Generalized Morrey spaces

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Abstract

Morrey spaces can complement the boundedness properties of operators that Lebesgue spaces cannot handle. Morrey spaces which we have been handling are called classical Morrey spaces. However, classical Morrey spaces are not totally enough to describe the boundedness properties. To this end, we need to generalize parameters $p$ and $q$, among others $p$.

1 Generalized Morrey spaces

Let $0 < q \leq p < \infty$. We defined the classical Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ to be the set of all measurable functions $f$ for which the quantity

$$
\|f\|_{\mathcal{M}^p_q} \equiv \sup_{x \in \mathbb{R}^n, r > 0} |Q(x, r)|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{Q(x, r)} |f(y)|^q dy \right)^{\frac{1}{q}}
$$

is finite, where $Q(x, r) = \{ y \in \mathbb{R}^n : |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \leq r \}$. To describe the endpoint case or to describe the intersection space, sometimes it is useful to generalize the parameter $p$: let us suppose that $p$ comes from the function $t^{-\frac{n}{p}}$. So, we envisage the situation where $t^{-\frac{n}{p}}$ is replaced by a general function $\varphi(t)^{-1}$. Likewise we will replace $q$ by a function $\Phi$ later in the next section.

1.1 Definition of generalized Morrey spaces

From the observation above, we are led to the following definition:

**Definition 1.1.** Let $0 < q < \infty$ and $\varphi : (0, \infty) \to [0, \infty)$ be a function which does not satisfy $\varphi \equiv 0$.

1. Define the generalized Morrey space $\mathcal{M}^{\varphi}_q(\mathbb{R}^n)$ to be the set of all measurable functions $f$ such that

$$
\|f\|_{\mathcal{M}^{\varphi}_q} \equiv \sup_{x \in \mathbb{R}^n, r > 0} \varphi(r) \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^q dy \right)^{\frac{1}{q}} < \infty.
$$

(1.1)
2. Define the generalized weak Morrey space $\text{wM}^p_q(\mathbb{R}^n)$ to be the set of all measurable functions $f$ such that
\[
\|f\|_{\text{wM}^p_q} \equiv \sup_{\lambda > 0} \|\lambda \chi_{(\lambda, \infty)}(|f|)\|_{\text{M}^p_q} < \infty.
\]

Although we tolerate the case where $\varphi(t) = 0$ for some $t > 0$, it turns out that there is no need to consider such possibility.

Before we go into more details, a clarifying remark may be in order.

Remark 1.2. Nakai defined the generalized Morrey space $\text{M}^p_q(\mathbb{R}^n)$ to be the set of all measurable functions $f$ such that
\[
\|f\|_{\text{M}^p_q} \equiv \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(r)} \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} < \infty. \tag{1.2}
\]

Here we follow the notation by Sawano, Sugano and Tanaka.

We can recover the Lebesgue space $L^q(\mathbb{R}^n)$ by letting $\varphi(t) \equiv t^{\frac{n}{q}}$ for $t > 0$ as we have mentioned. To compare Morrey spaces with generalized Morrey spaces, we call Morrey spaces classical Morrey spaces. In addition to the function $\varphi(t) = t^{\frac{n}{q}}$, we consider the following typical functions:

Example 1.3.

1. The function $\varphi \equiv 1$ generates $L^\infty(\mathbb{R}^n)$ thanks to the Lebesgue convergence theorem.

2. Let $0 < q < \infty$ and $a \in \mathbb{R}$. Define $\varphi(t) = t^{\frac{n}{q}} (\log(e + t))^a$ for $t > 0$. We remark that $\text{M}^p_q(\mathbb{R}^n) \neq \{0\}$ if and only if $a \leq 0$. In fact, we have
\[
\|f\|_{\text{M}^p_q} = \sup_{x \in \mathbb{R}^n} (\log(e + r))^a \|f\|_{L^q(Q(x, r))}.
\]

Thus, if $f$ is a measurable function such that $\|f\|_{\text{M}^p_q} < \infty$ and if $a > 0$, then we have $\|f\|_{L^q(Q(x, r))} = 0$ for any cube $Q(x, r)$. Thus, $f = 0$ a.e.. Conversely if $a \leq 0$, then $L^q(\mathbb{R}^n) \subset \text{M}^p_q(\mathbb{R}^n)$.

3. Let $0 < q \leq p_1 < p_2 < \infty$. Then $\varphi(t) = t^{\frac{n}{p_1}} + t^{\frac{n}{p_2}}$, $t > 0$ can be used to express the intersection of $\text{M}^p_{q_1}(\mathbb{R}^n) \cap \text{M}^p_{q_2}(\mathbb{R}^n)$. In general, for $0 < q < \infty$ and $\varphi_1, \varphi_2 : (0, \infty) \to (0, \infty)$ with equivalence of norms, $\text{M}^p_{q_1}(\mathbb{R}^n) \cap \text{M}^p_{q_2}(\mathbb{R}^n) = \text{M}^{p_1 + p_2}_{q_1 + q_2}(\mathbb{R}^n)$.

4. Let $0 < q \leq p_1 < p_2 < \infty$. Let $\varphi(t) = \chi_{Q \cap (0, \infty)}(t) t^{\frac{n}{p_1}} + \chi_{(0, \infty) \setminus Q}(t) t^{\frac{n}{p_2}}$ for $t > 0$. Then we have $\text{M}^p_q(\mathbb{R}^n) = \text{M}^{p_1}_q(\mathbb{R}^n) \cap \text{M}^{p_2}_q(\mathbb{R}^n)$ and for any $f \in L^0(\mathbb{R}^n)$
\[
\|f\|_{\text{M}^p_q} = \max\{\|f\|_{\text{M}^p_{q_1}}, \|f\|_{\text{M}^p_{q_2}}\}.
\]
5. We can consider the norm
\[
\sup_{x \in \mathbb{R}^n, r \in (0,1)} |Q(x,r)|^{\frac{1}{p'}} \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q'}}
\]
for \(0 < q < p < \infty\). In fact, we take \(\varphi(t) = t^{\frac{n}{q}} \chi_{[0,1]}(t)\).

6. Likewise we can consider the norm
\[
\sup_{x \in \mathbb{R}^n, r \in (1,\infty)} |Q(x,r)|^{\frac{1}{p'}} \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q'}}
\]
for \(0 < q < p < \infty\). In fact, we take \(\varphi(t) = t^{\frac{n}{q}} \chi_{[1,\infty)}(t)\).

7. Let \(0 < q < \infty\) The uniformly locally \(L^q\)-integrable space \(L^q_{uloc}(\mathbb{R}^n)\) is the set of all measurable functions \(f\) for which
\[
\sup_{x \in \mathbb{R}^n, r \in (0,1)} \left( \int_{Q(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} = \sup_{x \in \mathbb{R}^n} \left( \int_{Q(x,1)} |f(y)|^q \, dy \right)^{\frac{1}{q}}
\]
is finite. As before, if we let \(\varphi(t) = t^{\frac{n}{q}} \chi_{[0,1]}(t)\), then we obtain \(L^q_{uloc}(\mathbb{R}^n) = \mathcal{M}^q_{\ell}(\mathbb{R}^n)\). We can define the weak uniformly locally \(L^q\)-integrable space \(wL^q_{uloc}(\mathbb{R}^n)\) similarly. For \(f \in L^0(\mathbb{R}^n)\), the norm is given by
\[
\|f\|_{wL^q_{uloc}} \equiv \sup_{\lambda > 0} \lambda \|\chi(\lambda,\infty)|(f)|\|_{L^q_{uloc}}.
\]

The fifth example deserves a name. We define the small Morrey space as follows:

**Definition 1.4.** Let \(0 < q < p < \infty\). The small Morrey space \(m^p_q(\mathbb{R}^n)\) is the set of all measurable functions \(f\) for which the quantity
\[
\|f\|_{m^p_q} \equiv \sup_{x \in \mathbb{R}^n, r \in (0,1)} |Q(x,r)|^{\frac{1}{p'}} \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q'}}
\]
is finite. The weak small Morrey space \(wm^p_q(\mathbb{R}^n)\) is defined similarly. For \(f \in L^0(\mathbb{R}^n)\), the norm is given by
\[
\|f\|_{wm^p_q} \equiv \sup_{\lambda > 0} \lambda \|\chi(\lambda,\infty)|(f)|\|_{m^p_q}.
\]

**Example 1.5.** Let \(x \in \mathbb{R}^n\) and \(r > 0\). Then
\[
\|\chi_{Q(x,r)}\|_{\mathcal{M}^p_q} = \sup_{t > 0} \varphi(t) \min(t^{-n/q}, r^{-n/q}).
\]
In fact, simply observe that
\[
\|\chi_{Q(x,r)}\|_{\mathcal{M}^p_q} \equiv \sup_{R > 0} \varphi(R) \left( \frac{|Q(x,R) \cap Q(x,r)|}{|Q(x,R)|} \right)^{\frac{1}{q}} = \sup_{t > 0} \varphi(t) \min(t^{-n/q}, r^{-n/q}).
\]
The following min(1, q)-triangle inequality holds:

**Lemma 1.6.** Let $0 < q < \infty$ and $\varphi : (0, \infty) \to (0, \infty)$ be a function. Then

$$\|f + g\|_{\mathcal{M}_q^{\min(1, q)}} \leq \|f\|_{\mathcal{M}_q^{\min(1, q)}} + \|g\|_{\mathcal{M}_q^{\min(1, q)}}$$

for all $f, g \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

**Proof.** This is similar to classical Morrey spaces: Use the min(1, q)-triangle inequality for the Lebesgue space $L^q(\mathbb{R}^n)$.

**Proposition 1.7.** Let $0 < q < \infty$, and let $\varphi : (0, \infty) \to [0, \infty)$ be a function satisfying $\varphi(t_0) \neq 0$ for some $t_0 > 0$. Then $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is a quasi-Banach space and if $q \geq 1$, then $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is a Banach space.

**Proof.** The norm inequality follows from Lemma 1.6. The proof of the completeness is a routine, which we omit.

Proposition 1.7 guarantees that the (quasi-)norm of $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is complete. However, it may happen that $\mathcal{M}_q^\varphi(\mathbb{R}^n) = \{0\}$. We check that this extraordinary thing never happens if $\varphi$ satisfies a mild condition.

**Proposition 1.8.** Let $0 < q < \infty$, and let $\varphi : (0, \infty) \to [0, \infty)$ be a function satisfying $\varphi(t_0) \neq 0$ for some $t_0 > 0$. Then the following are equivalent:

(a) $L^\infty_c(\mathbb{R}^n) \subset \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

(b) $\mathcal{M}_q^\varphi(\mathbb{R}^n) \neq \{0\}$.

(c) $\sup_{t > 0} \varphi(t) \min(t^{-n/q}, 1) < \infty$.

**Proof.** It is clear that (a) implies (b).

Assume (b). Then there exists $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \setminus \{0\}$. We may assume that $f(0) \neq 0$ and $x = 0$ is the Lebesgue point of $|f|^q$. Then since $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, by the Lebesgue differential theorem,

$$\frac{1}{|Q(r)|} \int_{Q(r)} |f(y)|^q \, dy \sim 1$$

for all $0 < r < 1$. Thus,

$$\sup_{0 < t \leq 1} \varphi(t) < \infty.$$

Here the implicit constants depend on $f$. Since $\chi_{Q(1)} f \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \setminus \{0\}$, we conclude (c).

Finally, if (c) holds, then $\chi_{Q(x, 1)} \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$. Since $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is a linear space, $\chi_{Q(x, m)} \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$. Since any function $g \in L^\infty_c(\mathbb{R}^n)$ admits the estimate of the form $|g| \leq N \chi_{Q(x, m)}$ for some $m, N \in \mathbb{N}$, we conclude (a).

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**Example 1.9.** Let $0 < q \leq p < \infty$ and $B = (\beta_1, \beta_2) \in \mathbb{R}^2$. We write

$$\ell^B(r) = \ell^{(\beta_1, \beta_2)}(r) = \begin{cases} 
(1 + |\log r|)^{\beta_1} & (0 < r \leq 1), \\
(1 + |\log r|)^{\beta_2} & (1 < r < \infty).
\end{cases}$$

1. We set

$$\Phi(t) = \frac{t^n}{\ell^B(t)} \quad (t > 0).$$

Note that $\mathcal{M}_q^\varphi(\mathbb{R}^n) \neq \{0\}$ if and only if $\beta_2 \geq 0$. Indeed, according to Proposition 1.8 the case $\beta_2 < 0$ must be excluded in order that $\mathcal{M}_q^\varphi(\mathbb{R}^n) \neq \{0\}$. Conversely if $\beta_2 \geq 0$, then $L^\infty(\mathbb{R}^n) \subset \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

2. Let $0 < q < p < \infty$. We set

$$\Phi(t) = \frac{t^n}{\ell^B(t)} \quad (t > 0).$$

Then $L^\infty(\mathbb{R}^n) \subset \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

3. We set

$$\Phi(t) = \frac{1}{\ell^B(t)} \quad (t > 0).$$

For $a \in \mathbb{R}$, we set $f(x) = (1 + |x|)^{-a}$, $x \in \mathbb{R}^n$. Let us see that $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is close to $L^\infty(\mathbb{R}^n)$ if $\beta_1 \geq 0$.

(a) By the Lebesgue differentiation theorem, $\|f\|_{L^\infty} \leq 0$ if $\beta_1 < 0$, so that $f = 0$ a.e. So, if $\beta_1 < 0$, then $\mathcal{M}_q^\varphi(\mathbb{R}^n) = \{0\}$.

(b) Let $\beta_1 \geq 0 > \beta_2$. Then $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ if and only if $a < 0$.

(c) Let $\beta_1, \beta_2 \geq 0$. Then $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ if and only if $a \leq 0$.

We have the following scaling law:

**Lemma 1.10.** Let $0 < q_1, q < \infty$ and $\varphi : (0, \infty) \to (0, \infty)$ be a function. If $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$, then $|f|^q \in \mathcal{M}_{q/q_1}^\varphi(\mathbb{R}^n)$ with $\||f|^q\|_{\mathcal{M}_{q/q_1}^\varphi} = \|f\|_{\mathcal{M}_q^\varphi}$.

**Proof.** We calculate

$$\|\|f|^q\|_{\mathcal{M}_{q/q_1}^\varphi} = \sup_{Q=Q(a,r)} \frac{\varphi(r)^q \|f\|_{L^q(Q)}^q}{|Q|^{q/q_1}} = \|f\|_{\mathcal{M}_q^\varphi}$$

by using $\|\|f|^q\|_q = \|f\|_{q/q_1}$. 

The nesting property holds like classical Morrey spaces.

**Lemma 1.11.** Let $0 < q_1 \leq q_2 < \infty$ and $\varphi : (0, \infty) \to (0, \infty)$ be a function. Then $\mathcal{M}_{q_2}^\varphi(\mathbb{R}^n) \subset \mathcal{M}_{q_1}^\varphi(\mathbb{R}^n)$.

**Proof.** This is similar to classical Morrey spaces.
1.2 The class \( G_q \)

It will turn out demanding to consider all possible functions \( \varphi \). We will single out good functions. We answer the question of what functions are good.

**Definition 1.12.** An increasing function \( \varphi : (0, \infty) \rightarrow (0, \infty) \) is said to belong to the class \( G_q \) if \( t^{\frac{n}{q}} \varphi(t) \geq s^{\frac{n}{q}} \varphi(s) \) for all \( 0 < t \leq s < \infty \).

Remark that \( G_{q_1} \subset G_{q_2} \) if \( 0 < q_2 < q_1 < \infty \).

Here we list a series of the functions in \( G_q \).

**Example 1.13.** Let \( 0 < q < \infty \).

1. Let \( u \in \mathbb{R} \), and let \( \varphi(t) = t^u \) for \( t \geq 0 \). Then \( \varphi \) belongs to \( G_q \) if and only if \( 0 \leq u \leq \frac{n}{q} \).
2. Let \( 0 < u \leq \frac{n}{q} \), \( L \gg 1 \) and let \( \varphi(t) = t^u \log(L + t) \) for \( t \geq 0 \). Then \( \varphi \) belongs to \( G_q \).
3. If \( \varphi_1, \varphi_2 \in G_q \), then \( \varphi_1 + \varphi_2, \max(\varphi_1, \varphi_2) \in G_q \).
4. Let \( 0 \leq u \ll 1 \), and let \( \varphi(t) = t^u \log(e + t) \) for \( t \geq 0 \). Then \( \varphi \notin G_q \) because \( \varphi \) is not increasing.

We start with a simple observation that any function in \( G_q \) enjoys the doubling property:

**Proposition 1.14.** If \( \varphi \in G_q \) with \( 0 < q < \infty \), then \( \varphi(r) \leq \varphi(2r) \leq 2^{\frac{n}{q}} \varphi(r) \) for all \( r > 0 \).

**Proof.** The left inequality is a consequence of the fact that \( \varphi \) is increasing, while the right inequality follows from the fact that \( t \mapsto t^{-n/q} \varphi(t) \) is decreasing. \( \square \)

The next proposition justifies that we can naturally use the class \( G_q \).

**Proposition 1.15.** Let \( 0 < q < \infty \). Then for any function \( \varphi : (0, \infty) \rightarrow (0, \infty) \) satisfying \( \sup_{t>0} \varphi(t) \min(t^{-n/q}, 1) < \infty \), there exists a function \( \varphi^* : (0, \infty) \rightarrow (0, \infty) \in G_q \) such that \( \mathcal{M}^q_{\varphi}(\mathbb{R}^n) = \mathcal{M}^q_{\varphi^*}(\mathbb{R}^n) \) with equivalence of norms.

**Proof.** Let \( f \) be a measurable function. A simple geometric observation shows

\[
\frac{1}{|Q|} \int_Q |f(x)|^q \, dx \leq 2^n \sup_{Q' \in \mathcal{Q} : Q' \subset Q, \ell(Q') = t'} \left\{ \frac{1}{|Q'|} \int_{Q'} |f(x)|^q \, dx \right\}
\]

for any cube \( Q \in \mathcal{Q} \) and any positive number \( t' \leq \ell(Q) \). If we let

\[
\varphi_1(t) = \inf_{t' \in (0, t]} \varphi(t')
\]

for \( t > 0 \), then

\[
\varphi_1(t) \lesssim \frac{\varphi(t)}{t^{1/q}}
\]

and

\[
\varphi_2(t) = \sup_{t' \in (0, t]} \varphi(t')
\]

for \( t > 0 \), then

\[
\varphi_2(t) \lesssim t^{1/q} \varphi(t)
\]

for \( t > 0 \). Therefore, \( \varphi \) is controlled by \( \varphi_1 \) and \( \varphi_2 \).

The next proposition justifies that we can naturally use the class \( G_q \).

**Proposition 1.15.** Let \( 0 < q < \infty \). Then for any function \( \varphi : (0, \infty) \rightarrow (0, \infty) \) satisfying \( \sup_{t>0} \varphi(t) \min(t^{-n/q}, 1) < \infty \), there exists a function \( \varphi^* : (0, \infty) \rightarrow (0, \infty) \in G_q \) such that \( \mathcal{M}^q_{\varphi}(\mathbb{R}^n) = \mathcal{M}^q_{\varphi^*}(\mathbb{R}^n) \) with equivalence of norms.

**Proof.** Let \( f \) be a measurable function. A simple geometric observation shows

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for any cube \( Q \in \mathcal{Q} \) and any positive number \( t' \leq \ell(Q) \). If we let

\[
\varphi_1(t) = \inf_{t' \in (0, t]} \varphi(t')
\]

for \( t > 0 \), then

\[
\varphi_1(t) \lesssim \frac{\varphi(t)}{t^{1/q}}
\]

and

\[
\varphi_2(t) = \sup_{t' \in (0, t]} \varphi(t')
\]

for \( t > 0 \), then

\[
\varphi_2(t) \lesssim t^{1/q} \varphi(t)
\]

for \( t > 0 \). Therefore, \( \varphi \) is controlled by \( \varphi_1 \) and \( \varphi_2 \).
then
\[ \|f\|_{M_q^*} \leq \|f\|_{M_q^{*1}} \leq 2^{\frac{n}{q}} \|f\|_{M_q^*}. \]

Next, if we let
\[ \varphi^*(t) \equiv t^{-\frac{n}{q}} \inf_{t' \geq t} \varphi_1(t') t'^{\frac{n}{q}} \]
then \( \|f\|_{M_q^*} = \|f\|_{M_q^{*1}} \). In fact, it is trivial that \( \varphi^*(t) \leq \varphi(t) \) for all \( t > 0 \) and thus \( \|f\|_{M_q^*} \geq \|f\|_{M_q^{*1}} \). On the other hand, for any \( r > 0 \) and \( \varepsilon > 0 \), we can find \( r' \geq r \) such that
\[ (1 + \varepsilon) \varphi^*(r) \geq r^{-\frac{n}{q}} \varphi_1(r') r'^{\frac{n}{q}}. \]
Thus for any cube \( Q(x, r) \),
\[
\varphi(r) \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} \\
\leq (1 + \varepsilon) \varphi^*(r') \left( \frac{1}{|Q(x, r')|} \int_{Q(x, r')} |f(y)|^q \, dy \right)^{\frac{1}{q}} \\
\leq (1 + \varepsilon) \varphi^*(r') \left( \frac{1}{|Q(x, r')|} \int_{Q(x, r')} |f(y)|^q \, dy \right)^{\frac{1}{q}} \\
\leq (1 + \varepsilon) \|f\|_{M_q^{*1}}.
\]
Taking the supremum over \( x \) and \( r \), we have \( \|f\|_{M_q^*} \leq (1 + \varepsilon) \|f\|_{M_q^{*1}} \). Since \( \varepsilon > 0 \) is finite, we conclude \( \|f\|_{M_q^*} \leq \|f\|_{M_q^{*1}} \). \( \square \)

In view of Proposition 1.15, it follows that we can always suppose that \( \varphi \in G_q \). We may further suppose that there exists \( \delta > 0 \) such that
\[ \varphi(r) \leq \delta \quad (0 < r \leq 1) \quad \text{(1.3)} \]
and that
\[ r^{\frac{n}{p}} \varphi(r) > \delta \quad (1 < r < \infty). \quad \text{(1.4)} \]

As the next proposition shows, \( G_q \) is a good class in addition to the nice property that it naturally arises in generalized Morrey spaces.

**Proposition 1.16.** Let \( \varphi \in G_q \) with \( 0 < q < \infty \). Then we can find a continuous function \( \varphi^* \in G_q \) such that \( \varphi^* \) is strictly decreasing and that \( \varphi \sim \varphi^* \).

**Proof.** Since \( \varphi^q \in G_1 \), we may assume \( q = 1 \). Note that \( G_1 \) is the set of all nondecreasing functions \( \varphi : [0, \infty) \to [0, \infty) \) such that \( t \in (0, \infty) \mapsto \varphi(t)t^{-n} \in (0, \infty) \) is nonincreasing. Consider
\[
\varphi^*(t) \equiv t \int_t^\infty \left( \varphi(s) - \frac{e^s}{2(1 + e^s)} \inf \varphi \right) \frac{ds}{s^2} \quad (t > 0).
\]
Since \( \varphi \) is decreasing, the function \( \varphi^* \) is decreasing. Also, by the fact that \( r \varphi(r) \) is increasing, we see that \( \varphi^* \) and \( \varphi \) are equivalent. \( \square \)
Thus, we are led to the following definition:

**Definition 1.17.** The class $W$ stands for the set of all continuous functions $\varphi : (0, \infty) \to (0, \infty)$. That is, $W = C((0, \infty), (0, \infty))$.

We apply what we have obtained to small Morrey spaces.

**Example 1.18.** Let $0 < q < \infty$. Let us see how we modify $\psi$ in $M^\psi_q(\mathbb{R}^n)$ to obtain the equivalent space $M^\varphi_q(\mathbb{R}^n)$ with $\varphi \in \mathcal{G}_q$.

1. Let $\varphi(t) = \max(t^\frac{2}{a}, 1)$ and $\psi(t) = t^\frac{2}{a} \chi_{[0,1]}(t)$ for $t > 0$. Then $M^\psi_q(\mathbb{R}^n) = M^\varphi_q(\mathbb{R}^n)$.

2. Let $\varphi(t) = \max(t^a, 1)$ with $a \geq \frac{n}{q}$ and $\psi(t) = t^\frac{n}{q} \chi_{[0,1]}(t)$ for $t > 0$. Then $L^q_{uloc}(\mathbb{R}^n) = M^\psi_q(\mathbb{R}^n) = M^\varphi_q(\mathbb{R}^n)$.

Note that $\varphi \in \mathcal{G}_q \cap W$ in both cases.

So far we have shown that we may assume that $\varphi \in \mathcal{G}_q$. As a result, we may assume that $\varphi$ is doubling. This observation makes the definition of the norm $\| \cdot \|_{M^\varphi_q}$ more flexible.

**Remark 1.19.** In (1.1), we can replace balls with cubes; we will obtain an equivalent norms. So, we use the norm given by

$$
\|f\|_{M^\varphi_q} \equiv \sup_{x \in \mathbb{R}^n, r > 0} \varphi(r) \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.
$$

Related to this definition, we give some definitions related to the class $\mathcal{G}_q$. Although we show that it is sufficient to limit ourselves to $\mathcal{G}_q$, we still feel that this class is too narrow as the function of $\varphi(t) = t \log(e + t^{-1})$ shows. So, it is convenient to relax the condition on $\varphi$. The following definition will serve to this purpose.

**Definition 1.20.** Let $\ell > 0$.

1. A function $\varphi : (0, \ell] \to (0, \infty)$ is said to be almost decreasing if $\phi(s) \lesssim \phi(t)$ for all $0 \leq t < s \leq \ell$.

2. A function $\varphi : (0, \ell] \to (0, \infty)$ is said to be almost increasing if $\phi(s) \gtrsim \phi(t)$ for all $0 \leq t < s \leq \ell$.

3. A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost decreasing if $\phi(s) \lesssim \phi(t)$ for all $0 \leq t < s < \infty$.

4. A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost increasing if $\phi(s) \gtrsim \phi(t)$ for all $0 \leq t < s < \infty$. 

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The implicit constants in these inequality are called the almost decreasing/increasing constants.

**Example 1.21.**

1. Let $a, b$ be real parameters with $b \neq 0$. Let $\varphi_a(t) = t^a (\log(e+t))^b$ for $t > 0$. Then $\varphi_a$ is almost increasing for any $a > 0$. If $a < 0$, then $\varphi_a$ is almost decreasing.

2. The function $\varphi(t) = t + \sin 2t, t > 0$ is almost increasing but not increasing.

3. The function $\varphi(t) = 1 + e^t (1 - \cos t), t > 0$ is almost increasing because $\varphi(2\pi m) = 1$ and $\varphi(2\pi m + \pi) = 1 + 2e^{2\pi m + \pi}$ for all $m \in \mathbb{N}$.

4. Let $0 < p, q < \infty$ and $\beta_1, \beta_2 \in \mathbb{R}$. We write

$$\ell^B(r) \equiv \begin{cases} (1 + |\log r|)^{\beta_1} & (0 < r \leq 1), \\ (1 + |\log r|)^{\beta_2} & (1 < r < \infty). \end{cases}$$

We set

$$\varphi(t) = \frac{t^p}{\ell^B(t)} \quad (t > 0)$$

as we did in Example 1.9. We observe that $\varphi$ is almost increasing for all $p > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$. We also note that $t \mapsto \varphi(t)t^{-n/q}$ is almost decreasing if and only if $p = q$ and $\beta_1, \beta_2 \geq 0$ or $p > q$. Note that $\varphi$ is an equivalent to a function $\psi \in \mathcal{G}_q$ if and only if either $p = q$, $\beta_1, \beta_2 \geq 0$ or $p > q$ since we can neglect the effect of "log".

5. Let $E \subset (0, \infty)$ be a non-Lebesgue measurable set. Then $\varphi = 1 + \chi_E$ is almost increasing and almost decreasing. Note that $\varphi$ is not measurable.

6. A simple but still standard example is as follows: Let $m \in \mathbb{N}$ and define

$$\varphi(t) \equiv \frac{t^p}{l_m(t)} \quad (t > 0),$$

where $l_m(t)$ is given inductively by:

$$l_0(t) \equiv t, \quad l_m(t) \equiv \log(3 + l_{m-1}(t)) \quad (m = 1, 2, \ldots)$$

for $t > 0$. The for all $0 < q < \infty$ and $m \in \mathbb{N}$, $\varphi \in \mathcal{G}_q$.

As the following lemma shows, we can always replace an almost increasing function with an increasing function.

**Lemma 1.22.** If a function $\varphi : (0, \infty) \to (0, \infty)$ is almost increasing with the almost increasing constant $C_0 > 0$, then there exists an increasing function $\psi : (0, \infty) \to (0, \infty)$ such that $\varphi(t) \leq \psi(t) \leq C_0 \varphi(t)$ for all $t > 0$.

**Proof.** Simply set $\psi(t) = \sup_{0 < s \leq t} \varphi(s)$. 

\[ \square \]
Having set down the condition of \( \varphi \), we move on to the norm estimate of the function. The lemma below gives an estimate for the norm of \( \chi_B(R) \) in \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

**Lemma 1.23.** Let \( 0 < q < \infty \) and \( \varphi \in \mathcal{G}_q \). There exists a constant \( C > 0 \) such that \( \| \chi_Q(x,R) \|_{\mathcal{M}_q^\varphi} = \varphi(R) \) for all \( R > 0 \).

**Proof.** It is easy to see that \( \| \chi_Q(x,R) \|_{\mathcal{M}_q^\varphi} \geq \varphi(R) \). To prove the opposite inequality we consider

\[
\varphi(r) \frac{1}{|Q(y,r)|^{1/q}} \| \chi_{Q \cap \Omega}(x,R) \|_q
\]

for any cube \( Q = Q(y,r) \). When \( R \leq r \), then

\[
\varphi(r) \frac{\| \chi_{Q \cap \Omega}(x,R) \|_q}{|Q(y,r)|^{1/q}} \leq \varphi(r) \frac{\| \chi_{Q}(x,R) \|_q}{|Q(y,r)|^{1/q}} \leq \varphi(R) \frac{\| \chi_Q(x,R) \|_q}{|Q(y,R)|^{1/q}} = \varphi(R),
\]

since \( \varphi \in \mathcal{G}_q \). When \( R > r \), then

\[
\varphi(r) \frac{\| \chi_{Q \cap \Omega}(x,R) \|_q}{|Q(y,r)|^{1/q}} \leq \varphi(r) \leq \varphi(R)
\]

again by virtue of the fact that \( \varphi \in \mathcal{G}_q \).

A direct consequence of the above quantitative information is:

**Corollary 1.24.** Let \( 0 < q < \infty \) and \( \varphi : (0, \infty) \rightarrow (0, \infty) \) be a function in the class \( \mathcal{G}_q \). If \( N_0 > n/q \), then \( (1 + |\cdot|)^{-N_0} \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

**Proof.** Since \( \varphi \in \mathcal{G}_q \), we have \( \varphi(t)t^{-n/q} \leq \varphi(1) \) for all \( t \geq 1 \) and we have

\[
\| (1 + |\cdot|)^{-N_0} \|_{\mathcal{M}_q^\varphi} \lesssim \sum_{j=1}^{\infty} \frac{\| \chi_Q(j) \|_{\mathcal{M}_q^\varphi}}{\text{max}(1,j-1)^{N_0}} \leq \sum_{j=1}^{\infty} \frac{\varphi(j)}{\text{max}(1,j-1)^{N_0}} < \infty.
\]

Here for the second inequality we invoked Lemma 1.23.

Now we consider the role of \( \varphi \). We did not tolerate the case where \( p = \infty \) when we define \( \mathcal{M}_q^p(\mathbb{R}^n) \). If we define \( \mathcal{M}_q(\mathbb{R}^n) \) similar to \( \mathcal{M}_q^p(\mathbb{R}^n) \), then we have \( \mathcal{M}_q(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \). The next theorem concerns a situation close to this.

**Theorem 1.25.** Let \( 0 < q < \infty \) and \( \varphi \in \mathcal{G}_q \). Then, the following are equivalent:

1. \( \inf \varphi > 0 \),
2. \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \).

If these conditions are satisfied, then \( \mathcal{M}_q^\varphi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \cap \mathcal{M}_q^{\varphi-\inf \varphi}(\mathbb{R}^n) \) with equivalence of norms.
Proof. If \( \inf \varphi > 0 \), then by the Lebesgue differentiation theorem we get

\[
|f(x)|^q = \lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^q \, dy
\]

\[
\leq \frac{1}{\inf \varphi} \lim_{r \downarrow 0} \varphi(r) \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy
\]

\[
\leq \frac{1}{\inf \varphi} \|f\|_M^{q^*}
\]

for all \( f \in M^{q^*}_q(\mathbb{R}^n) \) and all Lebesgue points \( x \) of \( f \). Therefore, \( f \in L^\infty \).

Assume that \( M^{q^*}_q(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \). Then by the closed graph theorem, \( \|f\|_\infty \leq \|f\|_{M^{q^*}_q} \) for all \( f \in M^{q^*}_q(\mathbb{R}^n) \). If we choose \( f = \chi_{B(a,r)} \), then we have \( 1 \leq \varphi(r) \). This shows that \( \inf \varphi > 0 \).

Finally, by taking \( \varphi_1 = \inf \varphi \) and \( \varphi_2 = \varphi - \varphi_1 \), we obtain \( M^{q^*}_q = L^\infty(\mathbb{R}^n) \cap M^{q^*}_q \) with equivalence of norms from the general formula \( M^{q^*}_q + \varphi_2(\mathbb{R}^n) = M^{q^*_1}(\mathbb{R}^n) \cap M^{q^*_2}(\mathbb{R}^n) \) with equivalence of norms.

We also investigate the reverse inclusion to Theorem 1.25.

**Theorem 1.26.** Let \( 0 < q < \infty \) and \( \varphi \in \mathcal{G}_q \). Then, the following are equivalent:

1. \( \sup \varphi < \infty \),
2. \( M^{q^*}_q(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n) \).

**Proof.** If \( \sup \varphi < \infty \), then by letting \( \psi(t) \equiv \sup \varphi \) we have \( L^\infty(\mathbb{R}^n) = M^{q^*}_q(\mathbb{R}^n) \) with equivalence of norms. Since \( \varphi \leq \psi \), we have \( M^{q^*}_q(\mathbb{R}^n) \subset M^{q^*}_q(\mathbb{R}^n) \). Thus \( L^\infty(\mathbb{R}^n) \subset M^{q^*}_q(\mathbb{R}^n) \).

Conversely, if \( L^\infty(\mathbb{R}^n) \subset M^{q^*}_q(\mathbb{R}^n) \), then by the closed graph theorem the embedding norm is finite and hence \( \varphi(r) = \|\chi_{Q(r)}\|_{M^{q^*}_q} \leq \|\chi_{Q(r)}\|_{L^\infty} = 1 \) for all \( r > 0 \). Then \( \sup \varphi < \infty \). \( \square \)

By combining these two theorems, we obtain the following result.

**Corollary 1.27.** Let \( 0 < q < \infty \) and \( \varphi \in \mathcal{G}_q \). Then, the following are equivalent:

1. \( \log \varphi \in L^\infty(0,\infty) \), i.e., \( 0 < \inf \varphi \leq \sup \varphi < \infty \).
2. \( L^\infty(\mathbb{R}^n) = M^{q^*}_q(\mathbb{R}^n) \).

So, if \( \log \varphi \) grows or decays slowly we can say that \( M^{q^*}_q(\mathbb{R}^n) \) is close to \( L^\infty(\mathbb{R}^n) \).

The next example shows that when the support of the functions are torn apart, the norm does not increase even in the case of generalized Morrey spaces.
**Example 1.28.** Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$. Suppose that we have a collection of cubes $\{Q_j\}_{j=1}^{\infty} = \{Q(a_j, r_j)\}_{j=1}^{\infty}$ such that $\{3Q_j\}_{j=1}^{\infty} = \{Q(a_j, 3r_j)\}_{j=1}^{\infty}$ is disjoint and is contained in $Q = Q(a_0, r_0)$. Let $\{f_j\}_{j=1}^{\infty}$ be a collection of functions in $\mathcal{M}_q^f(\mathbb{R}^n)$ satisfying

\[
\|f_j\|_q \leq \varphi(r_0)^{-1}r_j^{\frac{n}{q}}; \quad (1.5) \\
\|f_j\|_{\mathcal{M}_q^f} \leq 1; \quad (1.6) \\
supp(f_j) \subset Q_j
\]

for each $j \in \mathbb{N}$. Then $f \equiv \sum_{j=1}^{\infty} f_j \in \mathcal{M}_q^f(\mathbb{R}^n)$. Since $f$ is supported in $Q$, we have only to consider cubes contained in $3Q$; if a cube $Q'$ intersects both $Q$ and $\mathbb{R}^n \setminus 3Q$, then its triple $3Q$ engulfs $Q' \cap Q$ and the radius of $Q$ is smaller than that of $Q'$. Thus we have

\[
\|f\|_{\mathcal{M}_q^f} \lesssim \sup_{x \in \mathbb{R}^n, r > 0, Q(x, r) \subset 3Q} \varphi(r) \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.
\]

We fix a cube $Q(x, r)$ and estimate

\[
\varphi(r) \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.
\]

We let

\[
J_1 \equiv \{ j \in \mathbb{N} : Q(x, r) \cap Q(a_j, r_j) \neq \emptyset, r_j \leq r \} \\
J_2 \equiv \{ j \in \mathbb{N} : Q(x, r) \cap Q(a_j, r_j) \neq \emptyset, r_j > r \}.
\]

Accordingly we set

\[
I_i \equiv \varphi(r) \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} \left| \sum_{j \in J_i} f_j(y) \right|^q \, dy \right)^{\frac{1}{q}}
\]

for $i = 1, 2$.

As for $I_1$, we use (1.5) and the fact that $\{Q(a_j, r_j)\}_{j=1}^{\infty}$ is disjoint:

\[
I_1 \leq \varphi(r) \left( \frac{1}{|Q(x, r)|} \sum_{j \in J_1} \varphi(r_0)^{-q}r_j^n \right)^{\frac{1}{q}} \leq \left( \frac{1}{|Q(x, r)|} \sum_{j \in J_1} r_j^n \right)^{\frac{1}{q}} \lesssim 1.
\]

As for $I_2$, we have only to consider only one summand, since $Q(a_j, 3r_j)$ engulfs $Q(x, r)$ if $j \in H_2$: we can deduce $I_2 \lesssim 1$ using (1.6).

As an application of Example 1.28, we present another example. Denote by $[t]$ the integer part of $t \in \mathbb{R}$. For positive sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ “$A_k \sim B_k$ as $k \to \infty$” means that $\{\log(A_k/B_k)\}_{k=1}^{\infty}$ is a bounded sequence.
Example 1.29. Let \( \varphi \in \mathcal{G}_q \) with \( 0 < q < \infty \). We fix \( a \in \mathbb{R}^n \). Let \( \{s_k\}_{k=1}^{\infty} \subset (0,1] \) be a sequence which decreases to 0.

Keeping in mind
\[
\inf_{0 < t \leq 1} \varphi(t)^{q/n} t^{-1} = \varphi(1)^{q/n}, \quad \inf_{0 < t \leq 1} \varphi(t)^{-q/n} = \varphi(1)^{-q/n},
\]
we let
\[
\ell_k \equiv [1 + \varphi(s_k)^{q/n} s_k^{-1}] \sim \varphi(s_k)^{q/n} s_k^{-1}
\]
\[
m_k \equiv [1 + \varphi(s_k)^{-q/n}] \sim \varphi(s_k)^{-q/n}.
\]
Then
\[
\ell_k s_k m_k \sim 1, s_k \downarrow 0 \quad (1.7)
\]
as \( k \to \infty \). Hence
\[
\varphi((\ell_k m_k)^{-1})^{-q} m_k^{-n} \sim \varphi(s_k)^{-q} s_k^{-n} \ell_k^{-n} \sim 1 \quad (1.8)
\]
as \( k \to \infty \).

Divide equally \( a + [0,1]^n \) into \( \ell_k^n \) cubes to have
\[
\{b_{k,j} + [0, \ell_k^{-1}]^n\}_{j=1,2,...,\ell_k^n}.
\]
Furthermore, we divide \( (0, \ell_k^{-1}]^n \) equally into \( m_k^n \) cubes to obtain a collection
\[
\{e_{k,j} + [0, m_k^{-1}]^n\}_{j=1,2,...,m_k^n}
\]
of cubes. Then
\[
b_{k,j} + (0, \ell_k^{-1}]^n = \bigcup_{i=1}^{m_k^n} (b_{k,j} + e_{k,i} + [0,(\ell_k m_k)^{-1}]^n).
\]
If we set
\[
g_{k,i,j,a} = \frac{1}{\varphi((\ell_k m_k)^{-1})} \chi_{b_{k,j} + e_{k,i} + [0,(\ell_k m_k)^{-1}]^n},
\]
and
\[
f_{a,i,k} = f_{k,i} = \sum_{j=1}^{\ell_k^n} \frac{\chi_{b_{k,j} + e_{k,i} + [0,(\ell_k m_k)^{-1}]^n}}{\varphi((\ell_k m_k)^{-1})}.
\]
Then, each \( f_{a,i,k} \) is supported in \( a + [0,1]^n \) and each \( g_{k,i,j,a} \) is supported in \( b_{k,j} + [0,(\ell_k)^{-1}]^n \). We can show by Example 1.28 that \( \{f_{a,i,k}\}_{a \in \mathbb{R}^n, k \in \mathbb{N}, i=1,2,...,\ell_k^n} \) forms a bounded set in \( M_q^\infty(\mathbb{R}^n) \). Let \( F_1, F_2, \ldots, F_{3^n} \) be a partition of \( \{1,2,\ldots,\ell_k^n\} \) such that \( \{b_{k,j} + [-((\ell_k)^{-1}), 2((\ell_k)^{-1})]^n\}_{j \in F_l} \) is not overlapping for each \( l' = 1,2,\ldots,3^n \). To check this we need to verify (1.5) and (1.6).

From Lemma 1.23, we have (1.5). Meanwhile,
\[
\|g_{k,i,j,a}\|_q = \varphi((\ell_k m_k)^{-1}) (\ell_k m_k)^{-n/q} \sim \ell_k^{-n/q}.
\]
Thus for each \( l' = 1, 2 \ldots, 3^n \)

\[
\left\{ \frac{1}{\varphi((\ell_km_k)^{-1})} \sum_{j \in F_{l'}} \chi_{b_{k,j} + e_{k,i} + I[0,(\ell_km_k)^{-1}]^n} \right\}_{a \in \mathbb{R}^n, k \in \mathbb{N}, i = 1, 2, \ldots, m^n}
\]

is a bounded set. So \( \{f_{a,i,k}\}_{a \in \mathbb{R}^n, k \in \mathbb{N}, i = 1, 2, \ldots, m^n} \) forms a bounded set in \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

From these examples, we obtain the following conclusion:

**Corollary 1.30.** Let \( 0 < q_1, q_2 < \infty \), and let \( \varphi_1 \in \mathcal{G}_{q_1} \) and \( \varphi_2 \in \mathcal{G}_{q_2} \). Assume in addition that \( \varphi_1 \) is unbounded. Then \( \mathcal{M}_q^{\varphi_1}(\mathbb{R}^n) \subset \mathcal{M}_q^{\varphi_2}(\mathbb{R}^n) \) if and only if \( q_1 \geq q_2 \) and \( \varphi_1 \gtrsim \varphi_2 \). In particular for \( 0 < q_1 \leq p_1 < \infty \) and \( 0 < q_2 \leq p_2 < \infty \), \( \mathcal{M}_q^{\varphi_1}(\mathbb{R}^n) \subset \mathcal{M}_q^{\varphi_2}(\mathbb{R}^n) \) if and only if \( q_1 \geq q_2 \) and \( p_1 = p_2 \).

**Proof.** The “if” part is trivial. Let us concentrate on the “only if” part.

Assume \( q_1 < q_2 \). We employ Example 1.29 with \( q = q_1 \) and \( \varphi = \varphi_1 \). Then

\[
\left\| \sum_{j \in F_{l'}} \chi_{b_{k,j} + e_{k,i} + I[0,(\ell_km_k)^{-1}]^n} \right\|_{L^{q_2}} \sim (\ell_km_k^{-n})^{\frac{1}{q_2}} = m_k^{-\frac{n}{q_2}}.
\]

As a result,

\[
\frac{1}{\varphi((\ell_km_k)^{-1})} \left\| \sum_{j \in F_{l'}} \chi_{b_{k,j} + e_{k,i} + I[0,(\ell_km_k)^{-1}]^n} \right\|_{L^{q_2}} \sim \frac{1}{\varphi((\ell_km_k)^{-1})m_k^{\frac{n}{q_2}}}
\]

\[
= m_k^{\frac{n}{q_1} - \frac{n}{q_2}} \to \infty,
\]

since \( \varphi_1 \) is unbounded. Thus, \( \mathcal{M}_q^{\varphi_1}(\mathbb{R}^n) \) contains a function which does not belong to \( L_{qc}^{q_2}(\mathbb{R}^n) \). Consequently, if \( \mathcal{M}_q^{\varphi_1}(\mathbb{R}^n) \subset \mathcal{M}_q^{\varphi_2}(\mathbb{R}^n) \), then \( q_1 \geq q_2 \). Finally, we must have \( \varphi_1 \gtrsim \varphi_2 \) from Lemma 1.23 if \( \mathcal{M}_q^{\varphi_1}(\mathbb{R}^n) \subset \mathcal{M}_q^{\varphi_2}(\mathbb{R}^n) \).

When \( \varphi_1 \) is bounded, the situation is different.

**Example 1.31.** When \( \varphi_1 \) is constant, we use Theorem 1.26 to have the following characterization. Let \( 0 < q_1, q_2 < \infty \), and let \( \varphi_1 = 1 \in \mathcal{G}_{q_1} \) and \( \varphi_2 \in \mathcal{G}_{q_2} \). Then \( L^{\infty}(\mathbb{R}^n) = \mathcal{M}_q^{\varphi_1}(\mathbb{R}^n) \subset \mathcal{M}_q^{\varphi_2}(\mathbb{R}^n) \) if and only if \( \sup \varphi_2 < \infty \).

For the later purpose we use the following characterization of \( \mathcal{M}_q^*(\mathbb{R}^n) \), which is defined to be the closure of \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \cap L^0_{qc}(\mathbb{R}^n) \) in \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

**Lemma 1.32.** For \( 0 < q \leq p < \infty \), we have

\[
\mathcal{M}_q^*(\mathbb{R}^n) = \left\{ f \in \mathcal{M}_q^\varphi(\mathbb{R}^n) : \lim_{R \to \infty} \| \chi_{\mathbb{R}^n \setminus B(R)} f \|_{\mathcal{M}_q^\varphi} = 0 \right\}.
\]

(1.10)
Proof. Similar to the case of $\mathcal{M}_q^\varphi(\mathbb{R}^n)$. \hfill $\Box$

We also have the following characterization of the space $\overline{\mathcal{M}_q^\varphi(\mathbb{R}^n)}$, the closure of $L^\infty(\mathbb{R}^n) \cap \mathcal{M}_q^\varphi(\mathbb{R}^n)$ in $\mathcal{M}_q^\varphi(\mathbb{R}^n)$:

**Lemma 1.33.** Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$. If $f \in \overline{\mathcal{M}_q^\varphi(\mathbb{R}^n)}$, then

$$
\lim_{R \to \infty} \| \chi_{\{|f| > R\}} f \|_{\mathcal{M}_q^\varphi} = 0.
$$

(1.11)

Proof. Similar to the case of $\overline{\mathcal{M}_q^\varphi(\mathbb{R}^n)}$. \hfill $\Box$

The generalized tilde subspace $\widetilde{\mathcal{M}_q^\varphi(\mathbb{R}^n)}$ denotes the completion of $L^\infty(\mathbb{R}^n) \cap \mathcal{M}_q^\varphi(\mathbb{R}^n)$ in $\mathcal{M}_q^\varphi(\mathbb{R}^n)$.

**Proposition 1.34.** Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$. Then $\widetilde{\mathcal{M}_q^\varphi(\mathbb{R}^n)} = \overline{\mathcal{M}_q^\varphi(\mathbb{R}^n)} \cap \overset{*}{\mathcal{M}_q^\varphi(\mathbb{R}^n)}$.

Proof. Similar to the case of classical Morrey spaces. \hfill $\Box$

Although we can define many other closed subspaces as we did for Morrey spaces, we content ourselves with these three definitions, which we actually use.

## 2 Boundedness properties of the operators in generalized Morrey spaces

Having set down the boundedness properties of generalized Morrey spaces, we are now interested in the boundedness properties of the operators.

### 2.1 Hardy–Littlewood maximal operator in generalized Morrey spaces and the class $\mathcal{Z}_1$

After we generalize the parameter $p$ in the space $\mathcal{M}_p^q(\mathbb{R}^n)$, we realize that the boundedness of the maximal operator is obtained due to the condition on $q \in (1, \infty)$.

**Theorem 2.1.** Let $1 < q < \infty$ and $\varphi \in \mathcal{G}_q$. Then

$$
\| Mf \|_{\mathcal{M}_q^\varphi} \lesssim \| f \|_{\mathcal{M}_q^\varphi}
$$

for all measurable functions $f$.

We observe that we did not require anything other than $\varphi \in \mathcal{G}_q$ as an evidence of the fact that the parameter $q$ play a central role for the boundedness of the Hardy-Littlewood maximal operator.
Proof. Once we assume \( \varphi \in G_q \), the proof of this theorem will be an adaptation of the classical case. Fix a cube \( Q(x,r) \). We need to prove
\[
\varphi(r) \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} Mf(y)^q \, dy \right)^\frac{1}{p} \lesssim \|f\|_{\mathcal{M}^p_q}.
\]
We let \( f_1 \equiv \chi_{Q(x,5r)}f \) and \( f_2 \equiv f - f_1 \). We need to prove
\[
\varphi(r) \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} Mf_1(y)^q \, dy \right)^\frac{1}{p} \lesssim \|f_1\|_{\mathcal{M}^p_q} \quad (2.1)
\]
and
\[
\varphi(r) \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} Mf_2(y)^q \, dy \right)^\frac{1}{p} \lesssim \|f_2\|_{\mathcal{M}^p_q}. \quad (2.2)
\]
The proof of (2.1) follows from the boundedness of the Hardy-Littlewood maximal operator and the fact that \( \varphi(3r) \simeq \varphi(r) \) for any \( r > 0 \). As for (2.2), we use
\[
M[\chi_{\mathbb{R}^n \setminus Q}f](y) \lesssim \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(z)| \, dz \quad (y \in Q)
\]
with \( Q = Q(x,r) \). Then by virtue of the fact that \( \varphi \) is increasing, we obtain
\[
\varphi(r) \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} Mf_2(y)^q \, dy \right)^\frac{1}{p} \lesssim \varphi(r) \times \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(z)| \, dz \\
\lesssim \sup_{Q \subset R \in \mathcal{Q}} \varphi(r) \times \frac{1}{|R|} \int_R |f(z)| \, dz \\
\lesssim \|f\|_{\mathcal{M}^p_q} \\
\lesssim \|f_2\|_{\mathcal{M}^p_q},
\]
where for the last inequality, we used the nesting property of \( \mathcal{M}^p_q(\mathbb{R}^n), 1 \leq q < \infty \). \( \square \)

As we have seen, any function \( \varphi : (0, \infty) \rightarrow (0, \infty) \) will do. So we have the following boundedness for small Morrey spaces.

**Example 2.2.** Let \( 1 < q \leq p < \infty \). Then \( M \) is bounded on \( m^p_q(\mathbb{R}^n) \) and hence \( L^q_{\text{uloc}}(\mathbb{R}^n) \). In fact, \( M \) is bounded on \( \mathcal{M}^p_q(\mathbb{R}^n) \sim m^p_q(\mathbb{R}^n) \) thanks to Theorem 2.1, where \( \varphi(t) = \max(t^{\frac{n}{q}}, 1), t > 0 \).

Similar to Theorem 2.1, we can prove the following theorem:

**Theorem 2.3.** Let \( 1 \leq q < \infty \) and \( \varphi \in \mathcal{G}_q \). Then
\[
\|Mf\|_{w\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}
\]
for all measurable functions \( f \).
Proof. Simply resort to the weak-(1, 1) boundedness of $M$ and modify the proof of Theorem 2.1. \hfill \square

We move on to the vector-valued inequality. Unlike the usual maximal inequality, we need the integral condition (2.3).

Theorem 2.4. Let $1 < q < \infty$, $1 < u \leq \infty$ and $\varphi \in \mathcal{G}_q$. Assume in addition that

$$
\int_r^\infty \frac{ds}{\varphi(s)s} \lesssim \frac{1}{\varphi(r)} \quad (r > 0).
$$

(2.3)

Then for all $\{f_j\}_{j=1}^\infty \subset \mathcal{M}_q^u(\mathbb{R}^n)$,

$$
\left\| \left( \sum_{j=1}^\infty Mf_j \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^u} \lesssim \left\| \left( \sum_{j=1}^\infty |f_j|^{iu} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^u}. \quad (2.4)
$$

Proof. When $u = \infty$, the result is clear from (2.1). The proof is essentially the same as the classical case except that we truly use (2.3). The proof of the estimate of the inner term remains unchanged except in that we need to generalize the parameter $p$ to the function $\varphi$. Going through a similar argument to the classical case, we will have

$$
\frac{\varphi(\ell(Q))}{|Q|} \int_Q \left( \sum_{j=1}^\infty Mf_{j,2}(y)^u \right)^{\frac{1}{u}} dy \lesssim \sum_{k=1}^\infty \frac{\varphi(\ell(Q))}{2^k|Q|} \int_{2^kQ} \left( \sum_{j=1}^\infty |f_j(z)|^{u} \right)^{\frac{1}{u}} dz.
$$

If we use the definition of the Morrey norm, we obtain

$$
\frac{\varphi(\ell(Q))}{|Q|} \int_Q \left( \sum_{j=1}^\infty Mf_{j,2}(y)^u \right)^{\frac{1}{u}} dy \lesssim \sum_{k=1}^\infty \frac{\varphi(\ell(Q))}{\varphi(2^k\ell(Q))} \left\| \left( \sum_{j=1}^\infty |f_j|^{iu} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^u}. \quad (2.5)
$$

Since $\varphi \in \mathcal{G}_q$, we obtain

$$
\sum_{k=1}^\infty \frac{\varphi(\ell(Q))}{\varphi(2^k\ell(Q))} \lesssim \int_{\ell(Q)}^\infty \frac{\varphi(\ell(t))}{\varphi(t)} dt.
$$

If we use (2.3) and $\varphi \in \mathcal{G}_q$, then we have

$$
\sum_{k=1}^\infty \frac{\varphi(\ell(Q))}{\varphi(2^k\ell(Q))} \lesssim 1.
$$

Inserting this estimate into (2.5), we obtain the counterpart to the classical case. \hfill \square

Since (2.3) is an important condition, we are interested in its characterization. In fact, we have the following useful one.
Theorem 2.5. Assume that $\varphi : (0, \infty) \to (0, \infty)$ is an almost increasing function. Then the following are equivalent:

1. $\varphi$ satisfies (2.3).

2. There exists $m_0 \in \mathbb{N}$ such that
   \[
   \varphi(2^{m_0}r) > 2\varphi(r) \tag{2.6}
   \]
   for all $r > 0$.

Proof. Assume that (2.3) holds. Let $m'_0$ be a positive integer such that $\varphi(2^{m'_0}r) \leq 2\varphi(r)$ for all $r > 0$. Thus since $\varphi$ is almost increasing,

\[
\frac{\log m'_0}{2\varphi(r)} \leq \int_r^{2m'_0-1} \frac{ds}{\varphi(s)s} \lesssim \varphi(r)
\]

Since $\varphi(r) > 0$, we have an upper bound for $m'_0$. Thus if we set $m_0 = m'_0 + 1$, we obtain the desired number $m_0$.

If (2.3) holds, then

\[
\int_0^\infty \frac{ds}{\varphi(s)s} = \sum_{j=1}^\infty \int_{2^{m_0(j-1)}r}^{2^{m_0}r} \frac{ds}{\varphi(s)s} \leq \sum_{j=1}^\infty \int_{2^{m_0(j-1)}r}^{2^{m_0}r} \frac{ds}{2^j\varphi(r)s} \lesssim \frac{1}{\varphi(r)}
\]

as required. \(\square\)

As we have mentioned, we need (2.3) for the vector-valued maximal inequality. We give an example showing that (2.3) is absolutely necessary: By no means (2.3) is artificial as the following proposition shows:

Proposition 2.6. Let $1 < q < \infty$, $1 < u < \infty$ and $\varphi \in \mathcal{G}_q$. Assume in addition that

\[
\left\| \left( \sum_{j=1}^\infty Mf_j \right)^u \right\|_{w\mathcal{M}_q^u} \lesssim \left\| \left( \sum_{j=1}^\infty |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^u}
\]

holds for all sequences of measurable functions. Then (2.3) holds.

Proof. Assume to the contrary; for all $m \in \mathbb{N} \cap [2, \infty)$, there would exist $r_m > 0$ such that $\varphi(2^mr_m) \leq 2\varphi(r_m)$ for $r = r_m$. Let us consider $f_j = \chi_{[1,j]}(m)\chi_{B(2^j r_m)\setminus B(2^{j-1} r_m)}$ for $j \in \mathbb{N}$. Observe

\[
\left\| \left( \sum_{j=1}^\infty |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^u} = \left\| \left( \sum_{j=1}^m \chi_{B(2^j r_m)\setminus B(2^{j-1} r_m)} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^u}
\]
As a result,
\[
\left\| \left( \sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^q_u} \leq \|\chi_{B(2^m r_m)}\|_{\mathcal{M}^q_u} \leq \varphi(2^m r_m) \lesssim \varphi(r_m).
\]

Let \( x \in B(r_m) \). For \( j > m \), we have \( Mf_j(x) = 0 \). Meanwhile for \( 1 \leq j \leq m \), we have
\[
Mf_j(x) \geq \frac{1}{|B(x, 2^{j+1} r_m)|} \int_{B(x, 2^{j+1} r_m)} f_j(y) \, dy \geq \frac{1}{|B(x, 2^{j+1} r_m)|} \int_{B(2^j r_m) \setminus B(2^{j-1} r_m)} \chi_{B(2^j r_m)}(y) \, dy = \frac{2^n - 1}{4^n} \geq \frac{1}{4^n}.
\]
Consequently,
\[
\left\| \left( \sum_{j=1}^{\infty} Mf_j u \right)^{\frac{1}{u}} \right\|_{w, \mathcal{M}^q_u} \geq \varphi(r_m) \sum_{j=1}^{m} \left( \frac{1}{4^n} \right)^u \sim \varphi(r_m) \left( \frac{m}{4^n} \right)^{\frac{1}{u}}.
\]
By our assumption, we have
\[
\varphi(r_m) m^{\frac{1}{u}} \lesssim \left\| \left( \sum_{j=1}^{\infty} Mf_j u \right)^{\frac{1}{u}} \right\|_{w, \mathcal{M}^q_u} \lesssim \left\| \left( \sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^q_u} \lesssim \varphi(r_m),
\]
or equivalently
\[
m \leq D
\]
where \( D \) does not depend on \( m \). This contradicts to the fact that \( m \in \mathbb{N} \cap [2, \infty) \) is arbitrary.

**Example 2.7.** Let \( 1 \leq q < p < \infty \). Then
\[
\left\| \left( \sum_{j=1}^{\infty} Mf_j u \right)^{\frac{1}{u}} \right\|_{w, m^q_u} \left( \{f_j\}_{j=1}^{\infty} \subset m^p_q(\mathbb{R}^n) \right)
\]
fails. In particular,
\[
\left\| \left( \sum_{j=1}^{\infty} Mf_j u \right)^{\frac{1}{u}} \right\|_{w, L^q_{uloc}} \left( \{f_j\}_{j=1}^{\infty} \subset L^q_{uloc}(\mathbb{R}^n) \right)
\]
fails. In fact, let \( \varphi(t) = \max(t^{\frac{n}{r}}, 1) \) as before. Then \( \varphi \) fails (2.3) because \( \int_{1}^{\infty} \frac{dr}{\varphi(r)} = \infty. \)
Proposition 2.6 led us to the conclusion that (2.3) is fundamental. The following proposition will be fundamental in the study of the boundedness of the operators in generalized Morrey spaces.

**Theorem 2.8.** If a nonnegative locally integrable function $\psi$ and a positive constant $D > 0$ satisfy (2.3), then

$$
\int_r^\infty \psi(t)t^\varepsilon \frac{dt}{t} \leq \frac{r^\varepsilon}{1 - D\varepsilon} \cdot \int_r^\infty \psi(t)t^{-1} dt \leq \frac{D}{1 - D\varepsilon} \cdot \psi(r)r^\varepsilon \quad (r > 0) \tag{2.7}
$$

for all $0 < \varepsilon < D^{-1}$.

**Proof.** Let

$$
\Psi(r) = \int_r^\infty \psi(t)t^{-1} dt \quad (r > 0).
$$

For $0 < r < R$

$$
\int_r^R \psi(t)t^\varepsilon \frac{dt}{t} = [-\psi(t)t^\varepsilon]_r^R + \int_r^R \psi(t)t^\varepsilon \frac{dt}{t} \leq \psi(r)r^\varepsilon + \varepsilon D \int_r^R \psi(t)t^\varepsilon \frac{dt}{t}.
$$

Therefore

$$
\int_r^R \psi(t)t^\varepsilon \frac{dt}{t} \leq \frac{1}{1 - \varepsilon D} \Psi(r)r^\varepsilon \leq \frac{D}{1 - \varepsilon D} \psi(r)r^\varepsilon.
$$

It remains to let $R \to \infty$. \qed

We change variables to have the following variant:

**Theorem 2.9.** Let $\psi : (0, \infty) \to (0, \infty)$ be a measurable function satisfying

$$
\int_0^r \psi(t) \frac{dt}{t} \leq D\psi(r) \quad (r > 0)
$$

for some $D > 0$ independent of $r > 0$. If $0 < \varepsilon < D^{-1}$, then

$$
\int_0^r \psi(t) \frac{dt}{t^{1+\varepsilon}} \leq \frac{1}{1 - D\varepsilon} r^{-\varepsilon} \int_r^\infty \eta(t) \frac{dt}{t} \leq \frac{D}{1 - D\varepsilon} r^{-\varepsilon} \psi(r).
$$

**Proof.** Set

$$
\eta(t) = \psi \left( \frac{1}{t} \right) \quad (t > 0).
$$

Then our assumption reads as:

$$
\int_r^\infty \eta(t) \frac{dt}{t} \leq D\eta(r) \quad (r > 0).
$$

Thus,

$$
\int_r^\infty \eta(t) \frac{dt}{t^{1-\varepsilon}} \leq \frac{1}{1 - D\varepsilon} r^{-\varepsilon} \int_r^\infty \eta(t) \frac{dt}{t} \leq \frac{D}{1 - D\varepsilon} r^\varepsilon \eta(r) \quad (r > 0)
$$

according to Theorem 2.8. If we express this inequality in terms of $\psi$, we obtain the desired result. \qed
In the next proposition, we further characterize and apply our key assumption (2.3).

**Proposition 2.10.** Let $\varphi$ be a nonnegative locally integrable function such that $\varphi(s) \leq \varphi(r)$ for all $r, s > 0$ with $\frac{1}{2} \leq \frac{r}{s} \leq 2$. There exists a constant $\varepsilon > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \lesssim \frac{r^\varepsilon}{\varphi(r)} \quad (t \geq r)$$

(2.8)

if and only if holds (2.3) or $\varphi$ satisfies (2.7) for some $\varepsilon > 0$. If one of these conditions is satisfied, then

$$\int_{r}^{\infty} \frac{ds}{\varphi(s)s^{1+\varepsilon}} \lesssim \frac{1}{\varphi(r)^u} \quad (r > 0)$$

(2.9)

for all $0 < u < \infty$, where the implicit constant depends only on $u$.

**Proof.** The implication $(2.3) \Rightarrow (2.7)$ follows from Proposition 2.8.

Assume (2.7). Then we have

$$\frac{t^\varepsilon}{\varphi(t)} \lesssim \int_{t}^{2t} \frac{dv}{v^{1-\varepsilon}} \lesssim \frac{r^\varepsilon}{\varphi(r)}$$

thanks to the doubling property of $\varphi$, proving (2.8).

If we assume (2.8), then we have

$$\int_{r}^{\infty} \frac{ds}{\varphi(s)s} = \int_{r}^{\infty} s^{\varepsilon} \frac{ds}{\varphi(s)s^{1-\varepsilon}} \lesssim \int_{r}^{\infty} \frac{r^{\varepsilon} ds}{\varphi(r)s^{1+\varepsilon}} = \frac{1}{\varepsilon \varphi(r)},$$

which implies (2.3). Note that (2.8) also implies (2.9) because $\varphi^u$ satisfies (2.8) as well. \hfill \Box

Let $0 < u < \infty$ be fixed. Inequality (2.9) is necessary for (2.3); simply apply Proposition 2.10 to $\varphi^u$.

**Example 2.11.** Let $\frac{1}{\varphi} \in \mathbb{Z}$. Then we have

$$\int_{r}^{\infty} \frac{1}{\varphi(s)s^{1-\varepsilon}} ds \lesssim \frac{r^\varepsilon}{\varphi(r)} \quad (r > 0).$$

Hence

$$\frac{s^\varepsilon}{\varphi(s)} \lesssim \frac{r^\varepsilon}{\varphi(r)}$$

whenever $0 < r \leq s < \infty$. As a result,

$$\int_{0}^{1} \varphi(r) \frac{dr}{r} < \infty.$$

We generalize condition (2.3) as follows:
Definition 2.12. Let $\gamma \in \mathbb{R}$.

1. The (upper) Zygmund class $\mathcal{Z}_\gamma$ is defined to be the set of all measurable functions $\varphi : (0, \infty) \to (0, \infty)$ for which $\lim_{r \downarrow 0} \varphi(r) = 0$ and
   \[ \int_0^r \varphi(t)t^{-\gamma-1} \, dt \lesssim \varphi(r)r^{-\gamma} \quad (r > 0), \]

2. The (lower) Zygmund class $\mathcal{Z}_\gamma$ is defined to be the set of all measurable functions $\varphi : (0, \infty) \to (0, \infty)$ for which $\lim_{r \downarrow 0} \varphi(r) = 0$ and
   \[ \int_r^\infty \varphi(t)t^{-\gamma-1} \, dt \lesssim \varphi(r)r^{-\gamma} \quad (r > 0). \]

Note that (2.3) reads as $\frac{1}{\varphi} \in \mathcal{Z}_0$.

Example 2.13. Let $\varphi(t) = t^p$ with $1 < p < \infty$.

1. $\varphi \in \mathcal{Z}_\gamma$ if and only if $p > \gamma$.
2. $\varphi \in \mathcal{Z}_\gamma$ if and only if $p < \gamma$.
3. $1 \notin \mathcal{Z}_\gamma \cup \mathcal{Z}_\gamma$.

We present an example of the functions in $\mathcal{M}_\varphi^q(\mathbb{R}^n)$.

Example 2.14. Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$. Define

\[ f \equiv \sum_{j=-\infty}^{\infty} \frac{\chi_{[2^{-j-1}, 2^{-j}]^n}}{\varphi(2^{-j})}, \quad g \equiv \sup_{j \in \mathbb{Z}} \frac{\chi_{[0, 2^{-j}]^n}}{\varphi(2^{-j})}. \]

We claim that the following are equivalent;

(a) $f \in \mathcal{M}_\varphi^q(\mathbb{R}^n)$,
(b) $g \in \mathcal{M}_\varphi^q(\mathbb{R}^n)$,
(c) $\frac{1}{\varphi} \in \mathcal{Z}_q$.

Let $0 < u < q$. Note that $f \leq g \leq 2^n M^{(u)} f$, where $M^{(u)}$ denotes the powered Hardy-Littlewood maximal operator. Observe that $M^{(u)}$ is bounded on $\mathcal{M}_\varphi^q(\mathbb{R}^n)$. Thus (a) and (b) are equivalent. Since $f$ is expressed as $f = f_0(\| \cdot \|_\infty)$, that is, there exists a function $f_0 : [0, \infty) \to \mathbb{R}$ such that $f(x) = f_0(\|x\|_\infty)$ for all $x \in \mathbb{R}^n$, where $\| \cdot \|_\infty$ denotes the $\ell^\infty$-norm, it follows that (a) and (c) are equivalent.
Example 2.15. Let \( 0 < q < \infty \) and \( \varphi \in \mathcal{G}_q \). Define a decreasing function \( \varphi^\dagger \) by:
\[
\varphi^\dagger(t) = \varphi(t) t^{-n/q}
\]
for \( t > 0 \). Define \( h = \sum_{j=-\infty}^{\infty} X_{[0,2^{-j} \mathbb{Z}^n]} \). Then \( \sum_{j=-\infty}^{\infty} \frac{1}{\varphi(2^{-j})} \lesssim \frac{1}{\varphi(2^{-l})} \) and that \( \sum_{j=-\infty}^{l} \frac{1}{\varphi(2^{-j})} \lesssim \frac{1}{\varphi^\dagger(2^{-l})} \) for all \( l \in \mathbb{Z} \) if and only if \( h \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

To verify this, we let \( f, g \) be as in Example 2.14. Suppose first \( h \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \). Then
\[
\varphi(2^{-l}) \left( \frac{1}{|0, 2^{-l} \mathbb{Z}^n|} \int_{|0, 2^{-l} \mathbb{Z}^n|} \left( \sum_{j=-\infty}^{\infty} \frac{1}{\varphi(2^{-j})} \right)^q \ dx \right)^\frac{1}{q} \leq \|h\|_{\mathcal{M}_q^\varphi}.
\]
Thus \( \sum_{j=-\infty}^{\infty} \frac{1}{\varphi(2^{-j})} \leq \frac{\|h\|_{\mathcal{M}_q^\varphi}}{\varphi(2^{-l})} \) for all \( l \in \mathbb{Z} \). This implies that \( f \leq g \leq h \lesssim f \), where \( f \) and \( g \) are defined in Example 2.14. Thus from Example 2.14, \( \sum_{j=-\infty}^{l} \frac{1}{\varphi(2^{-j})} \lesssim \frac{1}{\varphi(2^{-l})} \) holds as well.

Conversely, assume that
\[
\sum_{j=-\infty}^{l} \frac{1}{\varphi(2^{-j})} \lesssim \frac{1}{\varphi(2^{-l})} \tag{2.11}
\]
and
\[
\sum_{j=-\infty}^{\infty} \frac{1}{\varphi(2^{-j})} \lesssim \frac{1}{\varphi(2^{-l})} \tag{2.12}
\]
hold for all \( l \in \mathbb{Z} \). Then we have \( h \sim f \) from (2.12). Thus \( f \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \) by (2.11), from which it follows that \( h \in \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

We further present some examples of the functions in \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

Lemma 2.16. Let \( 0 < q < \infty \) and \( \varphi \in \mathcal{G}_q \cap \mathbb{Z}^{-\frac{n}{q}} \). Then the function \( \psi(x) = \varphi(|x|) \) belongs to \( \mathcal{M}_q^\varphi(\mathbb{R}^n) \).

Proof. First note that \( \varphi \in \mathbb{Z}^{-\frac{n}{q}} \) is equivalent to
\[
\frac{1}{r^n} \int_0^r \varphi(t) t^{q-1} \ dt \lesssim \varphi(r)^q \quad (r > 0).
\]
Note that \( \varphi(|\cdot|) \) is radial decreasing, so that for all \( a \in \mathbb{R}^n \) and \( r > 0 \),
\[
\left[ \frac{1}{|B(a, r)|} \int_{B(a, r)} \varphi(|x|)^q \ dx \right]^{\frac{1}{q}} \lesssim \left[ \frac{1}{|B(r)|} \int_{B(r)} \varphi(|x|)^q \ dx \right]^{\frac{1}{q}}. \tag{2.14}
\]
Combining (2.13) and (2.14) and using the spherical coordinate, we obtain the desired result. \( \square \)
2.2 Singular integral operators on generalized Morrey spaces

Let $T$ be a singular integral operator. To define the function $Tf$ for $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ we follow the same strategy as the one for $f \in \mathcal{M}_p^q(\mathbb{R}^n)$. To this end, we need to establish the following estimate:

**Lemma 2.17.** Let $1 < q < \infty$ and $\varphi \in \mathcal{G}_q$ satisfy $\frac{1}{p} \in \mathbb{Z}_0$. Then

$$
\|Tf\|_{\mathcal{M}_q^\varphi} \lesssim \|f\|_{\mathcal{M}_q^\varphi}
$$
for all $f \in L_c^\infty(\mathbb{R}^n)$.

We note that $\frac{1}{p} \in \mathbb{Z}_0$ appeared once again.

*Proof.* Let $Q$ be a fixed cube. Then we need to prove

$$
\varphi(\ell(Q)) \left( \frac{1}{|Q|} \int_Q |Tf(y)|^q \, dy \right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_q^\varphi}.
$$

To this end, we decompose $f$ according to $2Q$: $f_1 = \chi_{2Q}f$, $f_2 = f - f_1$. As for $f_1$, we have

$$
\varphi(\ell(Q)) \left( \frac{1}{|Q|} \int_Q |Tf_1(y)|^q \, dy \right)^{\frac{1}{q}} \leq \varphi(\ell(Q)) \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |Tf_1(y)|^q \, dy \right)^{\frac{1}{q}}
$$

$$
\lesssim \varphi(\ell(Q)) \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |f_1(y)|^q \, dy \right)^{\frac{1}{q}}
$$

$$
\lesssim \varphi(\ell(Q)) \left( \frac{1}{|Q|} \int_{2Q} |f(y)|^q \, dy \right)^{\frac{1}{q}}
$$

$$
\lesssim \|f\|_{\mathcal{M}_q^\varphi}.
$$

As for $f_2$, we use the size condition of $K$, the integral kernel of $T$, to have the local estimate:

$$
|Tf_2(y)| \lesssim \int_{\mathbb{R}^n \setminus 2Q} \frac{|f(y)| \, dy}{|y - c(Q)|^n} \lesssim \int_{\ell(Q)} \left( \frac{1}{\ell^{n+1}} \int_{B(c(Q),\ell)} |f(y)| \, dy \right) \, d\ell.
$$

By the definition of the norm and (2.3), we obtain

$$
|Tf_2(y)| \lesssim \int_{\ell(Q)} \frac{1}{r\varphi(r)} \, dr \cdot \|f\|_{\mathcal{M}_q^\varphi} \lesssim \frac{1}{\varphi(\ell(Q))} \|f\|_{\mathcal{M}_q^\varphi}.
$$

It remains to integrate this pointwise estimate. \hfill $\Box$

To carry over our program of proving the boundedness of the singular integral operators, we need to investigate the predual and its predual.

**Definition 2.18.** Let $1 < q < \infty$ and $\varphi \in \mathcal{G}_q$.

1. A $(\varphi,q)$-block is a measurable function $A$ supported on a cube $Q$ satisfying

$$
\|A\|_{L_{q'}^r} \lesssim \frac{\varphi(\ell(Q))}{|Q|^\frac{1}{q'}}.
$$
2. The block space $\mathcal{H}^\varphi_q(\mathbb{R}^n)$ is the set of all measurable functions $f$ for which it can be written

$$f = \sum_{j=1}^{\infty} \lambda_j A_j,$$

for some sequence $\{A_j\}_{j=1}^{\infty}$ of $(\varphi,q)$-blocks and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{R}^n)$. The norm $\|f\|_{\mathcal{H}^\varphi_q}$ is the infimum of $\|\{\lambda_j\}_{j=1}^{\infty}\|_{\ell^1}$ where $\{A_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ run over all expressions above.

**Example 2.19.** Let $A$ be a non-zero $L^q(\mathbb{R}^n)$-function supported on a cube $Q$. Then

$$B = \frac{\varphi(\ell(Q))}{|Q|^{\frac{1}{n}}} A$$

is a $(\varphi,q)$-atom.

**Proposition 2.20.** Let $1 < q < \infty$ and $\varphi \in \mathcal{G}_q$ be such that $\varphi(t) \geq t^\frac{n}{q}$ for all $t > 0$. Then a measurable function $f$ belongs to $\mathcal{H}^\varphi_q(\mathbb{R}^n)$ if and only if $f$ admits a decomposition:

$$f = \lambda_0 B + \sum_{j=1}^{\infty} \lambda_j A_j,$$

for some sequence $\{A_j\}_{j=1}^{\infty}$ of $(\varphi,q)$-blocks supported on cubes of volume less than or equal to 1 and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{R}^n)$ and $B \in L^q(\mathbb{R}^n)$ with unit norm. Furthermore the norm $\|f\|_{\mathcal{H}^\varphi_q}$ is the infimum of $\|\{\lambda_j\}_{j=0}^{\infty}\|_{\ell^1}$ where $\{A_j\}_{j=1}^{\infty}$, $B$ and $\{\lambda_j\}_{j=1}^{\infty}$ run over all expressions above.

**Proof.** This is because if $\ell(Q)$ is large, we can say that $A$ is a $(\varphi,q)$-block supported on $Q$ is and only if $2A$ has the $L^q(\mathbb{R}^n)$-norm 1 and $A$ is supported on $Q$.

The following lemma justifies the definition above.

**Lemma 2.21.** Let $1 < q < \infty$ and $\varphi \in \mathcal{G}_q$. If $A$ is a $(\varphi,q)$-block and $f \in \mathcal{M}^\varphi_q(\mathbb{R}^n)$, then $\|A \cdot f\|_{L^1} \leq \|f\|_{\mathcal{M}^\varphi_q}.$

It is easy to see that $\mathcal{H}^\varphi_q(\mathbb{R}^n)$ is a normed space. Similar to the classical case, we can prove the following theorem:

**Theorem 2.22.** Let $1 < q < \infty$, $\varphi \in \mathcal{G}_q$, and let $f \in \mathcal{H}^\varphi_q(\mathbb{R}^n)$. Then $f$ can be decomposed as

$$f = \sum_{Q \in \mathcal{D}} \lambda(Q) b(Q),$$

where $\lambda(Q)$ is a non-negative number with

$$\sum_{Q \in \mathcal{D}} \lambda(Q) \leq 2 \cdot 9^n \|f\|_{\mathcal{H}^\varphi_q},$$

and $b(Q)$ is a $(\varphi,q)$-block supported in $Q$.  

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Proof. Similar to the case of the predual of classical Morrey spaces. 

**Example 2.23.** Let $1 < q < \infty$, $\varphi \in G_q$, and let $f \in H_{\varphi}^q(\mathbb{R}^n)$. Then $f$ can be decomposed as

$$f = B + \sum_{Q \in \mathcal{D}, |Q| \leq 1} \lambda(Q)b(Q),$$

where $\lambda(Q)$ is a non-negative number with

$$\|B\|_{L_q'} + \sum_{Q \in \mathcal{D}} \lambda(Q) \lesssim 9^n \|f\|_{H_{\varphi}^q}$$

and $b(Q)$ is a $(\varphi, q)$-block supported in $Q$.

About the definition above, the following proposition is fundamental:

**Proposition 2.24.** Let $1 < q < \infty$ and $\varphi \in G_q$. Let $f \in M_{\varphi}^q(\mathbb{R}^n)$ and $g \in H_{\varphi}^q(\mathbb{R}^n)$. Then $\|f \cdot g\|_{L^1} \leq \|f\|_{M_{\varphi}^q}\|g\|_{H_{\varphi}^q}$.

**Proof.** Similar to the classical case. 

**Corollary 2.25.** Let $1 < q < \infty$ and $\varphi \in G_q$. Then every function in $H_{\varphi}^q(\mathbb{R}^n)$ is locally integrable.

**Proof.** Simply combine Proposition 2.24 and the fact that $\chi_Q \in M_{\varphi}^q(\mathbb{R}^n)$ for any cube $Q$. 

**Proposition 2.26.** Let $1 < q < \infty$ and $\varphi \in G_q$. Assume in addition that $\varphi$ satisfies (2.3). Suppose that $f$ and $f_k$, $(k = 1, 2, \ldots)$, are nonnegative, that each $f_k \in H_{\varphi}^q(\mathbb{R}^n)$, that $\|f_k\|_{H_{\varphi}^q} \leq 1$ and that $f_k \uparrow f$ a.e. Then $f \in H_{\varphi}^q(\mathbb{R}^n)$ and $\|f\|_{H_{\varphi}^q} \leq 1$.

**Proof.** By Theorem 2.22 $f_k$ can be decomposed as

$$f_k = \sum_{Q \in \mathcal{D}} \lambda_k(Q)b_k(Q),$$

where $\lambda_k(Q)$ is a non-negative number with

$$\sum_{Q \in \mathcal{D}} \lambda_k(Q) \leq 2 \cdot 9^n$$ \hspace{1cm} (2.15)

and $b_k(Q)$ is a $(\varphi, q)$-block supported in $Q$ and

$$\|b_k(Q)\|_{q'} \leq \frac{\varphi(f(Q))}{|Q|^{\frac{1}{q}}}. \hspace{1cm} (2.16)$$
Using (2.15), (2.16) and the weak-compactness of the Lebesgue space $L^{q'}(Q)$ we now apply a diagonalization argument and, hence, we can select an increasing sequence \( \{k_j\}_{j=1}^{\infty} \) of integers that satisfies the following:

\[
\lim_{j \to \infty} \lambda_{k_j}(Q) = \lambda(Q),
\]

\[
\lim_{j \to \infty} b_{k_j}(Q) = b(Q) \text{ in the weak-topology of } L^{q'}(Q),
\]

where \( b(Q) \) is a \((\varphi, q)\)-block supported in \( Q \). We set

\[
f_0 \equiv \sum_{Q \in D} \lambda(Q)b(Q).
\]

Then, by the Fatou theorem and (2.15),

\[
\sum_{Q \in D} \lambda(Q) \leq \liminf_{j \to \infty} \sum_{Q \in D} \lambda_{k_j}(Q) \leq 2 \cdot 9^n,
\]

which implies \( f_0 \in H^{\varphi}_{q'}(\mathbb{R}^n) \).

We will verify that

\[
\lim_{j \to \infty} \int_{Q_0} f_{k_j}(x) \, dx = \int_{Q_0} f_0(x) \, dx
\]

for all \( Q_0 \in D \). Once (2.20) is established, we will see that \( f = f_0 \) and hence \( f \in H^{\varphi}_{q'}(\mathbb{R}^n) \) by virtue of the Lebesgue differentiation theorem because at least we know that \( f_0 \) locally in \( L^{q'}(\mathbb{R}^n) \).

By the definition of \( f_{k_j} \), we have

\[
\int_{Q_0} f_{k_j}(x) \, dx = \sum_{l=-\infty}^{\infty} \sum_{Q \in D_l} \lambda_{k_j}(Q) \int_{Q_0} b_{k_j}(Q)(x) \, dx.
\]

Note that

\[
\|b_{k_j}(Q)\|_1 \leq |Q_0 \cap Q|^{\frac{1}{q'}} \|b_{k_j}(Q)\|_{q'} \leq \frac{\varphi(l(Q))|Q \cap Q_0|^{\frac{1}{q'}}}{|Q|^{\frac{1}{q'}}}
\]

for any cube \( Q \) containing \( Q_0 \). If

\[
\lim_{t \to \infty} \varphi(t)^{-\frac{n}{q'}} = 0,
\]

then for all \( \varepsilon > 0 \) there exists \( l \in \mathbb{N} \) such that

\[
\sum_{l=N}^{\infty} \sum_{Q \in D_l} \left| \lambda_{k_j}(Q) \int_{Q_0} b_{k_j}(Q)(x) \, dx \right| < \varepsilon.
\]

Thus, we are in the position of using the Lebesgue convergence theorem based on Example 2.11. Thus we obtain (2.20). If

\[
\lim_{t \to \infty} \varphi(t)^{-\frac{n}{q'}} > 0,
\]

we have...
then we go through a similar argument using Example 2.23 to obtain (2.20).

Since \( f_k \uparrow f \) a.e., we must have by (2.20)

\[
\int_{Q_0} f(x) \, dx = \int_{Q_0} f_0(x) \, dx
\]

for all \( Q_0 \in \mathcal{D} \). This yields \( f = f_0 \) a.e., by the Lebesgue differentiation theorem, and, hence, \( f \in \mathcal{H}^p_q(\mathbb{R}^n) \). Since we have verified \( f \in \mathcal{H}^p_q(\mathbb{R}^n) \), it follows that

\[
\|f\|_{\mathcal{H}^p_q} = \sup \left\{ \left| \int_{\mathbb{R}^n} f_k(x)g(x) \, dx \right| : k = 1, 2, \ldots, \|g\|_{\mathcal{M}^p_q} \leq 1 \right\} \leq 1.
\]

This completes the proof of the theorem.

\[\square\]

The proof of the following theorem is completely the same as the classical case once Proposition 2.26 is proved.

**Theorem 2.27.** Let \( 1 < q < \infty \), and let \( \varphi \in \mathcal{G}_q \) satisfy \( \frac{1}{\varphi} \in \mathbb{Z}_0 \).

1. The dual of \( \mathcal{H}^p_q(\mathbb{R}^n) \) is \( \mathcal{M}^p_q(\mathbb{R}^n) \).
2. The dual of \( \mathcal{M}^p_q(\mathbb{R}^n) \) is \( \mathcal{H}^p_q(\mathbb{R}^n) \).

**Proof.** The same as the classical case. \[\square\]

As we discussed for classical Morrey spaces, we have the following conclusion.

**Theorem 2.28.** Let \( 1 < q < \infty \) and let \( \varphi \in \mathcal{G}_q \) satisfy \( \frac{1}{\varphi} \in \mathbb{Z}_0 \). Then any singular integral operator, which is initially defined for \( L^\infty(\mathbb{R}^n) \)-functions, can be naturally extended to a bounded linear operator on \( \mathcal{M}^p_q(\mathbb{R}^n) \).

**Proof.** Similar to the classical case. \[\square\]

Recall that we did not use (2.3) for the proof of boundedness of the Hardy–Littlewood maximal operator. However for the proof of boundedness of the singular integral operators, (2.3) is absolutely necessary as the following proposition shows:

**Proposition 2.29.** Let \( 1 < q < \infty \) and \( \varphi \in \mathcal{G}_q \cap W \). If there exists a constant \( C > 0 \) such that \( \|R_1 f\|_{w\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q} \), then \( \frac{1}{\varphi} \in \mathbb{Z}_0 \), where \( R_1 \) denotes the 1-st Riesz transform.

**Proof.** Let \( V = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 2x_1 > |x| \} \). Assume that \( \frac{1}{\varphi} \notin \mathbb{Z}_0 \), so that for any \( m \in \mathbb{N} \cap [3, \infty) \) there exists \( r_m > 0 \) such that \( \varphi(2^m r_m) \leq 2\varphi(r_m) \). Then, consider
\[ f_m = \chi_{V \cap B(2^m-1r_m) \setminus B(2r_m)}. \] Let \( x \in V \cap B(r_m) \). If \( y \in V \cap B(2^m-1r_m) \setminus B(2r_m) \), then \( x - y \in V \) and \( r_m \leq |x - y| \leq 2^m r_m \). Thus,

\[
R_1 f_m(x) = \int_{\mathbb{R}^n} \frac{x_1 - y_1}{|x - y|^{n+1}} f_m(y) \, dy \\
= \int_{V \cap B(2^{m-1}r_m) \setminus B(2r_m)} \frac{y_1}{|y|^{n+1}} \, dy \\
\geq C \int_{V \cap B(2^{m-1}r_m) \setminus B(2r_m)} \frac{1}{|y|^n} \, dy \\
= C \log m. \tag{2.22}
\]

Since \( V \) is a cone, we have

\[ \chi_{B(r_m)} \lesssim M \chi_{V \cap B(r_m)}. \tag{2.23} \]

We use this estimate and the boundedness of \( M \) on \( M^p_q(\mathbb{R}^n) \) to obtain

\[ \varphi(r_m) \lesssim \| \chi_{B(r_m)} \|_{\mathcal{M}^p_q} \lesssim \| M \chi_{V \cap B(r_m)} \|_{\mathcal{M}^p_q} \lesssim \| \chi_{V \cap B(r_m)} \|_{\mathcal{M}^p_q}. \]

By using the inequality \( \log m \lesssim |R_1 f_m(x)| \) for \( x \in V \cap B(r_m) \) and the boundedness of \( R_1 \) on \( \mathcal{M}^p_q(\ell^n) \), we have

\[
\log(m) \varphi(r_m) \lesssim \| R_1 f_m \|_{w, \mathcal{M}^p_q} \\
\lesssim \| f_m \|_{\mathcal{M}^p_q} \\
\lesssim \| \chi_{B(2^m r_m)} \|_{\mathcal{M}^p_q} \\
\lesssim \varphi(2^m r_m) \\
\sim \varphi(r_m).
\]

This implies \( \log m \leq D \) where \( D \) is independent of \( m \), contradictory to the fact that \( m \geq 3 \) is arbitrary. Hence, there exists some \( m_0 \in \mathbb{N} \) such that \( \varphi(2^{m_0} r) > 2 \varphi(r) \). Thus the integral condition (2.3) holds.

We disprove that \( T \) can not be extended to a bounded linear operator on \( m^p_q(\mathbb{R}^n) \).

**Example 2.30.** Let \( 1 \leq q \leq p < \infty \). Then

\[
\| R_1 f \|_{wm^p_q} \lesssim \| f \|_{m^p_q} \quad (f \in L^\infty_c(\mathbb{R}^n))
\]

and

\[
\| R_1 f \|_{wL^q_{uloc}} \lesssim \| f \|_{L^q_{uloc}} \quad (f \in L^\infty_c(\mathbb{R}^n))
\]

fail. In fact, let \( \varphi(t) = \min(t^{\frac{q}{p}}, 1), \) \( t > 0 \) as in Example 1.3. Then \( \varphi \) fails (2.3) because

\[
\int_1^\infty \frac{dr}{\varphi(r)} = \infty.
\]

We end this section with extension to the vector-valued inequality.
Theorem 2.31. Let $1 < q < \infty$, $1 < u < \infty$ and $\varphi \in G_q$. Let also $T$ be a singular integral operator. Assume in addition that (2.3) holds. Then for all $\{f_j\}_{j=1}^\infty \subset \mathcal{M}_q^{u}(\mathbb{R}^n)$,

\[
\left\| \left( \sum_{j=1}^\infty |Tf_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^{u}} \lesssim \left\| \left( \sum_{j=1}^\infty |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^{u}}.
\]

Proof. Similar to Theorem 2.4.

\hfill \Box

2.3 Generalized fractional integral operators in generalized Morrey spaces

To consider the operator like $(1 - \Delta)^{-1}$, we are oriented to considering

\[ I_\rho f(x) = \int_{\mathbb{R}^n} f(y) \rho(|x - y|) \frac{dy}{|x - y|^n} \]

for any suitable function $f$ on $\mathbb{R}^n$, where $\rho : (0, \infty) \to (0, \infty)$ is a suitable measurable function. Generalized Morrey spaces allow us to consider more general fractional integral operators.

Let us discuss what condition we need in order to guarantee that $I_\rho$ enjoys some boundedness property. We always assume that $\rho$ satisfies the "Dini condition" for $I_\rho$.

\[ \int_0^1 \frac{\rho(s)}{s} ds < \infty, \quad (2.24) \]

so that $I_\rho \chi_Q(x)$ is finite for any cube $Q$ and $x \in \mathbb{R}^n$.

In addition, we also assume that $\rho$ satisfies the "growth condition": there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

\[ \sup_{\frac{1}{2} \leq s \leq r} \rho(s) \lesssim \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} ds, \quad r > 0. \quad (2.25) \]

Condition (2.29) is weaker than the usual doubling condition: there exists a constant $D > 0$ such that

\[ \frac{1}{D} \leq \frac{\rho(r)}{\rho(s)} \leq D \quad (2.26) \]

whenever $r > 0$ and $s > 0$ satisfy $r \leq 2s \leq 4r$.

Proposition 2.32. If $\rho : (0, \infty) \to (0, \infty)$ satisfies the doubling condition (2.26), then

\[ \sup_{\frac{1}{2} \leq s \leq r} \rho(s) \leq D^2 \int_{\frac{1}{2}}^r \rho(s) \frac{ds}{s}. \]
Proof. Keeping in mind \( \int_{r/2}^{r} \frac{ds}{s} = \log 2 < 1 \), we calculate

\[
\sup_{r/2 \leq s \leq r} \rho(s) \leq D \rho(r) = D \int_{r/2}^{r} \rho(r) \frac{ds}{s} \leq D^2 \int_{r/2}^{r} \rho(s) \frac{ds}{s}.
\]

\[\Box\]

Example 2.33. Let \( 0 < \alpha < \infty \).

1. \( \rho(t) = t^\alpha \), which generates \( I_\alpha \), satisfies the doubling condition.

2. \( \rho(t) = \frac{t^\alpha}{\log(e + t)} \) satisfies the doubling condition.

3. \( \rho(t) = t^\alpha e^{-t} \) satisfies the growth condition but fails the doubling condition.

4. Let \( 0 \leq \gamma < \infty \) and \( \beta_1, \beta_2 \in \mathbb{R} \). We set

\[
\ell^B(r) \equiv \begin{cases} 
(1 + |\log r|)^{\beta_1} & (0 < r \leq 1), \\
(1 + |\log r|)^{\beta_2} & (1 < r < \infty)
\end{cases}
\]

as before. Then \( \rho(t) = t^\gamma \ell^B(t) \) satisfies (2.24) if and only if \( \gamma = 0 > \beta_1 \) or \( \gamma > 0 \). Meanwhile (2.26) is always satisfied. Noteworthy is the fact that we can tolerate the case \( \gamma = 0 \) if \( \beta_1 < 0 \).

The above examples are natural in some sense but somewhat artificial because the second example and the third one do not appear naturally in the context of other areas of mathematics. Here we present some other examples related to partial differential equations.

Example 2.34. Note that the solution to \( (1 - \Delta)f = g \), where \( f \) is an unknown function and \( g \) is a give function is given by:

Definition 2.35. One defines \( (1 - \Delta)^{-s}f \) by

\[
(1 - \Delta)^{-s}f = G_s \ast f,
\]

where \( G_s \) is given by:

\[
G_s(x) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\eta(\varepsilon \xi) e^{ix \cdot \xi}}{(1 + |\xi|^2)^{\frac{3}{2}}} \, d\xi.
\]

The function \( G_s \) is called the Bessel kernel.

Although it is impossible to find \( I_{\rho \chi_{B(x,r)}}(y) \), \( y \in \mathbb{R}^n \), we still have a partial but important estimate.

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Lemma 2.36. There exists a constant $C > 0$ such that the inequality \( \tilde{\rho}(R/2) \lesssim I_{\rho} \chi_{B(R)}(x) \) holds whenever \( x \in B(R/2) \) and \( R > 0 \).

Proof. Take \( x \in B(R/2) \). We write the integral in full:
\[
I_{\rho} \chi_{B(R)}(x) = \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} \chi_{B(R)}(y) \, dy = \int_{B(R)} \frac{\rho(|x - y|)}{|x - y|^n} \, dy.
\]
A geometric observation shows that \( B(x, R/2) \subseteq B(R) \). Hence, we have
\[
I_{\rho} \chi_{B(R)}(x) \geq \int_{B(x,R/2)} \frac{\rho(|x - y|)}{|x - y|^n} \, dy = C \int_0^{R/2} \frac{\rho(s)}{s} \, ds.
\]
Note that we only use the spherical coordinates to obtain the last integral. \( \square \)

In the case of the radially symmetric functions, we can calculate \( I_{\rho} g_R(x) \) for \( x \) small.

Lemma 2.37. For every \( R > 0 \) and a measurable function \( \theta : (0, \infty) \rightarrow [0, \infty) \) satisfying the doubling condition
\[
\theta(s) \sim \theta(r) \quad (0 < r \leq s \leq 2r),
\]
the inequality
\[
C^{-1} \int_{2R}^{\infty} \frac{\theta(t)\rho(t)}{t} \, dt \leq I_{\rho} g_R(x) \leq \int_{2R/3}^{\infty} \frac{\theta(t)\rho(t)}{t} \, dt
\]
holds whenever \( x \in B \left( \frac{R}{3} \right) \), where \( g_R(x) \equiv \theta(|x|) \chi_{B(R)}(x) \).

Proof. We prove the right-hand inequality, the left-hand inequality being similar. A geometric observation shows that \( |x - y| \sim |y| \) for all \( x \in B \left( \frac{R}{3} \right) \) and \( y \in \mathbb{R}^n \setminus B \left( \frac{2R}{3} \right) \).

Since \( \theta \) satisfies (2.27), then
\[
I_{\rho} g_R(x) = \int_{\mathbb{R}^n \setminus B(R)} \frac{\theta(|y|)\rho(|x - y|)}{|x - y|^n} \, dy
\]
\[
\leq \int_{\mathbb{R}^n \setminus B(x,2R/3)} \theta(|y|)\rho(|x - y|) \, dy
\]
\[
= \int_{\mathbb{R}^n \setminus B \left( \frac{2R}{3} \right)} \theta(|x - y|)\rho(|y|) \, dy
\]
\[
\leq \int_{2R/3}^{\infty} \frac{\theta(t)\rho(t)}{t} \, dt \quad \text{for} \quad x \in B \left( \frac{R}{3} \right).
\]
It remains to write the most right-hand side in terms of the spherical coordinates. \( \square \)
For convenience, write
\[ \tilde{\rho}(r) \equiv \int_0^r \frac{\rho(t)}{t} \, dt. \tag{2.28} \]
Sometimes, we are interested in the case where matters are reduced to the classical fractional integral operators.

**Proposition 2.38.** Let \( \rho : (0, \infty) \to (0, \infty) \) be a measurable function satisfying (2.29). Then the following are equivalent:

(a) \( \rho(r) \lesssim r^\alpha \) for all \( r > 0 \).

(b) \( \tilde{\rho}(r) \lesssim r^\alpha \) for all \( r > 0 \).

**Proof.** Clearly (a) implies (b), since \( \int_0^r s^{\alpha-1} \, ds = \frac{1}{\alpha} r^\alpha \). Let us see (b) implies (a).

Combining (2.29) with (b), we obtain
\[ \sup_{r/2 < s \leq r} \rho(s) \lesssim \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} \, ds \lesssim \int_0^{k_2 r} \frac{\rho(s)}{s} \, ds \lesssim (k_2 r)^\alpha \sim r^\alpha. \] \hspace{1cm} (2.29)

Now we present three different criteria for the boundedness of \( I_\rho \). We prove the following three theorems on the boundedness of \( I_\rho \) on generalized Morrey spaces.

For the case of \( q = 1 \), we have the following simple result:

**Theorem 2.39.** Let \( 1 \leq p < \infty \), and let \( \varphi \in \mathcal{G}_p \) and \( \psi \in \mathcal{G}_1 \). Then \( I_\rho \) is bounded from \( \mathcal{M}_p^\varphi(\mathbb{R}^n) \) to \( \mathcal{M}_1^\psi(\mathbb{R}^n) \) if and only if
\[ \frac{1}{\varphi(r)} \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \frac{1}{\psi(r)} \quad (r > 0). \] \hspace{1cm} (2.30)

Note that the left-hand side of (2.30) equals
\[ \frac{1}{\varphi(r)} \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt = \int_0^\infty \frac{\rho(t)}{t \varphi(\max(r,t))} \, dt. \]

**Proof of Theorem 2.39 (Necessity).** Assume that \( I_\rho \) is bounded from \( \mathcal{M}_1^\varphi(\mathbb{R}^n) \) to \( \mathcal{M}_1^\psi(\mathbb{R}^n) \).

By Lemma 2.36 and the doubling property of \( \psi \), we obtain
\[ \tilde{\rho}(r) \lesssim \frac{1}{r^n} \int_{B(r/2)} I_\rho \chi_{B(r)}(x) \, dx \lesssim \frac{1}{r^n} \int_{B(r/2)} I_\rho \chi_{B(r)}(x) \, dx \lesssim \frac{1}{\psi(r)} \| I_\rho \chi_{B(r)} \|_{\mathcal{M}_1^\psi}. \]

Since \( \psi \in \mathcal{G}_1 \) and \( I_\rho \) is assumed bounded from \( \mathcal{M}_p^\varphi(\mathbb{R}^n) \) to \( \mathcal{M}_1^\psi(\mathbb{R}^n) \), it follows that
\[ \tilde{\rho}(r) \lesssim \frac{1}{\psi(r)} \| \chi_{B(r)} \|_{\mathcal{M}^\varphi_1}. \]
Since \( \|\chi_B(r)\|_{\mathcal{M}_p^\psi} \sim \varphi(r) \), we conclude

\[
\tilde{\rho}(r) \lesssim \frac{\varphi(r)}{\psi(r)}.
\]

Let \( g_r(x) = \frac{\chi_{B(\rho|x|)}(x)}{\rho(|x|)} \). By Lemma 2.37 with \( \theta = \frac{1}{\varphi} \), we have

\[
\int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \psi\left(\frac{r}{6}\right)^{-1} \|I_{\rho}g_r\|_{\mathcal{M}_1^\psi} \lesssim \psi(r)^{-1} \|g_r\|_{\mathcal{M}_1^\psi} \lesssim \psi(r)^{-1}.
\]

Thus Theorem 2.39 is proved. \( \blacksquare \)

**Proof of Theorem 2.39 (Sufficiency).** For a ball \( B(z, r) \), we let \( f_1 \equiv f \chi_{B(z, 2r)} \) and \( f_2 \equiv f - f_1 \). Then a geometric observation shows \( B(z, r) \subset B(y, 3r) \) for all \( y \in B(z, 2r) \). Hence by the Fubini theorem and the normalization,

\[
\int_{B(z,r)} |I_{\rho}f_1(x)| \, dx \leq \int_{B(z,r)} \left( \int_{B(z,2r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dy \right) \, dx
\]

\[
\leq \int_{B(z,2r)} \left( \int_{B(y,3r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dx \right) \, dy
\]

\[
= \int_{B(z,2r)} |f(y)| \, dy \times \int_{B(3r)} \frac{\rho(|x|)}{|x|^n} \, dx.
\]

By the use of the definition of the Morrey norm, we used (2.30) and the doubling condition of \( \psi \), we obtain

\[
\int_{B(z,r)} |I_{\rho}f_1(x)| \, dx \lesssim \tilde{\rho}(3r)\varphi(2r)^{-1}r^n
\]

\[
\lesssim \psi(3r)^{-1}r^n
\]

Thus the estimate for \( f_1 \) is valid. As for \( f_2 \), we let \( x \in B(z, r) \). Then we have

\[
|I_{\rho}f_2(x)| \leq \int_{B(z,2r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dy \leq \int_{B(x,r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dy
\]

and decomposing the right-hand side dyadically as we did in the proof of Theorem 2.43 for \( \sum_{II} \), we obtain

\[
|I_{\rho}f_2(x)| \leq \sum_{j=1}^\infty \int_{B(x,2^j r) \setminus B(x,2^{j-1} r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} \, dy \lesssim \int_{2k_1r}^\infty \frac{\rho(t)}{t \varphi(t)} \, dt.
\]

If we use (2.30) once again and the doubling condition of \( \psi \) of \( \psi \), then we obtain

\[
|I_{\rho}f_2(x)| \lesssim \psi(r)^{-1}.
\]

Thus the estimate for \( f_2 \) is valid as well. \( \blacksquare \)
In the following example, we consider why we need generalized Morrey spaces.

**Example 2.40.** Let \( s \in (0, n) \) and \( \kappa > 0 \). Define
\[
\psi(r) \equiv \frac{(1 + r)^s}{\max(1, \log r^{-1})} \quad (r > 0).
\]
Let
\[
\rho(r) = r^{n-s} \exp(-\kappa r), \quad \varphi(r) = r^{n-s}.
\]
Then if \( 0 < r < 1 \),
\[
\frac{1}{\varphi(r)} \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \sim \frac{1}{\psi(r)}.
\]
and if \( r \geq 1 \),
\[
\frac{1}{\varphi(r)} \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \frac{1}{\psi(r)}.
\]
Thus \( \| (1 - \Delta)^{-s/2} f \|_{\mathcal{M}^0_1} \lesssim_s \| f \|_{\mathcal{M}^{n-s}_1} \) for all \( f \in \mathcal{M}^{n-s}_1(\mathbb{R}^n) \). This calculation shows we cannot delete \( \max(1, \log r^{-1}) \).

Example 2.40 convince us that generalized Morrey spaces occur naturally.

**Example 2.41.** Let \( \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n} \) with \( 1 < p < s < \infty \) and \( 0 < \alpha < n \). Then \( I_\alpha \) does not map \( m^p_1(\mathbb{R}^n) \) to \( m^s_1(\mathbb{R}^n) \), since \( \rho(t) = t^\alpha \) and \( \varphi(t) = \max(t^{\frac{n}{p}}, 1) \) satisfies
\[
\int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt = \infty
\]
instead of
\[
\int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \frac{1}{\varphi(t)^{\frac{n}{p}}}.
\]
If we consider the truncated fractional maximal operator \( i_\alpha \) given by
\[
i_\alpha f(x) = \int_{|y| \leq 1} \frac{f(x - y)}{|y|^{n-\alpha}} \, dy,
\]
then \( i_\alpha \) maps \( m^p_1(\mathbb{R}^n) \) to \( m^s_1(\mathbb{R}^n) \), since
\[
\frac{1}{\varphi(r)} \int_0^r \frac{\rho(t)\chi(0,1)(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)\chi(0,1)(t)}{t \varphi(t)} \, dt \lesssim \frac{1}{\varphi(t)^{\frac{n}{p}}}.
\]

**Example 2.42.** Let \( 0 < s < n \). Define \( \varphi(r) \equiv r^s \) and \( \psi(r) \equiv (1 + r)^{-s} \ell_{-1,0}(r) \) for \( r > 0 \). Let \( \rho(r) \equiv r^n G_s(r) \), where \( G_s \) denotes the Bessel kernel, the kernel of \( (1 - \Delta)^{s/2} \).

Observe that \( \tilde{\rho}(r) \sim \min(r^s, 1) \) and hence \( \tilde{\rho}(r) = \min(1, r^{-s}) \). Note also that
\[
\int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \sim \begin{cases} \log(e/r) & (r < 1), \\ r^{n-s} G_s(r) & (r \geq 1). \end{cases}
\]
Then
\[ \frac{\tilde{\rho}(r)}{\varphi(r)} + \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \sim \frac{1}{\psi(r)} \quad (r > 0). \]

Hence it follows from Theorem 2.39 that \( \| I_\rho f \|_{M^\psi_1} \lesssim \| f \|_{M^\varphi_1}, \) extending Proposition 2.40. This triple \((\rho, \varphi, \psi)\) fulfills the assumption (2.30). However, \( \frac{\rho}{\varphi} \notin \mathbb{Z}_0 \) since
\[ \frac{\rho(r)}{\varphi(r)} = \mathcal{O}(1) \quad \text{as} \quad r \downarrow 0 \]
and (2.31) fails.

We give a result, which improves Example 2.40.

We move on to the Adams type estimate.

**Theorem 2.43.** Let \( 1 < p < q < \infty \) and \( \varphi \in \mathcal{G}_p. \)

1. The generalized fractional integral operator \( I_\rho \) is bounded from \( M^\varphi_p(\mathbb{R}^n) \) to \( M^{\varphi_p/q}_{q}(\mathbb{R}^n) \) if
\[ \frac{1}{\varphi(r)} \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \varphi(r)^{-p/q} \] for all \( r > 0. \) If \( \varphi \in \mathbb{Z}^{-\frac{p}{n}} \), then (2.31) is necessary for the boundedness of \( I_\rho \) from \( M^\varphi_p(\mathbb{R}^n) \) to \( M^{\varphi_p/q}_{q}(\mathbb{R}^n). \)

2. Assume \( \frac{\rho}{\varphi} \in \mathbb{Z}_0. \) Then \( I_\rho \) is bounded from \( M^\varphi_p(\mathbb{R}^n) \) to \( M^{\varphi_p/q}_{q}(\mathbb{R}^n) \) if and only if
\[ \tilde{\rho}(r) \lesssim \varphi(r)^{1-p/q} \quad (r > 0). \]

So, if \( \frac{\rho}{\varphi} \in \mathbb{Z}_0, \) condition (2.31) simplifies to (2.32).

**Remark 2.44.**

1. The first half of “only if” part (2.32) is clear from Theorem 2.39 with \( \psi = \varphi^{p/q}. \)

2. Once we assume \( \frac{\rho}{\varphi} \in \mathbb{Z}_0, \) it is easy to check that (2.32) implies (2.31). Indeed, if we use \( \frac{\rho}{\varphi} \in \mathbb{Z}_0 \) and \( \varphi \in \mathcal{G}_p, \) then we have
\[ \int_r^\infty \frac{\rho(s)}{s \varphi(s)} \, ds \lesssim \frac{\rho(r)}{\varphi(r)}. \]

Since \( \rho \) satisfies the growth condition, we have
\[ \int_r^\infty \frac{\rho(s)}{s \varphi(s)} \, ds \lesssim \frac{\tilde{\rho}(k_2 r)}{\varphi(r)}. \]

If we use (2.32) and the doubling condition on \( \varphi, \) then we obtain
\[ \int_r^\infty \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \varphi(r)^{-p/q}. \]

3. For the “if” part we only need the following estimate of Hedberg-type, see Lemma 2.45 below.
Proof of Theorem 2.43, necessity. According to Theorem 2.39, we have only to show
\[ \int_{2R}^{\infty} \frac{\rho(t)}{t^\varphi(t)} dt \lesssim \varphi(2R)^{-p/q}. \]

By virtue of Lemma 2.37, we obtain
\[ \int_{2R}^{\infty} \frac{\rho(t)}{t^\varphi(t)} dt \sim \left( \frac{1}{R^n} \int_B \left( \frac{R}{3} \right)^q I_\rho g(x)^q \, dx \right)^{\frac{1}{q}} \lesssim \varphi(R)^{-p/q} \| I_\rho g R \|_{\mathcal{M}_\varphi^{p/q}}. \]

Since $I_\rho$ is bounded, we obtain
\[ \int_{2R}^{\infty} \frac{\rho(t)}{t^\varphi(t)} dt \lesssim \varphi(R)^{-p/q} \| g R \|_{\mathcal{M}_\varphi^{p/q}} \lesssim \varphi(R)^{-p/q} \left\| \frac{1}{|\cdot|} \right\|_{\mathcal{M}_\varphi^{s}}. \]

Recall that we are assuming $\varphi \in \mathbb{Z}^{-\frac{n}{p}}$. Now we invoke Lemma 2.16 to conclude
\[ \int_{2R}^{\infty} \frac{\rho(t)}{t^\varphi(t)} dt \lesssim \varphi(R)^{-p/q} \lesssim \varphi(2R)^{-p/q}. \]

Thus necessity is proven. \qed

As we have mentioned, we want an estimate of Hedberg-type. We may ask ourselves whether $\inf_{r>0} \varphi(r)^{-p/q}$ can be removed, that is, we may assume $\sup \varphi = \infty$. However, it can happen that $\sup \varphi < \infty$ as example below shows.

**Lemma 2.45.** Let $1 \leq p < q < \infty$, and let $\varphi \in \mathcal{G}_p \cap W$ satisfy (2.31). If we normalize the norm of $f$ by $\| f \|_{\mathcal{M}_\varphi^{p}} = 1$, then
\[ |I_\rho f(x)| \lesssim [M f(x)]^{p/q} + \inf_{r>0} \varphi(r)^{-p/q} \] (2.33)
for $x \in \mathbb{R}^n$.

Once this estimate is satisfied, we can conclude the proof of Theorem 2.43 as follows: We choose an arbitrary ball $B = B(z, r)$. If we integrate Lemma 2.45, then we have
\[ \frac{1}{|B|} \int_B |I_\rho f(x)|^q \, dx \lesssim \frac{1}{|B|} \int_B [M f(x)]^p \, dx + \inf_{u>0} \varphi(u)^{-p} \]
If we multiply both sides by $\varphi(r)^p$, then we have
\[ \frac{\varphi(r)^p}{|B|} \int_B |I_\rho f(x)|^q \, dx \lesssim \left( \frac{\varphi(r)^p}{|B|} \int_B [M f(x)]^p \, dx + 1 \right) \lesssim 1 \]
by virtue of the boundedness of the maximal operator $M$ on $\mathcal{M}_\varphi^{s}(\mathbb{R}^n)$. The ball $B$ being arbitrary, we obtain the desired result.
Proof. Recall that $k_1$ and $k_2$ appeared in the condition (2.29) on $\rho$. Let
\[
\rho^*(r) = \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} \, ds.
\]
We have
\[
|I_\rho f(x)| \lesssim -\sum_{j=-\infty}^{-1} + \sum_{j=0}^\infty \rho^*(2^j r) \int |f(y)| \, dy
\]
for given $x \in \mathbb{R}^n$ and $r > 0$. Let $\Sigma_I$ and $\Sigma_{II}$ be the first and second summations above. Now we invoke the overlapping property:
\[
\sum_{j=-\infty}^{-1} \chi_{[2^j k_1 r, 2^j k_2 r]} \lesssim \chi_{(-\infty, 2^{-1} k_2 r]}, \quad \sum_{j=0}^\infty \chi_{[2^j k_1 r, 2^j k_2 r]} \lesssim \chi_{[k_1 r, \infty)}.
\]
(2.34)
As a result, we have
\[
\sum_{j=-\infty}^{-1} \rho^*(2^j r) \leq \sum_{j=-\infty}^{-1} 2^{j k_2 r} \rho(s) \, ds \lesssim \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} \, ds = \tilde{\rho}(k_2 r)
\]
and
\[
\sum_{j=0}^\infty \frac{\rho^*(2^j r)}{\varphi(2^j r)} \lesssim \int_{k_1 r}^{\infty} \left( \sum_{j=0}^\infty \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s \varphi(s)} \, ds \lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s \varphi(s)} \, ds.
\]
Thus thanks to (2.32)
\[
\Sigma_I \lesssim \sum_{j=-\infty}^{-1} \rho^*(2^j r) Mf(x) \lesssim \tilde{C}(k_2 r) Mf(x) \lesssim \varphi(r)^{1-p/q} Mf(x).
\]
Meanwhile
\[
\Sigma_{II} \lesssim \sum_{j=0}^\infty \frac{\rho^*(2^j r)}{\varphi(2^j r)} \| f \|_{M^p} \lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s \varphi(s)} \, ds.
\]
We use $\frac{\rho}{\varphi} \in Z_0$ or (2.31) now. If we use (2.31), then we have
\[
\int_{r}^{\infty} \frac{\rho(t)}{t \varphi(t)} \, dt \lesssim \varphi(r)^{-p/q}.
\]
By the doubling property of $\varphi$, we obtain $\Sigma_{II} \lesssim \varphi(r)^{-p/q}$. Hence,
\[
|I_\rho f(x)| \lesssim \varphi(r)^{1-p/q} \left( Mf(x) + \frac{1}{\varphi(r)} \right)
\]
(2.35)
for all $r > 0$.

First assume $Mf(x) \leq \inf_{r>0} \frac{1}{\varphi(r)}$. Then, the conclusion is immediate from (2.35).
Next, we assume $Mf(x) > \inf_{r>0} \frac{1}{\varphi(r)}$. Since $\|f\|_{\mathcal{M}^\varphi_p} = 1$, we have

$$1 \geq \varphi(r) \left( \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \geq \frac{\varphi(r)}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| \, dy.$$ 

Hence

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy \leq \frac{1}{\varphi(r)}$$

for all $r > 0$. This implies

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy \leq \sup_{R>0} \frac{1}{\varphi(R)}$$

for all $r > 0$. Since $r > 0$ and $x \in \mathbb{R}^n$ are arbitrary, it follows that $Mf(x) \leq \sup_{r>0} \frac{1}{\varphi(r)}$. We can thus find $R > 0$ such that $Mf(x) = 2\varphi(R)$ and, with this $R$, we can obtain the desired estimate. 

In order that $I_\rho$ be bounded from $\mathcal{M}^\varphi_p (\mathbb{R}^n)$ to $\mathcal{M}^{\varphi^{p/q}}_q (\mathbb{R}^n)$, we must have

$$\rho(r) \lesssim \varphi(r)^{-p/q}$$

according to Theorem 2.39 with $\psi = \varphi^{p/q}$.

We note that if $\rho(r) = r^\alpha$, with $0 < \alpha < n$, then $I_\rho = I_\alpha$ is the classical fractional integral operator, also known as the Riesz potential, which is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, where $1 < p, q < \infty$. The necessary part is usually proved by using the scaling arguments.

Theorem 2.43 characterizes the kernel function $\rho$ for which $I_\rho$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$. We have the following result:

**Corollary 2.46.** Let $1 < p < q < \infty$. The operator $I_\rho$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $\rho(r) \lesssim r^{\frac{n}{p} - \frac{n}{q}}$ for all $r > 0$.

For $\rho(r) = r^\alpha$, Corollary 2.46 further reads that the operator $I_\rho$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$, where $1 < p < q < \infty$.

With Theorems 2.39–2.43 we can characterize the function $\rho$ for which $I_\rho$ is bounded from one Morrey space to another.

The next corollary generalizes the previous characterization in Corollary 2.46.

**Corollary 2.47.** Assume that the parameters $p, q, s, t$ and $\alpha$ satisfy

$$1 < q \leq p < \infty, \quad 1 < t \leq s < \infty, \quad 0 < \alpha < n$$

and

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}.$$
Let \( \rho : (0, \infty) \to (0, \infty) \) be a function satisfying the growth condition. Then the generalized fractional integral operator \( I_\rho \) is bounded from \( \mathcal{M}^p_\ell(\mathbb{R}^n) \) to \( \mathcal{M}^*_\ell(\mathbb{R}^n) \) precisely when \( \rho(r) \lesssim r^\alpha \).

We show by examples that two statements in Theorem 2.43 are of independent interest. As before we write

\[
\ell^\varphi(r) = \begin{cases} 
(1 + |\log r|)^{\beta_1} & (0 < r \leq 1), \\
(1 + |\log r|)^{\beta_2} & (1 < r < \infty).
\end{cases}
\]

This function is used to describe the “log”-growth and “log”-decay properties. Also, we fix \( p \) and \( q \) so that \( 1 < p < q < \infty \). The key properties we are interested in are summarized in the following table:

<table>
<thead>
<tr>
<th>Example No.</th>
<th>( \ell^{\varphi} \in \mathbb{Z}_0 )</th>
<th>( \varphi \in \mathbb{Z}^{-\frac{n}{p}} )</th>
<th>(2.31)</th>
<th>(2.32)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 2.48</td>
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<td>Example 2.49</td>
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<tr>
<td>Example 2.50</td>
<td>-</td>
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</tr>
<tr>
<td>Example 2.51</td>
<td>-</td>
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<td>+</td>
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</tbody>
</table>

**Example 2.48.** Let \( \lambda < 0 \) satisfy \( 0 < \left( \frac{p}{q} - 1 \right) \lambda < n \) and \( -\frac{n}{p} < \lambda \). Take \( \mu_1, \mu_2 \) arbitrarily. Set \( \beta_i = \left( \frac{p}{q} - 1 \right) \mu_i \) for \( i = 1, 2 \). Define \( \varphi(r) = r^{-\lambda} \ell^{-\mu_1 - \mu_2} \) and \( \rho(r) = \varphi(r)^{1 - \frac{p}{q}} \) for \( r > 0 \). Then this pair \((\rho, \varphi)\) fulfills the assumptions \( \ell^{\varphi} \in \mathbb{Z}_0 \) and \( \varphi \in \mathbb{Z}^{-\frac{n}{p}} \) in Theorem 2.43. Indeed, for \( r > 0 \) we have \( \tilde{\rho}(r) \sim \rho(r) = \varphi(r)^{1 - \frac{p}{q}} \) and

\[
\int_r^\infty \frac{\rho(t)}{t^\varphi(t)} \, dt \sim \frac{\rho(r)}{\varphi(r)}.
\]

Example 2.49 is an endpoint case of the above example.

**Example 2.49.** Let \( \mu_1, \mu_2 \) satisfy \( \mu_1, \mu_2 \geq 0 \). Set \( \alpha = \frac{n}{p} - \frac{n}{q} \) and \( \beta_i = \left( \frac{p}{q} - 1 \right) \mu_i \) for \( i = 1, 2 \). Define \( \varphi(r) = r^{\frac{n}{p}} \ell^{-\mu_1 - \mu_2} \) for \( r > 0 \) and \( \rho = \varphi^{1 - \frac{p}{q}} \). We note that \( \tilde{\rho} \sim \rho \). Then this pair \((\rho, \varphi)\) fulfills the assumptions \( \ell^{\varphi} = \varphi^{-\frac{p}{q}} \in \mathbb{Z}_0 \) and (2.31) but \( \varphi \notin \mathbb{Z}^{-\frac{n}{p}} \) since \( \ell^{-\mu_1 - \mu_2} \notin \mathbb{Z}^0 \).

The next example concerns the case where the spaces are close to \( L^\infty(\mathbb{R}^n) \) and the smoothing order of \( I_\rho \) is “almost 0”.

**Example 2.50.** Let \( \mu_1, \mu_2 < 0 \). Set \( \beta_1 = \left( \frac{p}{q} - 1 \right) \mu_1 + 1 \in (1, \infty) \) and \( \beta_2 = \left( \frac{p}{q} - 1 \right) \mu_2 - 1 \in (-1, \infty) \). Define \( \rho = \ell^{\varphi} \) as we did in Example 1.9 and let \( \varphi = \ell_{\mu_1, \mu_2} \). Then this pair \((\rho, \varphi)\) fulfills \( \varphi \notin \mathbb{Z}^{-\frac{n}{p}} \) and assumption (2.31) but \( \ell^{\varphi} = \ell_{\beta_1 - \mu_1, \beta_2 - \mu_2} \notin \mathbb{Z}_0 \). More precisely, we have \( \tilde{\rho} \sim \ell_{\beta_1 - 1, \beta_2 + 1} \) since \( \beta_1 > 1 \), and

\[
\int_r^\infty \frac{\rho(t)}{t^\varphi(t)} \, dt \sim \ell_{\mu_1 + \beta_1 - 1, \mu_2 + \beta_2 + 1}(r) \quad (r > 0).
\]
We consider a case where the target space is close to $L^\infty(\mathbb{R}^n)$.

**Example 2.51.** Let $1 < p, q < \infty$. Let $\alpha, \beta_1, \mu_1, \mu_2$ satisfy $0 < \alpha < \frac{n}{p}$, $\mu_1 + \beta_1 < 1$, $\mu_2 < 0$. Set $\beta_2 \equiv \left(\frac{p}{q} - 1\right) \mu_2 - 1 \in (-1, \infty)$. Define $\rho(r) \equiv \min(1, r^\alpha) t^\beta(r)$ as we did in Example 1.9 and let $\varphi(r) \equiv \max(1, r^{-\alpha}) t^\mu_{\mu_1, \mu_2}(r)$ for $r > 0$. Then this pair $(\rho, \varphi)$ fulfills $\varphi \notin Z^{-\frac{n}{p}}$ and assumption (2.31) but $\rho \notin Z_0$. More precisely,

$$\frac{\rho(r)}{\varphi(r)} \sim t^{\mu_1 + \beta_1 + \beta_2 + 1}(r)$$

and

$$\int_r^{\infty} \frac{\rho(t)}{t \varphi(t)} dt \sim t^{\mu_1 + \beta_1 - 1, \mu_2 + \beta_2 + 1}(r)$$

for $r > 0$.

Based upon these preliminary results and Lemma 2.37, we will prove Theorems 2.43–2.39.

We remark that (2.31) includes (2.32). We prove an estimate. Once we prove Lemma 2.45 below, we can obtain the boundedness of $I^\rho$ from $M^{\rho \varphi}(\mathbb{R}^n)$ to $M^{\varphi \psi}(\mathbb{R}^n)$ as we will see below. Here we use the fact that the Hardy-Littlewood maximal operator $M$ is bounded on $M^{\rho \varphi}(\mathbb{R}^n)$, if $p > 1$ and $\varphi$ is almost decreasing; see Theorem 2.1.

### 2.4 Generalized fractional maximal operators in generalized Morrey spaces

We discuss the boundedness property of the generalized fractional maximal operator, defined by:

$$M_\rho f(x) = \sup_{r > 0} \frac{\rho(r)}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy \quad (x \in \mathbb{R}^n),$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\rho$ is a suitable function from $(0, \infty)$ to $[0, \infty)$.

**Example 2.52.** Let $0 \leq \alpha < n$.

1. If we let $\rho(t) = t^\alpha$, then we obtain the fractional maximal operator $M_\alpha$; $M_\rho = M_\alpha$.

2. If we let $\rho(t) = \min(t^\alpha, 1)$, then we obtain the local fractional maximal operator $m_\alpha$; $M_\rho = m_\alpha$, where

$$m_\alpha f(x) = \sup_{0 < r \leq 1} \frac{r^\alpha}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy$$

What $M_\rho$ is to $I^\rho$ is what $M_\alpha$ is to $I^\alpha$. So, we are interested in when $M_\rho$ is bounded from $M^{\rho \varphi}(\mathbb{R}^n)$ to $M^{\varphi \psi}(\mathbb{R}^n)$. We start with the following necessary condition:
Proposition 2.53. Let $1 \leq q < \infty$ and $(\varphi, \psi) \in \mathcal{G}_q \times \mathcal{G}_1$. Assume that $M_\rho$ is bounded from $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ to $\mathcal{W}_1^\psi(\mathbb{R}^n)$. Then $\rho \lesssim \frac{\varphi}{\psi}$. In particular, if $M_\rho$ is bounded from $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ to $\mathcal{M}_1^\psi(\mathbb{R}^n)$, then $\rho \lesssim \frac{\varphi}{\psi}$.

Proof. Let $R > 0$ be fixed. We utilize the pointwise estimate $\rho(R)\chi_B(R) \leq M_\rho \chi_B(2R)$, and the doubling condition of $\varphi$ to obtain

$$
\rho(R) \lesssim \frac{\|\rho(R)\chi_B(R)\|_{w,\mathcal{M}_1^\psi}}{\psi(R)} \lesssim \frac{\|\rho(R)\chi_B(2R)\|_{\mathcal{M}_1^\psi}}{\psi(R)} \lesssim \frac{\varphi(R)}{\psi(R)}.
$$

Our first result completely characterizes the boundedness of $M_\rho$ on generalized Orlicz–Morrey spaces.

Theorem 2.54. Let $0 < a < 1 < q < \infty$. Let $\varphi \in \mathcal{G}_q$. Assume that $\lim_{t \to 0} \varphi(t) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. Then, $M_\rho$ is bounded from $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ to $\mathcal{M}_{a^{-1}q}^\varphi(\mathbb{R}^n)$ if and only if $\rho$ and $\varphi$ satisfy the inequality

$$
\rho(R) \lesssim \varphi(R)^{1-a}
$$

for all $R > 0$.

The proof hinges on the following Hedberg inequality:

Lemma 2.55. Let $0 < a < 1 < q < \infty$. Let $\varphi \in \mathcal{G}_q$. Assume that $\lim_{t \to 0} \varphi(t) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. Then for any $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ with $\|f\|_{\mathcal{M}_q^\varphi} \leq 1$,

$$
M_\rho f(x) \lesssim M f(x)^a \quad (x \in \mathbb{R}^n).
$$

Once Lemma 2.55 is proved, we have only to resort to the scaling law (Lemma 1.10) and the boundedness of $M$ on $\mathcal{M}_q^\varphi(\mathbb{R}^n)$.

Proof of Lemma 2.55. Remark that both $\varphi$ is bijective. Let $R > 0$. By using the definition of $M$, we obtain

$$
\frac{\rho(R)}{|B(x,R)|} \int_{B(x,R)} |f(y)| \ dy \leq \rho(R) M f(x) \lesssim \varphi(R)^{1-a} M f(x)
$$

and

$$
\frac{\rho(R)}{|B(x,R)|} \int_{B(x,R)} |f(y)| \ dy \lesssim \rho(R) \frac{\|f\|_{\mathcal{M}_q^\varphi}}{\varphi(R)} \lesssim \varphi(R)^{-a}.
$$
Thus, it follows that
\[
\frac{\rho(R)}{|B(x, R)|} \int_{B(x, R)} |f(y)| \, dy \lesssim \min \left\{ \varphi(R)^{1-a} Mf(x), \varphi(R)^{-a} \right\} \\
\leq \sup_{t>0} \min \left\{ t^{1-a} Mf(x), t^{-a} \right\} \\
= Mf(x)^a.
\]
Since \( R > 0 \) being arbitrary, we obtain (2.37). 

The weak boundedness of \( M\rho \) can be characterized in a similar way.

**Corollary 2.56.** Let \( 0 < a < 1 \leq q < \infty \). Let \( \varphi \in \mathcal{G}_q \) satisfy \( \inf \varphi = 0 \) and \( \sup \varphi = \infty \). Then, \( M\rho \) is bounded from \( M\varphi_q(R^n) \) to \( wM\varphi_{a-1}(R^n) \) if and only if \( \rho \) and \( \varphi \) satisfy (2.36) for all \( R > 0 \).

We move on to the vector-valued inequality for \( M\rho \) on generalized Orlicz–Morrey spaces and generalized weak Orlicz–Morrey spaces.

**Theorem 2.57.** Let \( 0 < a < 1 < q < \infty \) and let \( 1 \leq u < \infty \). Let \( \varphi \in \mathcal{G}_q \). Assume that \( \lim_{t \downarrow 0} \varphi(t) = 0 \) and \( \lim_{t \to \infty} \varphi(t) = \infty \).

1. If \( \rho \) and \( \varphi \) satisfy (2.3) and (2.36), then for \( \{f_j\}_{j=1}^{\infty} \subset M\varphi_q(R^n) \)
\[
\left\| \left( \sum_{j=1}^{\infty} M\rho f_j^u \right)^{\frac{1}{u}} \right\|_{M\varphi_{a-1}} \lesssim \left\| \left( \sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{M\varphi_{a}}.
\]
2. Conversely, if
\[
\left\| \left( \sum_{j=1}^{\infty} M\rho f_j^u \right)^{\frac{1}{u}} \right\|_{wM\varphi_{a-1}} \lesssim \left\| \left( \sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{M\varphi_{a-1}}
\]
for \( \{f_j\}_{j=1}^{\infty} \subset M\varphi_q(R^n) \), then \( \rho \), \( \varphi \) and \( \psi \) satisfy (2.36). Moreover, under the assumption that \( \rho \sim \varphi/\psi \), inequality (2.38) holds if and only if \( \varphi \) satisfies (2.3).

**Proof.**

1. Using (2.36), we may assume that \( \rho = \varphi^{1-a} \). Then since \( \varphi \) is a doubling function and \( 0 < a < 1 \), we have
\[
M\varphi^{1-a} f_j \lesssim I_{\varphi^{1-a}} |f_j|.
\]
Thus,
\[
\left( \sum_{j=1}^{\infty} M_\rho f_j \right)^{\frac{1}{u}} \lesssim \left( \sum_{j=1}^{\infty} (I_{\varphi^{1-a}} |f_j|)^u \right)^{\frac{1}{u}} \lesssim I_{\varphi^{1-a}} \left[ \left( \sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right].
\]

It remains to resort to the boundedness of \(I_{\varphi^{1-a}}\) from \(Mq_\varphi(R^n)\) to \(M_{a-1}q_\varphi(R^n)\).

2. We let
\[
f_j = \begin{cases} f, & j = 1, \\ 0, & j \geq 1, \end{cases} \quad f \in M^q_\varphi(R^n).
\]
Then we have the boundedness of \(M_\rho\) on \(M^q_\varphi(R^n)\). Hence, by Theorem 2.54, we conclude that the inequality (2.36) holds.

Finally, under the assumption that \(\rho \sim \varphi/\psi\), we prove that inequality (2.38) holds if and only if \(\varphi\) satisfies (2.3). To do this, it is enough to show that (2.3) follows from (2.38). Now, assume that the integral condition (2.3) fails. Then, for any \(m \in \mathbb{N}\), there exists \(r_m > 0\) such that
\[
\varphi(2^m r_m) \leq 2 \varphi(r_m).
\]
Letting \(f_j = \chi_{[1,m]}(j) \chi_{B(2^m r_m) \setminus B(2^{j-1} r_m)}, j \in \mathbb{N}\), we have
\[
\|f_j\|_{M^q_\varphi(\ell^u)} \leq \left\| \chi_{B(2^m r_m)} \right\|_{M^q_\psi} \sim \varphi(2^m r_m) \leq 2 \varphi(r_m).
\]
(2.39)
Since \(\theta \in \mathcal{G}_n\) and \(\rho \sim \varphi/\psi = \varphi/\theta(\varphi)\), \(\rho(r) \leq \rho(s)\) for all \(r \leq s\). Due to this fact and the inequality \(M_\rho f_j \gtrsim \rho(2^j r_m) \chi_{B(r_m)}\), we have
\[
\|M_\rho f_j\|_{wM^q_1(\ell^u)} \gtrsim \left\| \sum_{j=1}^{\infty} \rho(2^j r_m)^u \chi_{B(r_m)} \right\|_{wM^q_1} \gtrsim \rho(2^m r_m) \varphi(2 r_m) m^{\frac{1}{u}} \gtrsim \varphi(r_m) m^{\frac{1}{u}}.
\]
(2.40)
We combine the inequalities (2.39) and (2.40) with the boundedness of \(M_\rho\) from \(wM^q_1(\ell^u)\) to \(M^q_1(\ell^u)\) to obtain \(m \leq D\) where \(D\) is independent of \(m\), contradicting the fact that \(m \in \mathbb{N}\) is arbitrary. Thus the integral condition (2.3) holds.