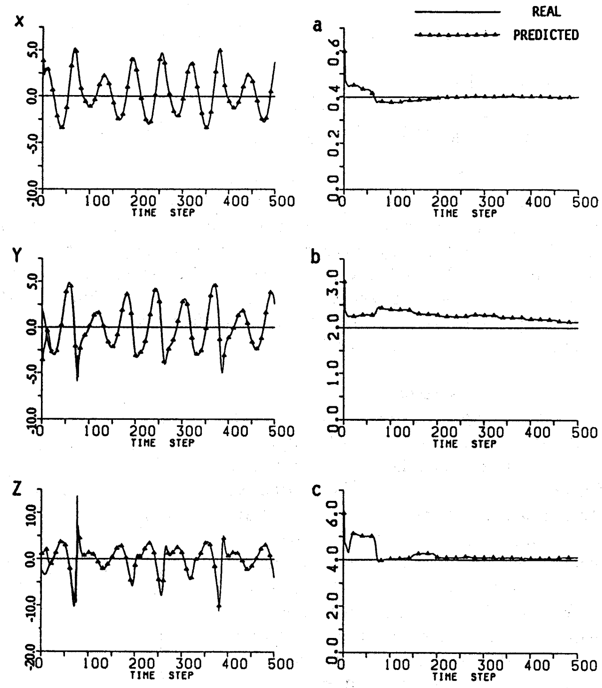

Figure 3. Series (left) and parameters (right) of reconstructing system evaluated from the time series x(t) by use of Extended Kalman Filter

3. SUMMARY, CONCLUSIONS AND FUTURE PROSPECTS

Recent findings indicate that many hydro-meteorological processes are characterized by a chaotic behavior, which may be generated by a system of deterministic non-linear equations. Information about this system can be obtained from observed time series of any system variable. The Extended Kalman Filter has been proven to be an efficient tool for investigating time series, and in the present study this technique is applied to chaotic time series from the Rössler equations in order to reconstruct the original system.

As starting-point models, both the original system and a more general form were used. The results indicate that the Extended Kalman Filter can efficiently be used for reconstructing original (or equivalent) systems from chaotic numerical time series, provided a proper system structure is assumed and a proper time series is available.

Regarding future research, the next step will be to test the possibility of using a generalized form of the function f(x,Y,Z), e.g., based on a third order Taylor series expansion (20 parameters). When this approach has been fully developed and tested on synthetic series, the main purpose of this study will start - to apply this technique to real observed time series of e.g. sunspots, temperature, and rainfall. This kind of researches might lead to the development of new methods to describe and predict hydro-meteorological processes.

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