VOLUME AND STRUCTURE OF HYPERBOLIC 3-MANIFOLDS

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Abstract. In this paper, we show that Gromov-Thurston's principle holds for hyperbolic 3-manifolds of infinite volume and with finitely generated fundamental group. As an application, we give a new proof of Ending Lamination Theorem. Our proof essentially relies only on Maximum Volume Law for hyperbolic 3-simplices.

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Let $f : M \longrightarrow M'$ be a proper degree-one map between oriented hyperbolic 3-manifolds of finite volume. In [Th1, Theorem 6.4], Thurston proved by using

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results of Gromov [Gr] that f is properly homotopic to an isometry if and only if $\mathrm{Vol}(M) = \mathrm{Vol}(M')$. This theorem suggests us Gromov-Thurston's principle on hyperbolic manifolds of dimension three (or more) that "*Volume determines the structure*". This principle is essentially supported by *Maximum Volume Law*, which says that a hyperbolic 3-simplex has the maximum volume $\boldsymbol{v}_3 = 1.01494\dots$ if and only if it is a regular ideal simplex, see [Th1, Chapter 7]. The main tool for connecting the rigidity with the volume is the smearing 3-cycle $z_M(\sigma)$ on M associated with a straight 3-simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$, which is introduced in [Th1, Chapter 6].

Now we consider the case when M is an oriented hyperbolic 3-manifold of infinite volume and with finitely generated fundamental group. Then, instead of the volume of M, we use the bounded 3-cocycle ω_M on M such that, for any singular 3-simplex $\tau : \Delta^3 \longrightarrow M$, $\omega_M(\tau)$ is the oriented volume of the straightened 3-simplex straight(τ) of τ. Suppose that any ends of M are incompressible and there exists an orientation and parabolic cusp-preserving homeomorphism $\varphi : M \longrightarrow M'$ to another oriented hyperbolic 3-manifold M'. Let Y be any infinite volume submanifold of M, possibly $Y = M$. Then, for the restriction $z_Y(\sigma)$ of $z_M(\sigma)$ on Y, the value of $\omega_M(z_Y(\sigma))$ is infinite. In such a case, we consider an expanding sequence of compact submanifolds X_n of Y with $\bigcup_{n=1}^{\infty} X_n = Y$ and substitute the restrictions $z_{X_n}(\sigma)$ for $z_Y(\sigma)$. The map φ is said to satisfy the *ω-upper bound condition* on Y if there exists a constant $c_0 > 0$ and submanifolds X_n as above such that

$$(\omega_M - \varphi^*\omega_{M'})(z_{X_n}(\sigma)) < c_0 \tag{0.1}$$

for any n and any straight simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ with $\mathrm{Vol}(\sigma) > 1$. Here we do not need the assumption that $(\omega_M - \varphi^*\omega_{M'})(z_{X_n}(\sigma)) > -c_0$. The lower bound '1' of $\mathrm{Vol}(\sigma)$ is chosen just as a constant such that $\boldsymbol{v}_3 - 1$ is a positive small number.

The following theorem is our main result.

Theorem A. *Let E be a neighborhood of a simply degenerate end of M. If φ satisfies the ω-upper bound condition on E, then the restriction $\varphi|_E$ is properly homotopic to a bi-Lipschitz map onto a simply degenerate end of M'.*

Next we consider the case that φ satisfies the ω-upper bound condition. Let $f_n : \Sigma(\sigma_n) \longrightarrow M$ $(n = 0, 1, 2, \dots)$ be pleated maps tending toward a simply degenerate end $\mathcal{E}$ of M and $\Sigma(\sigma_n)$ the surface Σ with the hyperbolic structure σ_n induced from that on M via f_n. Suppose that f_n realizes a measured lamination β_n, which is normalized so that $\mathrm{length}_{\sigma_0}(\beta_n)$ is equal to one. Then $\{\beta_n\}$ has a subsequence converging to a measured lamination ν in $\Sigma(\sigma_0)$. The support of ν is independent of the choice of the subsequence and called the *ending lamination* of $\mathcal{E}$. From the definition, β_n is arbitrarily close to ν in $\Sigma(\sigma_0)$ for all sufficiently large n. However, the realization ν_n of ν in $\Sigma(\sigma_n)$ is not necessarily close to β_n. So it would be possible to encounter unknown phenomena by observing the lamination ν_n with the 'moving' hyperbolic structures σ_n on Σ. In fact, the following theorem is proved by analyzing a limit lamination ν_∞ of ν_n in a geometric limit surface Σ_∞ of $\Sigma(\sigma_n)$.

Theorem B. *Suppose that E is a neighborhood of a simply degenerate end $\mathcal{E}$ of M and $E' = \varphi(E)$ is also a neighborhood of a simply degenerate end $\mathcal{E}'$ of M'. If $\mathcal{E}$ and $\mathcal{E}'$ have the same ending lamination, then either φ satisfies the ω-upper bound condition on E or φ^{-1} does on E'.*

Ending Lamination Theorem is a rigidity theorem for infinite volume hyperbolic 3-manifolds proved by Minsky partially collaborating with some authors, see [MM, Mi1, Mi2, BCM] and so on. In the original proof, the theory of curve complex is crucial. In particular, the Gromov hyperbolicity of curve complex [MM, Bow1] and Length Upper Bound Lemma for tight geodesics [Mi2, Bow2] are the two main pillars supporting the proof.

By Theorems A and B, we have an alternative proof of Ending Lamination Theorem without relying on the theory of curve complex.

Corollary C. *Suppose that $\varphi : M \longrightarrow M'$ preserves the end invariants, i.e. conformal structures on geometrically finite ends and ending laminations on simply degenerate ends. Then φ is properly homotopic to an isometry.*

This corollary says that Gromov-Thurston's principle is valid for hyperbolic 3-manifolds of infinite volume. For simplicity, we consider only the case when ends of hyperbolic 3-manifolds are incompressible. It would be possible to generalize our argument to the compressible end case by using the topological tameness theorem for hyperbolic 3-manifolds (Agol [Ag], Calegari-Gabai [CG]) and applying Canary's branched covering trick [Ca].

This paper is organized as follows. Section 1 recalls standard notations on hyperbolic geometry. Besides we construct normalized maps with certain bounded geometry by using pleated maps. Normalized maps have the advantage that they are embeddings to a hyperbolic 3-manifold M. Section 2 presents the decomposition of a neighborhood E of a simply degenerate end $\mathcal{E}$ of M by normalized maps tending toward $\mathcal{E}$, where the ubiquity of pleated maps in E are used essentially. In Section 3, smearing 3-chains $z_X(\sigma)$ supported on almost compact subsets X in E are defined. We consider there a continuous map $\psi : M \longrightarrow M'$ 'essentially' equal to a homeomorphism φ satisfying the ω-upper bound condition on E. For a small $\eta > 0$, a straight singular 3-simplex $\tau : \Delta^3 \longrightarrow M$ is *η-inefficient* if the volume of the 3-simplex obtained by straightening $\psi \circ \tau$ is not greater than $\boldsymbol{v}_3 - \eta$. It is shown that the ω-upper bound condition for φ on E implies that the η-inefficient 3-chains occupy only a small part of $\mathrm{supp}(z_X(\sigma))$ for any long blocks $X = N_{(n_0,n_1)}$ in E. In Section 4, we present the infinite volume version of results in [So2] for closed hyperbolic 3-manifolds. By using the notion of simplicial honeycombs, we will prove that the lift $\widetilde{\psi}$ of ψ to the universal covering $\mathbb{H}^3$ is approximated by the identity near the boundary S^2_∞ of $\mathbb{H}^3$ with respect to suitable coordinates on $\mathbb{H}^3$. In Section 5, we first construct a locally bi-Lipschitz map $\varphi^{(1)} : E_{\mathrm{thick}} \longrightarrow E' = \varphi(E)$ properly homotopic to $\varphi|_{E_{\mathrm{thick}}}$ and then extend $\varphi^{(1)}$ to a bi-Lipschitz map $\Phi_E : E \longrightarrow E'$, which proves Theorem A. In Section 6, we consider geometric limits of pleated maps, ending laminations and earthquakes and study their mutual relations. Let $f^{(\prime)}_\infty : \Sigma^{(\prime)}_\infty \longrightarrow E^{(\prime)}_\infty$ be a geometric limit of pleated maps $f^{(\prime)}_n : \Sigma^{(\prime)}_n \longrightarrow E^{(\prime)}$ tending toward the end $\mathcal{E}^{(\prime)}$ of $E^{(\prime)}$. Consider the realizations $\nu^{(\prime)}_n$ of the ending lamination $\nu^{(\prime)}$ of $\mathcal{E}^{(\prime)}$ in $\Sigma^{(\prime)}_n$ and their geometric limit $\nu^{(\prime)}_\infty$ in $\Sigma^{(\prime)}_\infty$. We investigate connections between ν_∞ and ν'_∞ under the assumption of $\nu = \nu'$ via φ. The main tool for comparing these laminations is supervising markings of Σ_n and Σ_∞ by a fixed hyperbolic surface $\Sigma^\natural$. As an application of these geometric limits, we will present Irreversibility Lemma (Lemma 6.9). In Section 7, by using the preceding lemma, we prove Volume Difference Boundedness Lemma (Lemma 7.1), which is a key to Theorem B.

1. Preliminaries

In this section, we present fundamental definitions and notations in forms suitable to our arguments. Refer to Thurston [Th1], Benedetti and Petronio [BP], Matsuzaki and Taniguchi [MT] and so on for other notations concerning hyperbolic geometry and to Hempel [He] for 3-manifold topology. For a subset A of a metric space $X = (X, d)$, the closure of A in X is denoted by $\overline{A}$. For any $r > 0$, the r-neighborhood $\{y \in X \mid d(y, \overline{A}) \leq r\}$ of $\overline{A}$ is denoted by $\mathcal{N}_r(A, X)$ or $\mathcal{N}_r(A)$ for short. In the case of $A = \{x\}$, we set $\mathcal{N}_r(\{x\}) = \mathcal{B}_r(x)$. For a constant c, $c(a_1, \dots, a_n)$ means that it depends on variables $a_1, \dots, a_n$.

A *Kleinian group* is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$. Throughout this paper, any Kleinian group Γ is supposed to be torsion-free, hence in particular the quotient map $p : \mathbb{H}^3 \longrightarrow \mathbb{H}^3/\Gamma = M$ is the universal covering. We always suppose that M has a uniquely determined hyperbolic structure with respect to which p is locally isometric and moreover M has the orientation compatible with the standard orientation on $\mathbb{H}^3$ via p.

Our definition of thin and thick parts of hyperbolic 3-manifolds are slightly different from standard ones.

Definition 1.1 (Thin and thick parts of hyperbolic 3-manifolds)**.** For a $\mu > 0$, the *pure μ-thin part* $M_{\text{p-thin}(\mu)}$ of M is the set of points $x \in M$ such that there exists a non-contractible loop l in M of length $\leq 2\mu$ and passing through x. The complement $M_{\text{p-thick}(\mu)} = M \setminus \mathrm{Int}M_{\text{p-thin}(\mu)}$ is called the *pure μ-thick part* of M. By the Margulis Lemma [Th1, Corollary 5.10.2], there exists a constant $\mu_* > 0$ independent of M, called a *Margulis constant*, such that, for any $0 < \mu \leq \mu_*$, each component of $M_{\text{p-thin}(\mu)}$ is either an equidistant tubular neighborhood of a simple closed geodesic, called a *Margulis tube*, in M or a parabolic cusp of type $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. The union $M_{\text{thin}(\mu)}$ of components of $M_{\text{p-thin}(\mu)}$ meeting $M_{\text{p-thin}(\mu/2)}$ non-trivially is called the *μ-thin part* of M and the complement $M_{\text{thick}(\mu)} = M \setminus \mathrm{Int}M_{\text{thin}(\mu)}$ is the *μ-thick part* of M. Then we have $M_{\text{p-thin}(\mu/2)} \subset M_{\text{thin}(\mu)} \subset M_{\text{p-thin}(\mu)}$. Let $M_{\text{cusp}(\mu)}$ be the union of cuspidal components of $M_{\text{thin}(\mu)}$ and $M_{\text{tube}(\mu)} = M_{\text{thin}(\mu)} \setminus M_{\text{cusp}(\mu)}$. In other words, $M_{\text{tube}(\mu)}$ is the union of Margulis tube components of $M_{\text{thin}(\mu)}$. The complement $M \setminus \mathrm{Int}M_{\text{cusp}(\mu)}$ is the *main part* of M and denoted by $M_{\text{main}(\mu)}$.

Remark 1.2. The pure μ-thin part $M_{\text{p-thin}(\mu)}$ may have a Margulis tube component with very small normal radius. In such a case, the boundedness of geometry on $M_{\text{p-thick}(\mu)}$ (see Subsection 1.1) would not be estimated by the constant μ. On the other hand, the normal radius of any component of $M_{\text{tube}(\mu)}$ with respect to our definition is greater than a constant $c(\mu) > 0$ depending only on μ.

For any $a, b > 0$, consider the subset $\widetilde{P}(a, b) = \{(z, t) \mid 0 \leq \mathrm{Re}(z) \leq a, t \geq b\}$ of $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$. Let $P(a, b)$ be the quotient space of $\widetilde{P}(a, b)$ by the action on $\mathbb{H}^3$ generated by the isometry $(z, t) \mapsto (z + \sqrt{-1}, t)$. A submanifold P of M is called a *finite parabolic cusp* if P is either a $\mathbb{Z} \times \mathbb{Z}$-component of $M_{\text{cusp}(\mu)}$ or isometric to $P(a, b)$ for some $a, b > 0$. We say that a subspace of M is *almost compact* if it is a union of a compact set and finitely many finite parabolic cusps of M.

Assumptions. Let Γ and Γ' be finitely generated non-abelian Kleinian groups. Suppose that there exists an orientation-preserving homeomorphism $\varphi : M = \mathbb{H}^3/\Gamma \longrightarrow M' = \mathbb{H}^3/\Gamma'$ which induces a bijection between the components of $M_{\text{cusp}(\mu)}$ and those of $M'_{\text{cusp}(\mu)}$. By Scott-McCullough's Core Theorem [Sc, MC], there exists a compact connected submanifold $C_{\rm main}$ of $M_{{\rm main}(\mu)}$ such that (i) the inclusion $C_{\rm main} \subset M_{{\rm main}(\mu)}$ is a homotopy equivalence, (ii) $C_{\rm main} \cap V$ is an annulus in ∂V for any $\mathbb{Z}$-cusp component V of $M_{{\rm cusp}(\mu)}$, and (iii) ∂V is a torus component of $\partial C_{\rm main}$ for any $\mathbb{Z} \times \mathbb{Z}$-cusp component V of $M_{{\rm cusp}(\mu)}$. In particular, the properties (ii) and (iii) imply that any end of $M_{{\rm main}(\mu)}$ contains no accidental parabolic cusps. A submanifold C of M is called a *finite core* if $C \cap M_{{\rm main}(\mu)} = C_{\rm main}$ and $C \cap V$ is a finite parabolic cusp for any component V of $M_{{\rm cusp}(\mu)}$. Throughout this paper, we suppose that any component Σ of ∂C is incompressible in C. Any end $\mathcal{E}$ of $M_{{\rm main}(\mu)}$ is simply called an end of M. The closure E of the component of $M \setminus \Sigma$ adjacent to $\mathcal{E}$ is said to be the *neighborhood* of $\mathcal{E}$ with respect to C. The end $\mathcal{E}$ is *geometrically finite* if one can choose C so that it is locally convex on a neighborhood of Σ in M. Otherwise $\mathcal{E}$ is *geometrically infinite*. According to Bonahon [Bo], any geometrically infinite end $\mathcal{E}$ is *simply degenerate*, that is, there exists a sequence of closed geodesics λ_n^* in E diverging toward $\mathcal{E}$ and freely homotopic in E to a simple closed curve λ_n in Σ. Note that E is homeomorphic to $\Sigma \times [0, \infty)$, see [Th1, Theorem 9.4.1] and [Bo, Corollaire C]. Fix a complete hyperbolic structure on Σ of finite area and realize each λ_n as a simple geodesic loop in Σ. Then the sequence of the normalized simple closed geodesics $r_n\lambda_n$ with $r_n = 1/{\rm length}_\Sigma(\lambda_n)$ has a subsequence converging to a measured lamination ν in Σ. The support ${\rm supp}(\nu)$ of ν is independent of the choice of the diverging sequence λ_n^* or that of the subsequence of $r_n\lambda_n$, which is called the *ending lamination* of $\mathcal{E}$, see [Th1, Section 9.3].

1.1. **Pleated maps, revisited.** First we review some results concerning pleated maps.

Let C be a finite core of M. Fix a Margulis constant $\mu_0 > 0$ such that C is disjoint from $M_{{\rm tube}(\mu_0)}$ and $M_{{\rm tube}(\mu_0)} \cap E$ is unknotted and unlinked in E in the sense of Otal [Ot] for any end neighborhood E with respect to C. Suppose that E is the neighborhood of a simply degenerate end $\mathcal{E}$ with respect to C. We set $E \cap M_{{\rm thick}(\mu_0)} = E_{{\rm thick}(\mu_0)}$, $E \cap M_{{\rm cusp}(\mu_0)} = E_{{\rm cusp}(\mu_0)}$ and so on. A proper homotopy equivalence $f : \Sigma(\sigma) \longrightarrow E$ is called a *pleated map* realizing a geodesic lamination λ in $\Sigma(\sigma)$ if f satisfies the following conditions, where σ is a hyperbolic structure on Σ.

- For any rectifiable path α in $\Sigma(\sigma)$, its image $f(\alpha)$ is also a rectifiable path in E with ${\rm length}_{\Sigma(\sigma)}(\alpha) = {\rm length}_E(f(\alpha))$.
- $f(l)$ is a geodesic in E for each leaf l of λ.
- For each component Δ of $\Sigma \setminus \lambda$, the restriction $f|_\Delta$ is a totally geodesic immersion into E

We say that the lamination λ is a *bending locus* of f or *realized* in E by f. In the case when Δ is a neighborhood of a cusp of Σ, the last condition is guaranteed by [Th1, Corollary 9.5.6]. Since ${\rm Area}_{\Sigma(\sigma)}(\lambda) = 0$, these conditions imply that, for any Borel subset A of Σ, ${\rm Area}_{\Sigma(\sigma)}(A) = {\rm Area}_E(f(A))$. If necessary adding finitely many simple geodesics to λ, we may assume that any pleated maps $f : \Sigma \longrightarrow E$ in this paper satisfy the following extra conditions.

- Each component Δ of $\Sigma \setminus \lambda$ is either a maximal ideal 2-simplex or a once-punctured mono-gon. In the former case, $f(\Delta) \cap E_{{\rm cusp}(\mu)} = \emptyset$ for a sufficiently small $\mu > 0$. In the latter case, $f(\Delta) \cap E_{{\rm cusp}(\nu)} \neq \emptyset$ for any $\nu > 0$.

Such a lamination λ is called *full* in Σ. Here Δ being a *maximal* 2-simplex means that Δ is isometric to an ideal 2-simplex in $\mathbb{H}^2$ such that all the vertices are points at infinity, or equivalently $\mathrm{Area}(\Delta) = \pi$.

For a pleated map $f : \Sigma(\sigma) \longrightarrow E$, set $Y(f) = f^{-1}(E_{\mathrm{thin}(\mu_0)})$ and $F(f) = f^{-1}(E_{\mathrm{thick}(\mu_0)})$. If necessary deforming f slightly, we may assume that each component of the boundary $\partial Y(f)$ is a (non-smooth) simple loop in Σ.

Lemma 1.3. *For any component Y_0 of $Y(f)$, the following* (1) *and* (2) *hold.*

(1) *The inclusion $\iota : Y_0 \longrightarrow \Sigma$ is π_1-injective.*
(2) *Y_0 is either a disk or an annulus or a once-punctured disk.*

Proof. (1) Let V be the component of $E_{\mathrm{thin}(\mu_0)}$ containing $f(Y_0)$. If the inclusion $\iota : Y_0 \longrightarrow \Sigma$ were not π_1-injective, then there would exist a component β of ∂Y_0 which bounds a disk D in $\Sigma \setminus \mathrm{Int}Y_0$. Since the inclusion $V \subset E$ is π_1-injective and since $f|_\beta$ is contractible in E, $f|_\beta$ is contractible also in V. It follows that $f|_D : D \longrightarrow E$ is homotopic rel. β to a map into V. Any component α of $D \cap \lambda$ is an arc such that $f(\alpha)$ is geodesic in E which is homotopic into V rel. $\partial\alpha$. Since V is locally convex in E, $f(\alpha)$ itself is contained in V. Since the restriction of f on any component Δ of $D \setminus \lambda$ is totally geodesic and $f(\partial\Delta) \subset V$, $f(\Delta)$ is contained in V and hence $f(D) \subset V$. So we have $D \subset Y_0$, a contradiction. It follows that $\iota : Y_0 \longrightarrow \Sigma$ is π_1-injective.

(2) Since $f \circ \iota : Y_0 \longrightarrow V$ is π_1-injective and $\pi_1(V)$ is isomorphic to $\mathbb{Z}$, $\pi_1(Y_0)$ is either trivial or isomorphic to $\mathbb{Z}$. Thus Y_0 is either a disk or an annulus or a once-punctured disk. □

Let Λ_f be the core of $\Sigma(\sigma)_{\mathrm{tube}(\mu_0)}$ consisting of simple geodesic loops. Now we consider the case when a pleated map $f : \Sigma(\sigma) \longrightarrow E$ realizes Λ_f, that is, the bending locus of f contains Λ_f. Suppose that μ_0 is sufficiently small compared with a fixed Margulis constant μ_*. For any components V_0 of $E_{\mathrm{tube}(\mu_0)}$ and V_* of $E_{\mathrm{tube}(\mu_*)}$ with $V_0 \subset V_*$, $\mathrm{dist}(\partial V_*, V_0)$ is greater than an arbitrarily large constant $r > 0$. Let Y_* be the component of $f^{-1}(E_{\mathrm{tube}(\mu_*)})$ with $f(Y_*) \cap E_{\mathrm{tube}(\mu_0)} \neq \emptyset$. If Y_* were a disk, then Y_* would contain a hyperbolic disk of radius r. This contradicts that $\mathrm{Area}(Y_*) < \mathrm{Area}(\Sigma(\sigma)) = -2\pi\chi(\Sigma)$ if r is large or equivalently μ_0 is small. As in Lemma 1.3, it follows that Y_* is an annulus, which has a topological core l_0. Then we have a pleated map $f_1 : \Sigma \longrightarrow E$ such that $f_1(\Lambda_{f_1}) = f(\Lambda_f) \cup \lambda_1$, where λ_1 is the geodesic core of a component of $E_{\mathrm{tube}(\mu_0)}$ freely homotopic to $f(l_0)$ in E. By repeating this process finitely many times, we have a pleated map $g : \Sigma(\tau) \longrightarrow E$ satisfying the following conditions.

(Y1) For the geodesic core l of any component of $\Sigma(\tau)_{\mathrm{tube}(\mu_0)}$, $g(l)$ is a geodesic loop in E.
(Y2) $\Sigma(\tau)_{\mathrm{thin}(\mu_0)}$ is a core of $Y(g)$, or equivalently, $F(g) := g^{-1}(E_{\mathrm{thick}(\mu_0)})$ is a core of $\Sigma(\tau)_{\mathrm{thick}(\mu_0)}$.

Abbreviations and uniform constants. From now on, we work under a fixed μ_0 and set $E_{\mathrm{thin}(\mu_0)} = E_{\mathrm{thin}}$, $E_{\mathrm{thick}(\mu_0)} = E_{\mathrm{thick}}$, $E_{\mathrm{cusp}(\mu_0)} = E_{\mathrm{cusp}}$ and $E_{\mathrm{main}(\mu_0)} = E_{\mathrm{main}}$ and so on. Moreover we say that a constant c is *uniform* if c depends only on the Euler characteristic $\chi(\Sigma)$ of Σ and μ_0. For example, a uniform constant $c = c(k, l)$ means that c is a constant depending only on $\chi(\Sigma), \mu_0$ and k, l.

For any element γ of $\mathrm{PSL}_2(\mathbb{C})$ and $x \in \mathbb{H}^3$, $\mathrm{tl}(\gamma, x) = \mathrm{dist}_{\mathbb{H}^3}(x, \gamma x)$ is the *translation length* of γ with respect to x. The *infimum translation length* $\mathrm{tl}(\gamma)$ of γ is defined by $\inf\{\mathrm{tl}(\gamma, x) \,|\, x \in \mathbb{H}^3\}$. In particular, if γ is parabolic, then $\mathrm{tl}(\gamma) = 0$.

Lemma 1.4. *Let $f : \Sigma(\sigma) \longrightarrow E$ be a pleated map satisfying* (Y1) *and* (Y2)*. For any component F of $\Sigma(\sigma)_{\mathrm{thick}}$, let x_F be a fixed point of $F \cap F(f)$. Then there exists a generator system $\gamma_1, \ldots, \gamma_u$ of $\pi_1(F, x_F)$ with $u \le u_0$ and such that*

$$\mu_0 < \mathrm{tl}(\gamma_j) \le \mathrm{tl}(\gamma_j, \widetilde{x}_F) < l_0$$

for some uniform constant $l_0 > 0$ and $u_0 \in \mathbb{N}$, where $\gamma_j \in \pi_1(F, x_F)$ is identified with the element of Γ uniquely determined from γ_j and a point $\widetilde{x}_F$ with $p(\widetilde{x}_F) = x_F$.

Proof. Since $\mathrm{diam}(F)$ is uniformly bounded, it is not hard to show that there exists a positive integer u_0 depending only on $\chi(\Sigma)$ and oriented closed curves $c_1, \ldots, c_u$ with $u \le u_0$ in F passing through x_F and satisfying the following conditions.

- The elements $\gamma_1, \ldots, \gamma_u$ of $\pi_1(F, x_F)$ represented by $c_1, \ldots, c_u$, respectively, form a generator system of $\pi_1(F, x_F)$.
- $\mathrm{length}_{\Sigma(\sigma)}(c_j) < l_0$ $(j = 1, \ldots, u)$ for a uniform constant $l_0 > 0$.
- Any c_j is not freely homotopic in F to a loop cyclically covering a simple loop in F.

The second condition shows that $\mathrm{tl}(\gamma_i, \widetilde{x}_F) < l_0$. The third condition implies that $f(c_j)$ is not freely homotopic into any component of $E_{\mathrm{thin}(\mu_0)}$. This shows $\mu_0 < \mathrm{tl}(\gamma_i)$. □

Bounding volume. Let C be a connected oriented 3-manifold such that the boundary ∂C is a disjoint union of smooth surfaces of finite type. Suppose that $\zeta : \partial C \longrightarrow E$ is a proper continuous and piecewise smooth map which is extended to a proper continuous and piecewise smooth map $Z : C \longrightarrow E$. Then the *bounding volume* $\mathrm{Vol}^{\mathrm{bd}}(\zeta)$ of ζ is defined by

$$\mathrm{Vol}^{\mathrm{bd}}(\zeta) = \int_C Z^*(\Omega_E),$$

where Ω_E is the volume form on E. It is a standard fact in homology theory that $\mathrm{Vol}^{\mathrm{bd}}(\zeta)$ is independent of the choice of the proper extension Z. Consider the case of $C = \Sigma \times [0, 1]$ and that $f_0 = \zeta|_{\Sigma \times \{0\}}$, $f_1 = \zeta|_{\overline{\Sigma} \times \{1\}}$ are proper homotopy equivalences. Here $\overline{\Sigma}$ means that it has the orientation opposite to that on Σ. Then we set $\mathrm{Vol}^{\mathrm{bd}}(\zeta) = \mathrm{Vol}^{\mathrm{bd}}(f_0, f_1)$. From the definition, $\mathrm{Vol}^{\mathrm{bd}}(f_1, f_0) = -\mathrm{Vol}^{\mathrm{bd}}(f_0, f_1)$ holds.

Lemma 1.5. *Let P be either a* 3-*ball or a solid torus and $\zeta : \partial P \longrightarrow E$ a continuous map satisfying the conditions as above. Then $|\mathrm{Vol}^{\mathrm{bd}}(\zeta)| \le \mathrm{Area}(\partial P)$, where $\mathrm{Area}(\partial P)$ is the (absolute) area of ∂P with respect to the metric on ∂P induced from that on E via ζ.*

Proof. First we consider the case that P is a 3-ball. Then a proper extension $Z : P \longrightarrow E$ of ζ has a lift $\widetilde{Z} : P \longrightarrow \mathbb{H}^3$. We identify P with the unit ball $\mathbb{D}^3 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, \|\boldsymbol{x}\| \le 1\}$ by an orientation-preserving diffeomorphism. Fix a point $v_0 \in \mathbb{H}^3$. Let $\widetilde{X} : \mathbb{D}^3 \longrightarrow \mathbb{H}^3$ be the map extending $\widetilde{\zeta} = \widetilde{Z}|_{\partial P}$ such that, for any $x \in \partial \mathbb{D}^3$, $\widetilde{X}|_{[\mathbf{0}, x]}$ is the affine map onto the geodesic segment in $\mathbb{H}^3$ connecting v_0

with $Z(x)$, where $[\mathbf{0}, x]$ is the straight segment in $\mathbb{D}^3$ connecting the origin $\mathbf{0}$ with x. Then we have

$$\mathrm{Vol}^{\mathrm{bd}}(\zeta) = \int_P Z^*(\Omega_E) = \int_P \widetilde{Z}^*(\Omega_{\mathbb{H}^3}) = \int_{\mathbb{D}^3} \widetilde{X}^*(\Omega_{\mathbb{H}^3}).$$

For any straight 2-simplex Δ in $\mathbb{H}^3$, let $v_0 * \Delta$ be the 3-simplex in $\mathbb{H}^3$ obtained by suspending Δ with v_0. Then it is well known that $\mathrm{Vol}(v_0 * \Delta) \le \mathrm{Area}(\Delta)$. This shows that $|\mathrm{Vol}^{\mathrm{bd}}(\zeta)| \le \mathrm{Area}(\partial P)$.

Next we consider the case that P is a solid torus. Let D be a meridian disk of P. Consider the cyclic n-fold covering $p_n : P_n \longrightarrow P$. Cutting open P_n along a lift D_n of D, we get a 3-ball C_n. By the former result on 3-balls,

$$|\mathrm{Vol}(P_n)| = |\mathrm{Vol}(C_n)| \le \mathrm{Area}(\partial C_n) = \mathrm{Area}(\partial P_n) + 2\mathrm{Area}(D_n).$$

Since $|\mathrm{Vol}(P_n)| = n|\mathrm{Vol}(P)|$, $\mathrm{Area}(\partial P_n) = n\mathrm{Area}(\partial P)$ and $\mathrm{Area}(D_n) = \mathrm{Area}(D)$, it follows that

$$\begin{aligned}|\mathrm{Vol}^{\mathrm{bd}}(\zeta)| &= \frac{1}{n}|\mathrm{Vol}(P_n)| \le \frac{1}{n}(\mathrm{Area}(\partial P_n) + 2\mathrm{Area}(D_n)) \\ &= \mathrm{Area}(\partial P) + \frac{2}{n}\mathrm{Area}(D).\end{aligned}$$

The required inequality is obtained by letting $n \to \infty$. □

Bounded geometry. For metric spaces (X, d_X) and (Y, d_Y), we say that a homeomorphism $h : X \longrightarrow Y$ is *K-bi-Lipschitz* for $K \ge 1$ if

$$\frac{1}{K} d_X(x_0, x_1) \le d_Y(h(x_0), h(x_1)) \le K d_X(x_0, x_1)$$

for any $x_0, x_1 \in X$. Here we consider the case that X_n ($n \in \mathbb{N}$) and Y_0 are complete Riemannian manifolds of the same dimension with the base points x_n and y_0 respectively. A sequence $\{(X_n, x_n)\}$ is said to *converge geometrically* to a (Y_0, y_0) if there exist sequences $\{R_n\}$, $\{K_n\}$ with $R_n \nearrow \infty$ and $K_n \searrow 1$ and a K_n-bi-Lipschitz map $h_n : \mathcal{B}_{R_n}(x_n, X_n) \longrightarrow \mathcal{B}_{R_n}(y_0, Y)$ for each $n \in \mathbb{N}$.

We will apply a standard argument of bounded geometry together with the theory of geometric convergence. As a typical example, consider the geometric convergence of pleated maps $f_n : \Sigma(\sigma_n) \longrightarrow M_n$. Take a base point y_n of $\Sigma(\sigma_n)$ in a component F_n of $\Sigma(\sigma_n)_{\mathrm{thick}}$. Then $f_n(y_n)$ is contained in the thick part $M_{n,\mathrm{thick}(\mu)}$ for some Margulis constant μ less than μ_0. Otherwise, since the diameter of the component F_n is uniformly bounded, $f_n(F_n)$ would be contained in the component V of $E_{\mathrm{thin}(\mu)}$ for some n. Then the non-abelian group $f_{n*}(\pi_1(F_n))$ would be a subgroup of the abelian group $\pi_1(V)$, a contradiction. Thus $\{(M_n, f_n(y_n))\}$ has a subsequence converging geometrically to a hyperbolic 3-manifold (M_∞, y_∞), see [Th1, Corollary 9.1.7]. By the Ascoli-Arzelà Theorem, we may assume that $f_n|_{F_n} : F_n \longrightarrow M_n$ converges to a sub-pleated map $f_\infty|_{F_\infty} : F_\infty \longrightarrow M_\infty$ up to marking. This suggests us that, in many cases, it suffices only to consider the situation of $f_\infty|_{F_\infty}$ to know common geometric properties on $f_n|_{F_n}$ ($n = 1, 2, \dots$). A similar argument works for a sequence of proper least area maps to thick parts of hyperbolic 3-manifolds. However, we should remind that one can not apply such an argument to obtain common geometric properties on thin parts.

1.2. **Combined pleated maps and normalized maps.** Let $g : \Sigma \longrightarrow E$ be a pleated map satisfying the conditions (Y1) and (Y2) in Subsection 1.1. Then, for a component F of $F(g)$, we say that the sub-pleated map $g|_F$ is *unwrapped* if $g|_F$ is properly homotopic in E_{thick} to an embedding. A proper homotopy equivalence $f : \Sigma \longrightarrow E$ is called a *combined pleated map* if $f|_F$ is an unwrapped sub-pleated map for each component F of $F(f) = f^{-1}(E_{\mathrm{thick}})$ and and $f|_Y$ is either a ruled annulus or a totally geodesic once-punctured disk for each component Y of $Y(f) = f^{-1}(E_{\mathrm{thin}})$. Note that, for two components F_1, F_2 of $F(f)$, $f|_{F_1}$ and $f|_{F_2}$ are not necessarily assumed to be restrictions of the same pleated map.

Now we define a proper homotopy equivalence embedding associated with a combined pleated map $f : \Sigma \longrightarrow E$. For any component F of $F(f)$, consider an embedding $h_F : F \longrightarrow E_{\mathrm{thick}}$ satisfying one of the following two conditions.

- The intersection $f(\partial F) \cap E_{\mathrm{tube}} (= f(F) \cap E_{\mathrm{tube}})$ is non-empty. By modifying slightly the hyperbolic metric on E_{thick} in a small collar neighborhood of $\partial E_{\mathrm{thick}}$ in E_{thick}, we have a new metric such that $\partial E_{\mathrm{thick}}$ is locally convex in E_{thick}. By Freedman-Hass-Scott [FHS], $f|_F$ is properly homotopic in E_{thick} to a least area embedding h_F. Then we say that h_F is a least area map of *type I*. The least area property implies that $h_{F_1}(F_1) \cap h_{F_2}(F_2)$ is empty for any distinct components F_i $(i = 1, 2)$ of $F(f)$ with $f(\partial F_i) \cap E_{\mathrm{tube}} \neq \emptyset$.
- The intersection $f(\partial F) \cap E_{\mathrm{tube}}$ is possibly either empty or non-empty. Modify the metric on E_{thick} the 1-neighborhood $\mathcal{N}_1(f(F), E_{\mathrm{thick}})$ of $f(F)$ in E_{thick} such that the boundary $\partial \mathcal{N}_1(f(F))$ is locally convex in $\mathcal{N}_1(f(F))$. Again by Freedman-Hass-Scott [FHS], there exists an embedding $h_F : F \longrightarrow \mathcal{N}_1(f(F))$ which has least area among all piecewise smooth maps $h'_F : F \longrightarrow \mathcal{N}_1(f(F))$ properly homotopic to $f|_F$ *in* E_{thick}. Then we say that h_F is a least area map of *type II.*

Definition 1.6 (Normalized maps)**.** An embedding $\widehat{f} : \Sigma \longrightarrow E$ is called a *normalized map* associated with the combined pleated map f if the following two conditions hold.

- For any component F of $F(f)$, $\widehat{f}|_F$ is a least area map either of type I or II.
- For any component Y of $Y(f)$, $\widehat{f}(Y)$ is either a least area annulus or a totally geodesic once-punctured disk embedded in E_{thin}.

If $\widehat{f}|_F$ is a least area map of type I for all components F of $F(f)$, then we say that $\widehat{f}$ is a normalized map of *type I.*

Lemma 1.7. *Let* $\widehat{f} : \Sigma(\widehat{\sigma}) \longrightarrow E$ *be a normalized map. Then the following* (1)–(3) *hold.*

(1) *There exists a uniform constant* $a_0 > 0$ *with* $\mathrm{Area}_{\widehat{\sigma}}(\Sigma) \leq a_0$*.*
(2) *There exists a uniform constant* $d_0 > 0$ *with* $\mathrm{diam}_{\widehat{\sigma}}(F) \leq d_0$ *for any component* F *of* $F(\widehat{f})$*.*
(3) *For any* $d > 0$*, there exists a uniform constant* $v_0(d) > 0$ *with* $\mathrm{Vol}(\mathcal{N}_d(\widehat{f}(\Sigma))) < v_0(d)$*.*

Proof. (1) For any component F of $F(\widehat{f})$, $\mathrm{Area}_{\widehat{\sigma}}(F) \leq \mathrm{Area}_{\sigma_f}(F) \leq -2\pi\chi(\Sigma)$. Since $F \subset \Sigma(\widehat{\sigma})_{\mathrm{thick}}$, a standard argument of bounded geometry on least area maps shows that there exists a uniform constant $\widehat{l} > 0$ with $\mathrm{length}_{\widehat{\sigma}}(b) \leq \widehat{l}$ for any component b of $\partial F(\widehat{f})$. It follows that $\mathrm{Area}_{\widehat{\sigma}}(A) \leq 2\widehat{l}$ for any component A of $A(\widehat{f})$, where $A(\widehat{f})$ is the union of annulus components of $Y(\widehat{f})$. From these facts, we have a required uniform constant $a_0 > 0$.

(2) The assertion (2) follows immediately from the assertion (1) and $\mathrm{length}_{\widehat{\sigma}}(b) \leq \widehat{l}$ for any component of ∂F.

(3) Again by an argument of bounded geometry, we know that $\mathrm{Vol}(\mathcal{N}_d(\widehat{f}(F)))$ is less than a uniform constant $v_0'(d) > 0$ for any component F of $F(\widehat{f})$. Since $\mathrm{Area}_{\widehat{\sigma}}(A) \leq a_0$ for any component A of $A(\widehat{f})$, one can have a uniform constant $v_0''(d) > 0$ with $\mathrm{Vol}(\mathcal{N}_d(\widehat{f}(A))) < v_0''$ by using an argument similar to that in [Th1, Proposition 8.12.1], where the π_1-injectivity of $\widehat{f}|_A$ in E_{thin} is crucial. By these facts, one can have a uniform constant v_0 satisfying the condition (3). □

2. Decomposition of neighborhoods of simply degenerate ends by normalized maps

Let $\mathcal{E}$ be a simply degenerate end of M and E the neighborhood of $\mathcal{E}$ with respect to a finite core of M. In this section, we consider a decomposition of E by normalized maps in E tending toward the end $\mathcal{E}$. For any proper homotopy equivalence $f : \Sigma \longrightarrow E$, the closure of the component of $E \setminus f(\Sigma)$ adjacent to $\mathcal{E}$ is denoted by $E^+(f)$. Let $f_0, f_1 : \Sigma \longrightarrow E$ be two proper embeddings which are homotopy equivalences with $f_0(\Sigma) \neq f_1(\Sigma)$ (possibly $f_0(\Sigma) \cap f_1(\Sigma) \neq \emptyset$). Then $f_0 < f_1$ means that $E^+(f_0) \supset f_1(\Sigma)$. A sequence $\{f_n\}$ of homotopy equivalence embeddings in E is said to be *monotone increasing* if $f_n < f_{n+1}$ for any n.

Let $f : \Sigma \longrightarrow E$ be a combined pleated map. A component F of $F(f)$ is *maximal* if any non-contractible simple loop l in F such that $f(l)$ is homotopic in E_{thick} to a loop in $\partial E_{\mathrm{tube}}$ is homotopic in F to a component of ∂F. A combined pleated map is *maximal* if $f|_F$ is maximal for any component F of $F(f)$. Fix a maximal combined pleated map $f_0 : \Sigma \longrightarrow E$. Let $W(f_0)$ be the union of components of E_{tube} meeting $f_0(F(f_0))$ non-trivially. Suppose that $E^+(f_0) \cap (E_{\mathrm{tube}} \setminus W(f_0)) \neq \emptyset$. Let $V_1, \dots, V_k$ be the components of $E_{\mathrm{tube}} \setminus W(f_0)$ contained in $E^+(f_0)$ and nearest to $f_0(\Sigma)$. That is, for $i = 1, \dots, k$, there exists a non-contractible simple loop l_i in Σ such that $f_0(l_i)$ is freely homotopic in $E_{\mathrm{thick}} \cup W(f_0)$ to a loop in ∂V_i. Since E_{tube} is unknotted and unlinked in E by Otal [Ot], $l_1, \dots, l_k$ are taken to be mutually disjoint in Σ. From the maximality of f_0, any l_i is not homotopic in Σ to any loop in $F(f_0)$ or $A(f_0)$. Let $G(f_0, l_i)$ be the union of components of $F(f_0)$ or $A(f_0)$ intersecting l_i homotopically essentially and $P(f_0, l_i)$ the union of components of $A(f_0)$ meeting $\partial G(f_0, l_i)$ non-trivially. We say that $G(f_0, l_i)$ is *minimal* if there are no loop l_j with $j \in \{1, \dots, k\}$ and $G(f_0, l_j) \subsetneq G(f_0, l_i)$. By renumbering l_i's, we may assume that $G(f_0, l_1)$ is minimal and $G(f_0, l_1)$ contains l_i if and only if $i = 1, \dots, k_0$ for some $k_0 \leq k$. From the minimality of $G(f_0, l_1)$, $G(f_0, l_1) = G(f_0, l_i)$ for $i = 2, \dots, k_0$. Then there exists a maximal combined pleated map $f_1 : \Sigma \longrightarrow E$ such that $f_1|_{\Sigma \setminus G(f_0, l_1) \cup P(f_0, l_1)} = f_0|_{\Sigma \setminus G(f_0, l_1) \cup P(f_0, l_1)}$ and $W(f_1) = W'(f_0, l_1) \cup V_1 \cup \cdots \cup V_{k_0}$, where $W'(f_0, l_1)$ is the union of components of $W(f_0)$ meeting $f_0(A(f_0) \setminus G(f_0, l_1))$ non-trivially, see Figure 2.1. If $E^+(f_1) \cap (E_{\mathrm{tube}} \setminus W(f_1)) \neq \emptyset$, one can define a maximal combined pleated map f_2 from f_1 similarly. Repeating this process as much as possible, we have a sequence $\{f_{\widehat{m}}\}_{\widehat{m}=1}^{\widehat{m}_+}$ (possibly $\widehat{m}_+ = \infty$) of maximal combined pleated maps in $E^+(f_0)$.

Let $\widehat{f}_{\widehat{m}}$ be a normalized map of type I derived from $f_{\widehat{m}}$ such that $\widehat{f}_{\widehat{m}}|_{F_1} = \widehat{f}_{\widehat{n}}|_{F_2}$ if $f_{\widehat{m}}|_{F_1} = f_{\widehat{n}}|_{F_2}$ for components F_1 of $F(f_{\widehat{m}})$ and F_2 of $F(f_{\widehat{n}})$. By [FHS], $\widehat{f}_{\widehat{m}}(F_1) \cap \widehat{f}_{\widehat{n}}(F_2) = \emptyset$ if $f_{\widehat{m}}|_{F_1} \neq f_{\widehat{n}}|_{F_2}$. From our construction of a sequence $\{f_{\widehat{m}}\}_{\widehat{m}=1}^{\widehat{m}_+}$, the

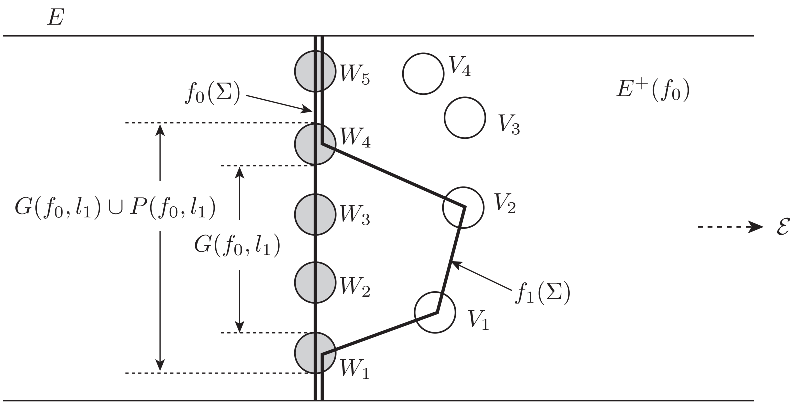


Figure 2.1. The case of $k_0 = 2$ and $G(f_0, l_1) = G(f_0, l_2)$. $W(f_0) = W_1 \cup \cdots \cup W_5$. $W'(f_0, l_1) = W_1 \cup W_4 \cup W_5$.

normalized sequence $\{\widehat{f}_{\widehat{m}}\}_{\widehat{m}=1}^{\widehat{m}_+}$ is monotone increasing. Set

$$\widehat{\mathcal{F}} = \bigcup_{\widehat{m}=1}^{\widehat{m}_+} \widehat{f}_{\widehat{m}}(\Sigma) \cap E_{\mathrm{thick}}. \tag{2.1}$$

Let R be the closure of a component of $E_{\mathrm{thick}} \setminus \widehat{\mathcal{F}}$, and let $\partial_1 R = \partial R \cap E_{\mathrm{thin}}$ and $\partial_0 R = \overline{\partial R \setminus \partial_1 R}$. If any neighborhood of the end $\mathcal{E}$ of E intersects E_{tube} non-trivially, then $\partial_0 R$ is contained in $\widehat{f}_{\widehat{m}}(\Sigma) \cup \widehat{f}_{\widehat{m}+1}(\Sigma)$ for some $\widehat{m}$. See Figure 2.3. If R is compact, then R contains a properly embedded compact surface H, called a *vertical core* of R, with $\partial H \subset \partial_1 R$ which admits a homeomorphism $h : H \times [-1, 1] \longrightarrow R$ with $h(H \times \{0\}) = H$ and $h(H \times \{-1, 1\}) \supset \partial_0 R$. If $\mathcal{E}$ has a neighborhood disjoint from E_{tube}, then there exists a unique component of $E_{\mathrm{thick}} \setminus \widehat{\mathcal{F}}$ the closure R_∞ of which is not compact. Then R_∞ is homeomorphic to $\Sigma_{\mathrm{main}} \times [0, \infty)$.

Let H' be a compact connected subsurface of H such that each component of $\partial H'$ is non-contractible in H, and let $\eta : H' \longrightarrow R$ be an embedding with $\eta(\partial H') \subset W(\widehat{f}_{\widehat{m}}) \cup W(\widehat{f}_{\widehat{m}+1})$ and such that $\eta(H')$ is isotopic in R to $h(H' \times \{0\})$ by a (possibly non-proper) isotopy. Then we have the following:

Claim 2.1. At least one of $\eta(\partial H') \cap \partial(W(\widehat{f}_{\widehat{m}}) \setminus W(\widehat{f}_{\widehat{m}+1}))$ and $\eta(\partial H') \cap \partial(W(\widehat{f}_{\widehat{m}+1}) \setminus W(\widehat{f}_{\widehat{m}}))$ is empty.

Otherwise, $\partial H'$ would contain components λ_0', λ_1' with $\eta(\lambda_0') \subset \partial(W(\widehat{f}_{\widehat{m}}) \setminus W(\widehat{f}_{\widehat{m}+1}))$ and $\eta(\lambda_1') \subset \partial(W(\widehat{f}_{\widehat{m}+1}) \setminus W(\widehat{f}_{\widehat{m}}))$. Let λ_i $(i = 0, 1)$ be a simple loop in $A(\widehat{f}_{\widehat{m}+i})$ such that $\widehat{f}_{\widehat{m}+i}(\lambda_i)$ is homotopic to $\eta(\lambda_i')$ in $W(\widehat{f}_{\widehat{m}+i})$ and $A(\lambda_0)$ the component of $A(\widehat{f}_{\widehat{m}})$ containing λ_0. Then $G(\widehat{f}_{\widehat{m}}, \lambda_1)$ is contained in $G(\widehat{f}_{\widehat{m}}, l_1) \setminus \mathrm{Int}A(\lambda_0)$. See Figure 2.2. This contradicts the minimality of $G(\widehat{f}_{\widehat{m}}, l_1)$.

Suppose that R has a point x with $\mathrm{dist}_R(x, \partial_0 R) > d_1 + 3$, where d_1 is a uniform constant with $\mathrm{diam}(F) \leq d_1$ for any hyperbolic structure σ on Σ and any component F of $\Sigma(\sigma)_{\mathrm{thick}}$. From the ubiquity of pleated maps, there exists a sub-pleated map $q : F' \longrightarrow E_{\mathrm{thick}}$ meeting the 1-neighborhood of x in R, see for example the proofs

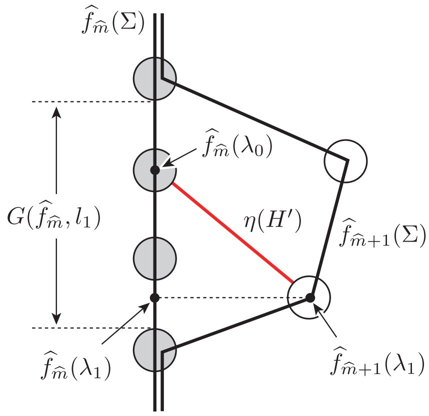


Figure 2.2

of Proposition 9.5.12 and Theorem 9.5.13 in [Th1]. Since $\mathrm{dist}_{E_{\mathrm{thick}}}(q(F'), \partial_0 R) > 2$, this implies that $q(F')$ is contained in R and F' is a subsurface of F. By Claim 2.1, there exists a normalized map $\widehat{f} : \Sigma \longrightarrow E$ with $\widehat{f}_{\widehat{m}} < \widehat{f} < \widehat{f}_{\widehat{m}+1}$, $\widehat{f}(F') \subset \mathcal{N}_1(q(F')) \cap E_{\mathrm{thick}}$ and such that $\widehat{f}(F(\widehat{f}) \setminus F')$ is contained in either $\widehat{f}_{\widehat{m}}(\Sigma)$ or $\widehat{f}_{\widehat{m}+1}(\Sigma)$. Figure 2.3 illustrates the case of $\widehat{f}(F(\widehat{f}) \setminus F') \subset \widehat{f}_{\widehat{m}}(\Sigma)$.

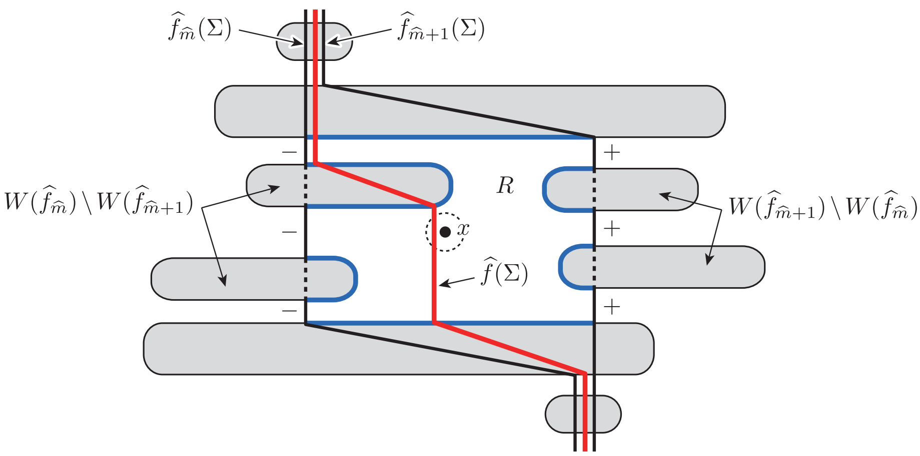


Figure 2.3. The union of blue segments and blue curves represents $\partial_1 R$. The union of vertical segments labelled with '+' or '−' is $\partial_0 R$.

Repeating the same argument for all such R and the closures of components of $R \setminus \widehat{f}(F)$, we have a monotone increasing sequence $\{\widehat{f}_n\}_{n=0}^{\infty}$ of normalized maps containing the original $\{\widehat{f}_{\widehat{m}}\}_{\widehat{m}=1}^{\widehat{m}_+}$ as a subsequence and tending toward the end $\mathcal{E}$ of E as $n \to \infty$. The union $\widehat{\mathcal{G}} = \bigcup_{n=0}^{\infty} \widehat{f}_n(\Sigma) \cap E_{\mathrm{thick}}$ contains $\widehat{\mathcal{F}}$. For any normalized maps $\widehat{g}_0, \widehat{g}_1 : \Sigma \longrightarrow E$ with $\widehat{g}_0 < \widehat{g}_1$, we write $E(\widehat{g}_0, \widehat{g}_1) = E^+(\widehat{g}_0) \setminus \mathrm{Int}E^+(\widehat{g}_1)$.

Moreover set $N_n = E(\widehat{f}_n, \widehat{f}_{n+1})$ and $N_{n,\mathrm{thick}} = N_n \cap E_{\mathrm{thick}}$. Let R_n be the closure of $\overline{\mathrm{Int}N_{n,\mathrm{thick}}}$. The boundary ∂R_n consists of $\partial_1 R_n = \partial R_n \cap E_{\mathrm{thin}}$ and $\partial_0 R_n = \overline{\partial R_n \setminus \partial_1 R_n}$. Note that $\partial_0 R_n$ is contained in $\widehat{f}_n(\Sigma) \cup \widehat{f}_{n+1}(\Sigma)$. We say that R_n is the *main part* of N_n, see Figure 2.4. Then the following (R1) and (R2) hold.

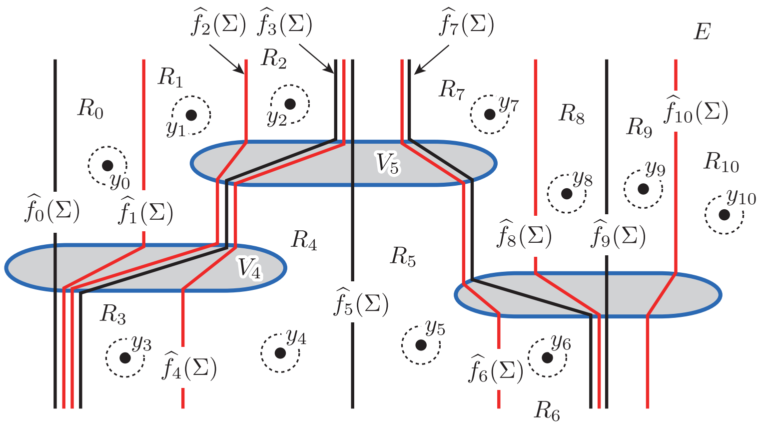


Figure 2.4

(R1) For any point x of R_n, $\mathrm{dist}_{R_n}(x, \partial_0 R_n) \leq d_0 + 3$.
(R2) If at least one of $\partial_0^- R_n = \partial_0 R_n \cap \widehat{f}_n(\Sigma)$ and $\partial_0^+ R_n = \partial_0 R_n \cap \widehat{f}_{n+1}(\Sigma)$ is disjoint from the union $\widehat{\mathcal{F}}$ defined as (2.1), then $\mathrm{dist}_{R_n}(\partial_0^- R_n, \partial_0^+ R_n) \geq 1$.

Summarizing the arguments as above, we have the following lemma.

Lemma 2.2. *For any $n \in \mathbb{N} \cup \{0\}$, there exist constants satisfying the following conditions:*

(1) *a uniform constant $d_2 > 0$ with $\mathrm{diam}(R_n) < d_2$,*
(2) *a uniform constant $V_1 > 0$ with $\mathrm{Vol}(N_n) < V_1$,*
(3) *a constant $r_0 > 0$ independent of $n \in \mathbb{N} \cup \{0\}$ such that R_n contains an embedded hyperbolic 3-ball B_n of radius r_0.*

If y_n is the center of B_n, then $B_n = \mathcal{B}_{r_0}(y_n) \subset R_n \subset N_n$. We regard that y_n is the base point of R_n and of N_n, see Figure 2.4.

Proof. (1) The assertion follows immediately from Lemma 1.7 (2) and (R1).

(2) By (1), $\mathrm{Vol}(R_n) = \mathrm{Vol}(N_{n,\mathrm{thick}})$ is uniformly bounded. The closure of the intersection $\mathrm{Int}N_n \cap E_{\mathrm{tube}}$ consists of at most $-3\chi(\Sigma)/2$ solid tori V. The boundary ∂V contains $\widehat{f}_n(A_n)$ and $\widehat{f}_{n+1}(A_{n+1})$ for some components A_i of $A(\widehat{f}_i)$ for $i = n, n+1$, where possibly one of A_n and A_{n+1} is empty. See V_4 and V_5 in Figure 2.4. Moreover the closure of $\partial V \setminus \widehat{f}_n(A_n) \cup \widehat{f}_{n+1}(A_{n+1})$ consists of at most two annuli with uniformly bounded diameter by (R1). It follows from this fact together with Lemma 1.7 (1) that $\mathrm{Area}(\partial V)$ is uniformly bounded. By Lemma 1.5, we have $\mathrm{Vol}(V) \leq \mathrm{Area}(\partial V)$. Again by using (R1), one can show that the volume of any component of $N_n \cap E_{\mathrm{cusp}}$ is uniformly bounded. This shows (2).

(3) If at least one component of $\partial_0 R_n$ is disjoint from $\widehat{\mathcal{F}}$, then the assertion follows from (R2). Otherwise, $\partial_0 R$ is contained in $\widehat{\mathcal{F}}$ and hence all components of $\partial_0 R$ are least area surfaces in E_{thick} which are not properly homotopic to each other. Such surfaces are not accumulate in E_{thick}. Thus the existence of $r_0 > 0$ as above is proved by an argument using a standard argument of bounded geometry. □

3. Smearing chains on 3-manifolds

3.1. **Definition and fundamental properties of smearing chains.** Suppose that $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold satisfying Assumptions in Section 1. Then the quotient map $p : \mathbb{H}^3 \longrightarrow M$ is a locally isometric universal covering. Let Δ^n be a regular k-simplex of edge length 1 in the Euclidean k-space. A singular k-simplex $\sigma : \Delta^k \longrightarrow M$ is called *straight* if its lift $\widetilde{\sigma} : \Delta^k \longrightarrow \mathbb{H}^3$ to $\mathbb{H}^3$ is *straight*, that is, $\widetilde{\sigma}$ is the affine map with respect to the Euclidean structure on Δ^3 and the quadratic model on $\mathbb{H}^3$. For any singular k-simplex $\widetilde{\sigma} : \Delta^k \longrightarrow \mathbb{H}^3$, let $\mathrm{straight}(\widetilde{\sigma}) : \Delta^k \longrightarrow \mathbb{H}^3$ be the straight map with $\mathrm{straight}(\widetilde{\sigma}(v_j)) = \widetilde{\sigma}(v_j)$ for all vertices v_j ($j = 0, 1, \dots, k$) of Δ^k. We note that the image $\mathrm{straight}(\widetilde{\sigma})(\Delta^k)$ is a (possibly degenerate) straight k-simplex in $\mathbb{H}^3$. For a singular k-simplex $\sigma : \Delta^k \longrightarrow M$, the map $\mathrm{straight}_M(\sigma) = p \circ \mathrm{straight}(\widetilde{\sigma}) : \Delta^k \longrightarrow M$ is called the k-simplex obtained by *straightening* σ, where $\widetilde{\sigma} : \Delta^k \longrightarrow \mathbb{H}^3$ is a lift of σ.

The *oriented volume* of a C^1 singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$ is defined by

$$\mathrm{Vol}(\sigma) = \int_{\Delta^3} \sigma^*(\Omega_M),$$

where Ω_M is the volume form on M. We say that σ is *non-degenerate* if $\mathrm{Vol}(\sigma) \neq 0$, and *positive* (resp. *negative*) if $\mathrm{Vol}(\sigma) > 0$ (resp. $\mathrm{Vol}(\sigma) < 0$).

Let ω_M be the 3-cocycle on M defined by

$$\omega_M(\sigma) = \mathrm{Vol}(\mathrm{straight}_M(\sigma))$$

for any singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$. Note that $|\omega_M(\sigma)|$ is less the volume $\boldsymbol{v}_3$ of a regular ideal 3-simplex in $\mathbb{H}^3$ for any singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$.

For any smooth manifold N, let $C^1(\Delta^k, N)$ be the topological space of C^1-maps $\Delta^k \longrightarrow N$ with C^1-topology. We denote by $\mathscr{C}_k(N)$ the $\mathbb{R}$-vector space consisting of Borel measures μ on $C^1(\Delta^k, N)$ with the bounded total variation $\|\mu\| < \infty$. An element of $\mathscr{C}_k(N)$ is called a *k-chain*. The boundary operator $\partial_k : \mathscr{C}_k(N) \longrightarrow \mathscr{C}_{k-1}(N)$ is defined naturally. Thus we have the chain complex $(\mathscr{C}_*(N), \partial_*)$.

Now we consider the case of $N = M$. Take the base point x_0 of $\mathbb{H}^3$ and suppose that $y_0 = p(x_0)$ is the base point of M. Let μ_{Haar} be a left-right invariant Haar measure on $\mathrm{PSL}_2(\mathbb{C})$, which is normalized so that, for any bounded Borel subset U of $\mathbb{H}^3$,

$$\mu_{\mathrm{Haar}}(\{\alpha \in \mathrm{PSL}_2(\mathbb{C}) \,|\, \alpha x_0 \in U\}) = \mathrm{Vol}(U). \tag{3.1}$$

From the invariance of μ_{Haar}, we know that the quotient map $q : \mathrm{PSL}_2(\mathbb{C}) \longrightarrow P(M) = \Gamma\backslash\mathrm{PSL}_2(\mathbb{C})$ induces the measure $\widehat{\mu}_{\mathrm{Haar}}$ on the quotient space $P(M)$. That is, $\widehat{\mu}_{\mathrm{Haar}}(q(\mathcal{A}))$ is equal to $\mu_{\mathrm{Haar}}(\mathcal{A})$ for any measurable subset $\mathcal{A}$ of $\mathrm{PSL}_2(\mathbb{C})$ with $\mathcal{A} \cap \gamma\mathcal{A} = \emptyset$ if $\gamma \in \Gamma \setminus \{1\}$. For any point $x \in \mathbb{H}^3$ and $a \in P(M)$, $a \bullet x$ denotes the point of M defined by $p(\alpha x)$ for an $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ with $q(\alpha) = a$. Note that the point does not depend on the choice of $\alpha \in q^{-1}(a)$. Thus the map

$$\bullet : P(M) \times \mathbb{H}^3 \longrightarrow M$$

is well-defined. For any singular 3-simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ and $a \in P(M)$, the singular 3-simplex $a \bullet \sigma : \Delta^3 \longrightarrow M$ is defined by $p \circ (\alpha\sigma)$ for an $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ with $q(\alpha) = a$.

Let $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ be a non-degenerate straight 3-simplex. Suppose that $\mathrm{smear}_M(\sigma)$ is the Borel measure on $C^1(\Delta^3, M)$ introduced in [Th1, Section 6.1], which satisfies the following conditions.

- The support $\mathrm{supp}(\mathrm{smear}_M(\sigma))$ is $\{a \bullet \sigma \,|\, a \in P(M)\}$.
- For any closed non-empty subset $\mathcal{X}$ of $P(M)$,

$$\mathrm{smear}_M(\sigma)(\{a \bullet \sigma \,|\, a \in \mathcal{X}\}) = \widehat{\mu}_{\mathrm{Haar}}(\mathcal{X}). \tag{3.2}$$

We denote the inner center of the straight 3-simplex $\sigma(\Delta^3)$ in $\mathbb{H}^3$ by $o(\sigma)$. For any non-empty almost compact subset X of M, the restriction of $\mathrm{smear}_M(\sigma)$ to $\{a \bullet \sigma \,|\, a \in P(M) \text{ with } a \bullet o(\sigma) \in X\}$ is denoted by $\mathrm{smear}_X(\sigma)$. By (3.1) and (3.2), its total variation is

$$\|\mathrm{smear}_X(\sigma)\| = \mathrm{Vol}(X). \tag{3.3}$$

In particular, $\mathrm{smear}_X(\sigma)$ is an element of $\mathscr{C}_3(M)$. Set $\sigma_- = \rho \circ \sigma$ for an orientation-reversing isometry ρ on $\mathbb{H}^3$ with $\rho(o(\sigma)) = o(\sigma)$. Consider the element $z_X(\sigma)$ of $\mathscr{C}_3(M)$ defined by

$$z_X(\sigma) = \frac{1}{2}(\mathrm{smear}_X(\sigma) - \mathrm{smear}_X(\sigma_-)). \tag{3.4}$$

Then, by (3.2) and (3.3), we have $\|z_X(\sigma)\| = \mathrm{Vol}(X)$ and

$$z_X(\sigma)(\{a \bullet \sigma \,|\, a \in P(M) \text{ with } a \bullet o(\sigma) \in X\}) = \frac{1}{2}\mathrm{Vol}(X).$$

For a Borel measure ω on $C^1(\Delta^3, M)$, let $\mathrm{supp}^{(2)}(w)$ be the subset of $C^1(\Delta^2, M)$ defined by

$$\mathrm{supp}^{(2)}(w) = \big\{\tau|_D \,\big|\, \tau \in \mathrm{supp}(w) \text{ and } D \in (\Delta^3)^{(2)}\big\}, \tag{3.5}$$

where $(\Delta^3)^{(2)}$ is the set of 2-faces of Δ^3. By the definition, $\mathrm{supp}(\partial_3 w) \subset \mathrm{supp}^{(2)}(w)$.

Lemma 3.1. *For any almost compact subset X of M, $\mathrm{supp}(\partial_3 z_X(\sigma))$ is contained in $\mathrm{supp}^{(2)}(z_{\mathcal{N}_2(\partial X, M)}(\sigma))$, where $\partial X = \overline{X} \setminus \mathrm{Int}X$. In particular, $\|\partial_3 z_X(\sigma)\| \le 4\mathrm{Vol}(\mathcal{N}_2(\partial X, M))$.*

Proof. The volume of any straight 3-simplex Δ in $\mathbb{H}^3$ is less than $\boldsymbol{v}_3 = 1.014916\ldots$. On the other hand, since the volume of a 3-ball in $\mathbb{H}^3$ of radius one is $\pi(\sinh 2 - 2) = 5.11093\ldots$, the radius of the inscribed ball in Δ is less than one. Let D be any element of $(\Delta^3)^{(2)}$. For any $a \bullet \sigma$ with $a \bullet o(\sigma) \in X$, there exists $b \in P(M)$ with $b \bullet o(\sigma_-) \in \mathcal{N}_2(X, M)$ and such that $a \bullet \sigma|_D = b \bullet \sigma_-|_D$. Similarly, we have $a \bullet \sigma_-|_D = b \bullet \sigma|_D$. Moreover, if $a \bullet o(\sigma) \in X \setminus \mathcal{N}_2(\partial X, M)$, then $b \bullet o(\sigma_-)$ is an element of X. This shows $\mathrm{supp}(\partial_3 z_X(\sigma)) \subset \mathrm{supp}^{(2)}(z_{\mathcal{N}_2(\partial X, M)})$. Since $\|z_{\mathcal{N}_2(\partial X, M)}(\sigma)\| = \mathrm{Vol}(\mathcal{N}_2(\partial X))$ and Δ^3 has four 2-faces, $\|\partial_3 z_X(\sigma)\| \le 4\mathrm{Vol}(\mathcal{N}_2(\partial X, M))$ holds. □

Since the image $\tau(\Delta^3)$ of any element $\tau = a \bullet \sigma \in \mathrm{supp}\{z_X(\sigma)\}$ has 'long tails', $\tau(\Delta^3)$ is not necessarily contained in X even if $a \bullet o(\sigma)$ is an element of $\mathrm{Int}X$ with $\mathrm{dist}(a \bullet o(\sigma), \partial X)$ large. So we sometimes need to treat the body (inner part) and tails (outer part) of $\tau(\Delta^3)$ separately as in the next section.

There exists $r = r(\Sigma) > 0$ such that, for any complete hyperbolic structure σ on Σ with $\mathrm{Area}(\Sigma(\sigma)) < \infty$, $\Sigma(\sigma)$ contains a disjoint union $\mathcal{H} = \lambda_1 \sqcup \cdots \sqcup \lambda_m$ of mutually disjoint simple closed geodesics satisfying the following conditions.

- For each component λ_j, $\mathrm{length}_\sigma(\lambda_j) < r$.
- $\mathcal{H}$ contains the geodesic cores of all components of $\Sigma(\sigma)_{\mathrm{tube}}$.
- The Euler characteristic of each component of $\Sigma(\sigma) \setminus \mathcal{H}$ is -1. In other words, $\mathcal{H}$ is a maximal disjoint union of simple closed geodesics in $\Sigma(\sigma)$.

We say that $\mathcal{H}$ is an *r-hoop family* of $\Sigma(\sigma)$. If our Margulis constant $\mu_0 > 0$ is sufficiently small, then the length of any simple closed geodesic in $\Sigma(\sigma)$ crossing components of $\Sigma_{\mathrm{tube}}(\sigma)$ is greater than r. So the second condition always holds. One can fix a constant $r > 0$ depending only on the topological type of Σ such that $\Sigma(\sigma)$ admits an r-hoop family $\mathcal{H} = \lambda_1 \sqcup \cdots \sqcup \lambda_m$. Then we say that $\mathcal{H}$ is just a hoop family of $\Sigma(\sigma)$.

Let M be a hyperbolic 3-manifold satisfying the conditions in Assumptions of Section 1. Suppose that $\mathcal{E}$ is a simply degenerate end of M and E is the neighborhood of $\mathcal{E}$ with respect to a finite core C. Since M has only finitely many parabolic cusps, one can choose the finite core C so that, for any pleated map $f : \Sigma(\sigma_f) \longrightarrow E$ in E and any hoop family $\lambda_1 \sqcup \cdots \sqcup \lambda_m$ of $\Sigma(\sigma_f)$, $f(\lambda_j)$ $(j = 1, \dots, m)$ does not correspond to any parabolic cusps of M. From now on, we denote a hoop family of $\Sigma(\sigma_f)$ by $\mathcal{H}(f)$. We say that f is *hoop-realizing* if f realizes a hoop family $\mathcal{H}(f)$. This means that any component λ_j of $\mathcal{H}(f)$ is not only a geodesic loop in Σ but also the image $f(\lambda_j)$ is a geodesic loop in E. Let $f_i : \Sigma \longrightarrow E$ $(i = 0, 1)$ be pleated maps satisfying the following conditions.

(F1) f_i is hoop-realizing and unwrapped in the sense of Subsection 1.2.
(F2) $\mathcal{N}_4(f_0(\Sigma)) \cap \mathcal{N}_4(f_1(\Sigma)) \cap E_{\mathrm{main}} = \emptyset$, and $f_1(\Sigma)$ is contained in the component of $E \setminus f_0(\Sigma)$ adjacent to $\mathcal{E}$.

Let $\widehat{f}_i : \Sigma \longrightarrow E$ be a normalized map contained in a small neighborhood of $f_i(\Sigma)$ in E, see Definition 1.6. Then an $\widehat{r}$-hoop family $\mathcal{H}(\widehat{f}_i)$ of $\Sigma(\widehat{f}_i)$ is defined similarly for some constant $\widehat{r} = \widehat{r}(\Sigma) \geq r(\Sigma)$.

By Lemma 1.7 (2), one can define an (ideal) triangulation τ_i $(i = 0, 1)$ on Σ satisfying the following conditions, where Σ is supposed to have the piecewise Riemannian metric induced from that on E via $\widehat{f}_i$.

(T1) Each element v of $\tau_i^{(0)}$ is either a point of $\mathcal{H}(\widehat{f}_i)$ or an ideal point of Σ.
(T2) $\bigcup \tau_i^{(1)}$ contains $\mathcal{H}(\widehat{f}_i)$.
(T3) For any component l of $\mathcal{H}(\widehat{f}_i)$, $l \cap \bigcup \tau_i^{(0)}$ consists of just two points.
(T4) The cardinality of τ_i is uniformly bounded.
(T5) There exists a uniform constant $d_3 > 0$ such that the d_3-neighborhood of any point x of $F(\widehat{f}_i) = \widehat{f}_i^{-1}(E_{\mathrm{thick}})$ contained in $\mathrm{star}(v)$ for some $v \in \tau_i^{(0)}$, where $\mathrm{star}(v)$ is the union $\bigcup_\alpha \mathrm{Int} D_\alpha$ for all elements D_α of τ_i with v as a common vertex.

Let $\mathcal{H}(\widehat{f}_i) \cap \widehat{f}_i^{-1}(E_{\mathrm{tube}}) = \mathcal{H}(\widehat{f}_i)_{\mathrm{tube}}$. We consider the unions of closed curves

$$\widehat{\mathcal{H}}_i = \widehat{f}_i(\mathcal{H}(\widehat{f}_i)) \quad \text{and} \quad \widehat{\mathcal{H}}_{i,\mathrm{tube}} = \widehat{f}_i(\mathcal{H}(\widehat{f}_i)_{\mathrm{tube}}) \tag{3.6}$$

in E. A singular 2-simplex $\sigma : \Delta^2 \longrightarrow \widehat{f}_i(\Sigma)$ is called a 2-simplex *with respect to* $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ if, for any edge e of Δ^2, either $\sigma(e)$ is an element of $\widehat{f}_i(\tau_i^{(0)} \cup \tau_i^{(1)})$ (possibly an ideal vertex) or the restriction $\sigma|_e$ is an immersion into $\widehat{\mathcal{H}}_i$ connecting two points of $\widehat{f}_i(\tau_i^{(0)})$. Then $\sigma|_e$ is called a 1-simplex *with respect to* $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$. Note that $\widehat{f}_i(\Sigma)$ is not necessarily a closed surface. So any simplicial

2-cycle on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ is supposed to represent a class of the locally finite homology group $H_2^{\mathrm{loc.f.}}(\widehat{f}_i(\Sigma),\mathbb{R})$.

We set $\widehat{X}=E(\widehat{f}_0,\widehat{f}_1)$, which is the closure of the component of $E\setminus\widehat{f}_0(\Sigma)\cup\widehat{f}_1(\Sigma)$ lying between $\widehat{f}_0(\Sigma)$ and $\widehat{f}_1(\Sigma)$ as is defined in Section 2. The following connecting lemma given in [So4, Lemma 5.1] plays an important role in the proof of Theorem A.

Lemma 3.2 (Connecting Lemma [So4])**.** *Suppose that $\sigma:\Delta^3\longrightarrow\mathbb{H}^3$ is a straight 3-simplex with $\mathrm{Vol}(\sigma)>1$. Then there exists a 3-chain z on M satisfying the following conditions.*

(1) *$z=z_{\widehat{X}}(\sigma)+\widehat{a}$, where $\widehat{a}$ is a 3-chain on M with $\|\widehat{a}\|\le b_0$ for some uniform constant $b_0>0$.*
(2) *For $i=0,1$, there exists a simplicial 2-cycle $w(\tau_i)$ on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ representing the fundamental class of $\widehat{f}_i(\Sigma)$ and satisfying*
$$\partial_3 z=\mathrm{Vol}(\sigma)(w(\tau_1)-w(\tau_0)).$$

3.2. **Inefficiency of smearing 3-chains.** Let $\varphi:M\longrightarrow M'$ be a homeomorphism between hyperbolic 3-manifolds satisfying the conditions in Assumptions of Section 1 and $\psi:M\longrightarrow M'$ a continuous map properly homotopic to φ. Afterwards ψ will be chosen so that it satisfies (P1) and (P2) below. Suppose that $p:\mathbb{H}^3\longrightarrow M$ and $p':\mathbb{H}^3\longrightarrow M'$ are the universal coverings. Take the base points y_0 of M and y_0' of M' so that $\psi(y_0)=y_0'$ and points x_0, x_0' of $\mathbb{H}^3$ with $p(x_0)=y_0$ and $p'(x_0')=y_0'$. Consider the lift $\widetilde{\psi}:\mathbb{H}^3\longrightarrow\mathbb{H}^3$ of ψ to the universal coverings with $\widetilde{\psi}(x_0)=x_0'$. We note that $\widetilde{\psi}$ is equivariant with respect to the isomorphism $\psi_*:\pi_1(M,y_0)\longrightarrow\pi_1(M',y_0')$. That is, for any $\gamma\in\pi_1(M,y_0)$, $\widetilde{\psi}\circ\gamma=\psi_*(\gamma)\circ\widetilde{\psi}$ holds. Here the covering transformation on $\mathbb{H}^3$ determined uniquely by $\gamma\in\pi_1(M,y_0)$ (resp. $\psi_*(\gamma)\in\pi_1(M',y_0')$) is also denoted by γ (resp. $\psi_*(\gamma)$). Let $\sigma:\Delta^3\longrightarrow\mathbb{H}^3$ be a non-degenerate straight 3-simplex. For any $\eta>0$ and $\alpha\in\mathrm{PSL}_2(\mathbb{C})$, a 3-simplex $\alpha\sigma:\Delta^3\longrightarrow\mathbb{H}^3$ is *η-efficient* (resp. *η-inefficient*) with respect to $\widetilde{\psi}$ if $\iota(\sigma)\mathrm{Vol}(\mathrm{straight}(\widetilde{\psi}\circ\alpha\sigma))>\boldsymbol{v}_3-\eta$ (resp. $\iota(\sigma)\mathrm{Vol}(\mathrm{straight}(\widetilde{\psi}\circ\alpha\sigma))\le\boldsymbol{v}_3-\eta$), where $\iota(\sigma)=\mathrm{Vol}(\sigma)/|\mathrm{Vol}(\sigma)|$. Let $\alpha\sigma:\Delta^3\longrightarrow\mathbb{H}^3$ be any η-efficient straight 3-simplex in $\mathbb{H}^3$ with respect to $\widetilde{\psi}$. Note that the η-efficiency is an open condition. We say that a non-degenerate straight 3-simplex $\tau:\Delta^3\longrightarrow M$ is *η-efficient* with respect to ψ if its lift to the universal covering is η-efficient with respect to $\widetilde{\psi}$, otherwise τ is *η-inefficient*.

For any closed subset X of M, let $C^{\eta}_{\mathrm{ineffi}}(\sigma;X)$ be the subset of $P(M)$ consisting of elements a such that $a\bullet o(\sigma)\in X$ and $a\bullet\sigma$ is η-inefficient. We denote the restriction of $z_X(\sigma)$ to 3-simplices $a\bullet\sigma$ with $a\in C^{\eta}_{\mathrm{ineffi}}(\sigma;X)$ by $z^{\eta}_{X,\mathrm{ineffi}}(\sigma)$. Let $z^{\eta}_{X,\mathrm{effi}}(\sigma)$ be the restriction of $z_X(\sigma)$ to the closure of $\mathrm{supp}(z_X(\sigma))\setminus\mathrm{supp}(z^{\eta}_{X,\mathrm{ineffi}}(\sigma))$.

Let E be the neighborhood of a simply degenerate end of M with respect to a finite core of M. Suppose that $\{\widehat{f}_n\}_{n=0}^{\infty}$ is the monotone increasing sequence of normalized maps in E as in Section 2 and $N_n=E(\widehat{f}_n,\widehat{f}_{n+1})$. For any n_0, $n_1\in\mathbb{N}\cup\{0\}$ with $n_0<n_1$, we denote $E(\widehat{f}_{n_0},\widehat{f}_{n_1})$ by $N_{(n_0,n_1)}$, that is, $N_{(n_0,n_1)}=\bigcup_{n=n_0}^{n_1-1}N_n$. For the d-neighborhood of $\mathcal{N}_d(N_{(n_0,n_1)})$ with $d\ge 0$, we set $z_{\mathcal{N}_d(N_{(n_0,n_1)})}(\sigma)=z(n_0,n_1;d)(\sigma)$ or $z(n_0,n_1;d)$ shortly. For any $d\ge 0$, let $C^{\eta}_{\mathrm{ineffi}}(\sigma,;n_0,n_1;d)$ be the subset of $P(M)$ consisting of elements a such that $a\bullet o(\sigma)\in\mathcal{N}_d(N_{(n_0,n_1)})$ and $a\bullet\sigma$ is η-inefficient. We denote the restriction of $z(n_0,n_1;d)$ to 3-simplices $a\bullet\sigma$ with

$a \in C^\eta_{\rm ineffi}(\sigma,;n_0,n_1;d)$ by $z^\eta_{\rm ineffi}(n_0,n_1;d)$. Let $z^\eta_{\rm effi}(n_0,n_1;d)$ be the restriction of $z(n_0,n_1;d)$ to the closure of $\mathrm{supp}(z(n_0,n_1;d))\setminus \mathrm{supp}(z^\eta_{\rm ineffi}(n_0,n_1;d))$.

Now we consider the case of $X = N_{(n_0,n_1)}$, that is, the case of $d=0$. Note that $N_{(n_0,n_1)}$ is an almost compact submanifold of M for any n_0, $n_1 \in \mathbb{N}\cup\{0\}$ with $n_0 < n_1$. See Section 1 for the definition of almost compact subspaces. Let τ_{n_i} be a triangulation on Σ such that $\widehat f_{n_i}(\tau_{n_i})$ is a triangulation satisfying the conditions (T1)–(T5) given in Section 3. We set $\widehat{\mathcal H}_E = \bigcup_{n=0}^\infty \widehat{\mathcal H}_n$, see (3.6) for $\widehat{\mathcal H}_n$. Let $\mathcal N(\widehat{\mathcal H}_E)$ be a neighborhood of $\widehat{\mathcal H}_E$ in M consisting of mutually disjoint tubular neighborhoods with $\mathrm{Vol}(\mathcal N(\widehat{\mathcal H}_E)) = \sum_{n=0}^\infty \mathcal N(\widehat{\mathcal H}_n) < \infty$. Then the normal radius of any components of $\mathcal N(\widehat{\mathcal H}_n)$ converges to zero as $n\to\infty$. Suppose that $\psi : M \longrightarrow M'$ is a continuous map satisfying the following conditions.

(P1) $\psi|_{M\setminus \mathcal N(\widehat{\mathcal H}_E)} = \varphi|_{M\setminus \mathcal N(\widehat{\mathcal H}_E)}$.

(P2) For each component l of $\widehat{\mathcal H}_E$, $\psi(l)$ is a closed geodesic in M'.

Consider a piecewise totally geodesic map $f'^*_{n_i} : \Sigma \longrightarrow M'$ properly homotopic to $\psi\circ \widehat f_{n_i} : \Sigma \longrightarrow M'$ and satisfying the following conditions.

- For any $v \in \tau^{(0)}_{n_i}$, $f'^*_{n_i}(v) = \psi\circ\widehat f_{n_i}(v)$.
- For any $e \in \tau^{(1)}_{n_i}$, $f'^*_{n_i}(e)$ is a geodesic segment in E' homotopic to $\psi\circ\widehat f_{n_i}(e)$ rel. ∂e.
- For any $\Delta \in \tau^{(2)}_{n_i}$, $f'^*_{n_i}(\Delta)$ is a totally geodesic triangle in E' bounded by $f'^*_{n_i}(\partial\Delta)$.

Lemma 3.3. *With the notation as above, there exists a constant $C>0$ independent of n_0 and n_1 such that* $\mathrm{Vol}^{\rm bd}(f'^*_{n_0}, f'^*_{n_1}) < \mathrm{Vol}(E(\widehat f_{n_0},\widehat f_{n_1})) + C$.

Proof. Let $\sigma : \Delta^3 \longrightarrow \mathbb H^3$ be any straight simplex in $\mathbb H^3$ with $\mathrm{Vol}(\sigma) > 1$. Suppose that $\widehat a_{n_0,n_1}$ is the connecting 3-chain given in Lemma 3.2 (1) associated with $\widehat X = E(\widehat f_{n_0},\widehat f_{n_1})$. Then $\|\widehat a_{n_0,n_1}\| \le b_0$ and $\partial_3 z_{n_0,n_1} = \mathrm{Vol}(\sigma)(w(\tau_{n_1}) - w(\tau_{n_0}))$ holds for the 3-chain $z_{n_0,n_1} = z(n_0,n_1;0) + \widehat a_{n_0,n_1}$ in E, where $w(\tau_{n_j})$ $(j=0,1)$ is the 2-cycle on $\widehat f_{n_j}(\Sigma)$ as in Lemma 3.2 (2). There exists the fundamental 2-cycle $S(\tau_{n_j})$ on Σ with respect to τ_{n_j} mod $\mathcal H(\widehat f_{n_j})_{\rm tube}$ such that $\widehat f_{n_j *}(S(\tau_{n_j})) = w(\tau_{n_j})$. Then $\mathrm{straight}(\psi_*(z_{n_0,n_1}))$ is a locally finite 3-chain on M' with

$$\begin{aligned}
&\partial_3\,\mathrm{straight}(\psi_*(z_{n_0,n_1}))\\
&\qquad = \mathrm{Vol}(\sigma)(\mathrm{straight}(\psi\circ\widehat f_{n_1})_*(S(\tau_{n_1})) - \mathrm{straight}(\psi\circ\widehat f_{n_0})_*(S(\tau_{n_0})))\\
&\qquad = \mathrm{Vol}(\sigma)((f'^*_{n_1})_*(S(\tau_{n_1})) - (f'^*_{n_0})_*(S(\tau_{n_0}))).
\end{aligned}$$

Here the equality $\mathrm{straight}(\psi\circ\widehat f_{n_j})_*(S(\tau_{n_j})) = (f'^*_{n_j})_*(S(\tau_{n_j}))$ is proved by the fact that $f'^*_{n_j}$ is a piecewise totally geodesic map defined as above. Then we have

$$\begin{aligned}
\omega_{M'}(\psi_*(z(n_0,n_1;0)+\widehat a_{n_0,n_1})) &= \mathrm{Vol}(\mathrm{straight}(\psi_*(z_{n_0,n_1})))\\
&= \mathrm{Vol}(\sigma)\mathrm{Vol}^{\rm bd}(f'^*_{n_0}, f'^*_{n_1}).
\end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}
\omega_{M'}(\psi_*(z(n_0,n_1;0)+\widehat a_{n_0,n_1})) &\le \boldsymbol v_3(\|z(n_0,n_1;0)\| + \|\widehat a_{n_0,n_1}\|)\\
&\le \boldsymbol v_3(\mathrm{Vol}(E(\widehat f_{n_0},\widehat f_{n_1})) + b_0).
\end{aligned}$$

By letting $\mathrm{Vol}(\sigma)\to \boldsymbol v_3$, one can have a required inequality. □

Now we recall the definition of ω-upper bound condition (0.1) for φ on E, where $\{X_n\}$ is an expanding sequence of compact submanifolds of E with $\bigcup_{n=1}^{\infty} X_n = E$. For any almost compact submanifold Y of M and any $\varepsilon > 0$, there exists a compact submanifold Y' of Y with $\mathrm{Vol}(Y \setminus Y') < \varepsilon$. Thus the compactness condition for X_n can be replaced by the almost compactness condition. Moreover any continuous map ψ with the properties (P1) and (P2) also satisfies the ω-upper bound condition if we replace the constant c_0 by $c_0 + 4\boldsymbol{v}_3\mathrm{Vol}(\mathcal{N}(\widehat{\mathcal{H}}_E))$, where we used the fact that $\mathrm{straight}(\psi \circ \sigma') \neq \mathrm{straight}(\varphi \circ \sigma')$ for $\sigma' \in \mathrm{supp}(z_{X_n}(\sigma))$ occurs only when at least one of the four vertices $\sigma'(v_i)$ $(i = 0, \dots, 3)$ is contained in $\mathcal{N}(\widehat{\mathcal{H}}_E)$. Hence the property for φ is equivalent to the existence of a constant $c_0 > 0$ satisfying the following condition for ψ. For any almost compact submanifold X of E, there exists an almost compact submanifold $\widehat{X}$ with $\widehat{X} \supset X$ and satisfying

$$(0.1)' \qquad \omega_{M'}(\psi_*(z_{\widehat{X}}(\sigma))) > \omega_M(z_{\widehat{X}}(\sigma)) - c_0$$

for any straight simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ with $\mathrm{Vol}(\sigma) > 1$.

The following lemma is the infinite volume version of Lemma 1 in Soma [So2]. Here the η-inefficiency is the condition with respect to the continuous map ψ.

Lemma 3.4. *Suppose that $\psi : M \longrightarrow M'$ satisfies the ω-upper bound condition $(0.1)'$ on E and $0 < \varepsilon < \boldsymbol{v}_3 - 1$. If $\mathrm{Vol}(\sigma) > \boldsymbol{v}_3 - \varepsilon$, then*

$$\|z^{\eta}_{\mathrm{ineffi}}(n_0, n_1; 0)\| \leq \frac{\varepsilon V_1(n_1 - n_0)}{\eta} + \frac{b_1}{\eta}$$

for any n_0, $n_1 \in \mathbb{N} \cup \{0\}$ with $n_0 < n_1$, where V_1 is the constant given in Lemma 2.2 (2) and $b_1 = b_1(c_0) > 0$ is a uniform constant.

Proof. Suppose that $X = N(n_0, n_1)$ and $\widehat{X}$ is an almost compact submanifold of E with $\widehat{X} \supset X$ and satisfying $(0.1)'$ for any straight simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ with $\mathrm{Vol}(\sigma) > 1$. Let $\widehat{Y}$ be the closure of $\widehat{X} \setminus X$ in E. Since $|\mathrm{Vol}(\tau)| = \mathrm{Vol}(\sigma)$ for any $\tau \in \mathrm{supp}(z(n_0, n_1; 0))$,

$$(3.8) \qquad \begin{aligned} \omega_M(z_{\widehat{X}}(\sigma)) &= \mathrm{Vol}(\sigma)\|z_{\widehat{X}}(\sigma)\| = \mathrm{Vol}(\sigma)(\|z_{\widehat{Y}}(\sigma)\| + \|z(n_0, n_1; 0)\|) \\ &= \mathrm{Vol}(\sigma)(\mathrm{Vol}(\widehat{Y}) + \mathrm{Vol}(E(\widehat{f}_{n_0}, \widehat{f}_{n_1}))). \end{aligned}$$

By Lemma 3.2 and (3.7),

$$(3.9) \qquad \begin{aligned} \omega_{M'}(\psi_*(z_{\widehat{X}}(\sigma))) &= \omega_{M'}(\psi_*(z_{\widehat{Y}}(\sigma))) + \omega_{M'}(\psi_*(z(n_0, n_1; 0))) \\ &\leq \boldsymbol{v}_3\|z_{\widehat{Y}}(\sigma)\| + \omega_{M'}(\widehat{a}_{n_0,n_1}) + \boldsymbol{v}_3\mathrm{Vol}^{\mathrm{bd}}(f'^*_{n_0}, f'^*_{n_1}) \\ &\leq \boldsymbol{v}_3\mathrm{Vol}(\widehat{Y}) + b_0\boldsymbol{v}_3 + \boldsymbol{v}_3\mathrm{Vol}^{\mathrm{bd}}(f'^*_{n_0}, f'^*_{n_1}). \end{aligned}$$

By $(0.1)'$, (3.8) and (3.9) with $\mathrm{Vol}(\sigma) \to \boldsymbol{v}_3$, we have that

$$(3.10) \qquad \mathrm{Vol}(E(\widehat{f}_{n_0}, \widehat{f}_{n_1})) \leq \mathrm{Vol}^{\mathrm{bd}}(f'^*_{n_0}, f'^*_{n_1}) + b_0 + c_0\boldsymbol{v}_3^{-1}.$$

Now we suppose that $\mathrm{Vol}(\sigma) > \boldsymbol{v}_3 - \varepsilon$ for a fixed $0 < \varepsilon < \boldsymbol{v}_3 - 1$. We have first

$$\begin{aligned} \omega_{M'}(\psi_*(z(n_0, n_1; 0))) &= \omega_{M'}(\psi_*(z^{\eta}_{\mathrm{effi}}(n_0, n_1; 0))) + \omega_{M'}(\psi_*(z^{\eta}_{\mathrm{ineffi}}(n_0, n_1; 0))) \\ &\leq \boldsymbol{v}_3\|z^{\eta}_{\mathrm{effi}}(n_0, n_1; 0)\| + (\boldsymbol{v}_3 - \eta)\|z^{\eta}_{\mathrm{ineffi}}(n_0, n_1; 0)\| \\ &= \boldsymbol{v}_3\|z(n_0, n_1; 0)\| - \eta\|z^{\eta}_{\mathrm{ineffi}}(n_0, n_1; 0)\| \\ &= \boldsymbol{v}_3\mathrm{Vol}(E(\widehat{f}_{n_0}, \widehat{f}_{n_1})) - \eta\|z^{\eta}_{\mathrm{ineffi}}(n_0, n_1; 0)\|. \end{aligned}$$

On the other hand, by (3.7) and (3.10),

$$\begin{aligned}\omega_{M'}(\psi_*(z(n_0,n_1;0))) &= \mathrm{Vol}(\sigma)\mathrm{Vol}^{\mathrm{bd}}(f'^*_{n_0},f'^*_{n_1}) - \omega_{M'}(\psi_*(\widehat{a}_{n_0,n_1}))\\ &\geq (\boldsymbol{v}_3-\varepsilon)(\mathrm{Vol}(E(\widehat{f}_{n_0},\widehat{f}_{n_1}))-b_0-c_0\boldsymbol{v}_3^{-1})-b_0\boldsymbol{v}_3\\ &\geq (\boldsymbol{v}_3-\varepsilon)\mathrm{Vol}(E(\widehat{f}_{n_0},\widehat{f}_{n_1}))-b_1,\end{aligned}$$

where b_1 is the uniform constant defined by

$$b_1=\boldsymbol{v}_3(b_0+c_0\boldsymbol{v}_3^{-1})+b_0\boldsymbol{v}_3=2b_0\boldsymbol{v}_3+c_0.$$

Thus we have

$$\eta\|z^{\eta}_{\mathrm{ineffi}}(n_0,n_1;0)\|\leq\varepsilon\mathrm{Vol}(E(\widehat{f}_{n_0},\widehat{f}_{n_1}))+b_1.$$

By Lemma 2.2 (2), $\mathrm{Vol}(E(\widehat{f}_{n_0},\widehat{f}_{n_1}))\leq(n_1-n_0)V_1$. This completes the proof. □

Corollary 3.5. *With the assumptions as in Lemma 3.4, for any $d\geq 0$, there exists a uniform constant $e(d)>0$ satisfying*

$$\widehat{\mu}_{\mathrm{Haar}}(C^{\eta}_{\mathrm{ineffi}}(\sigma;n_0,n_1;d))\leq\frac{2\varepsilon V_1(n_1-n_0)}{\eta}V_1+\frac{2e(d)}{\eta}.$$

Proof. By (3.2),

$$\begin{aligned}\|z^{\eta}_{\mathrm{ineffi}}(n_0,n_1;d)\| &= \frac{1}{2}(\mathrm{smear}_M(\sigma)+\mathrm{smear}_M(\sigma_-))(C^{\eta}_{\mathrm{ineffi}}(\sigma;n_0,n_1;d)\bullet\sigma)\\ &= \frac{1}{2}\widehat{\mu}_{\mathrm{Haar}}(C^{\eta}_{\mathrm{ineffi}}(\sigma;n_0,n_1;d)).\end{aligned}$$

On the other hand, by Lemma 1.7 (3),

$$\begin{aligned}\|z^{\eta}_{\mathrm{ineffi}}(n_0,n_1;d)\| &\leq \|z^{\eta}_{\mathrm{ineffi}}(n_0,n_1;0)\|+\|z_{\mathcal{N}_d(\widehat{f}_{n_0}(\Sigma))}(\sigma)\|+\|z_{\mathcal{N}_d(\widehat{f}_{n_1}(\Sigma))}(\sigma)\|\\ &\leq \|z^{\eta}_{\mathrm{ineffi}}(n_0,n_1;0)\|+\boldsymbol{v}_3(\mathrm{Vol}(\mathcal{N}_d(\widehat{f}_{n_0}(\Sigma)))+\mathrm{Vol}(\mathcal{N}_d(\widehat{f}_{n_1}(\Sigma))))\\ &\leq \|z^{\eta}_{\mathrm{ineffi}}(n_0,n_1;0)\|+2\boldsymbol{v}_3v_0(d).\end{aligned}$$

By Lemma 3.4, $e(d):=b_1+2\boldsymbol{v}_3v_0(d)$ is a required uniform constant. □

4. Simplicial honeycombs (infinite volume version)

In this section, we first recall the notion of simplicial honeycombs which is introduced in [So2] for hyperbolic 3-manifolds of finite volume and show that it is applicable also to the case of infinite volume. Similar tools are used also in [So1]. However, in [So1], the author needed the Cannon-Thurston map to define them. Here we do not rely on the Cannon-Thurston map. We will prove by using simplicial honeycombs that the lift $\widetilde{\psi}$ of ψ to $\mathbb{H}^3$ is well approximated by the identity near the boundary S^2_∞ of $\mathbb{H}^3$ with respect to suitable coordinates on $\mathbb{H}^3$ (Lemma 4.10) if ψ satisfies the ω-upper bound condition on a simply degenerate end of M.

4.1. Simplicial honeycombs revisited.

Throughout this section, we work with a number $J>4$, which will be fixed in Subsection 5.1. The number is a uniform constant $J(r_0)$ depending only on $r_0>0$ given in Lemma 2.2. We may assume that $r_0<1$.

For any element z of the complex plane $\mathbb{C}$, we denote by $B_a(z)$ the disk in $\mathbb{C}$ of radius $a>0$ and centered at z. The set of vertices of a triangle T on $\mathbb{C}$ is denoted by $v(T)$. We denote by $\mathbf{0}$ the origin of $\mathbb{C}$ and by $4J$ the point $4J+0\sqrt{-1}$ of $\mathbb{C}$. For any $z\in\mathbb{C}\setminus\{\mathbf{0}\}$, let $\widehat{T}_z$ be the regular triangle in $\mathbb{C}$ centered at $\mathbf{0}$ and with

$z \in v(\widehat{T}_z)$. Take $\delta > 0$ sufficiently smaller than r_0. For any $z \in B_\delta(4J)$ and a given $m \in \mathbb{N}$, we divide $\widehat{T}_z$ into 9^m regular sub-triangles $T_{z,1}, T_{z,2}, \ldots, T_{z,9^m}$ of the same size with $\mathbf{0} \in v(T_{z,i})$ for $i = 1, \ldots, 6$. Let $V^{(m)}(\widehat{T}_z)$ be the union $\bigcup_{i=1}^{9^m} v(T_{z,i})$. Then $B_\delta(4J)$ is the control disk for $V^{(m)}(\widehat{T}_z)$'s. See Figure 4.1. The length of each edge

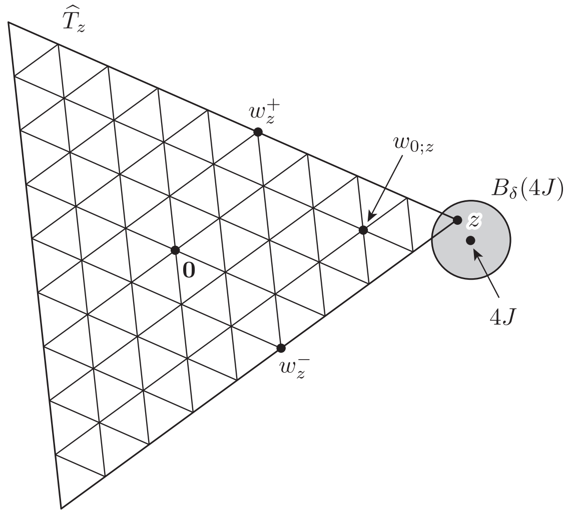


Figure 4.1. The case of $m = 2$.

of $T_{z,i}$ is $3^{-m}\sqrt{3}|z| < 3^{-m} \cdot 5\sqrt{3}J$, which is called the *fineness* of $V^{(m)}(\widehat{T}_z)$. For any $z \in B_\delta(4J)$, let $w_{0;z}$, w_z^+, w_z^- be the specified points of $V^{(m)}(\widehat{T}_z)$ defined by

$$w_{0;z} = \frac{2}{3}z, \quad w_z^+ = \frac{1}{9}\left(3 + 4\sqrt{-1}\right) z, \quad w_z^- = \frac{1}{9}\left(3 - 4\sqrt{-1}\right) z.$$

We set $z = x_{[\mathbb{C}]}$, $t = x_{[\mathbb{R}]}$ for a point $x = (z,t) \in \mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$. For a subset A of $\mathbb{H}^3$, we denote the subset $\{x_{[\mathbb{C}]} \,|\, x \in A\}$ of $\mathbb{C}$ by $A_{[\mathbb{C}]}$. For $i = 1, 2, \ldots, 9^m$ and $0 < t \le s < 1$, let $\Delta_{z,i,t}^{(s)}$ be the straight simplex in $\mathbb{H}^3$ with four vertices v_0, v_1, v_2, v_3 with $v_0 = (\mathbf{0}, 1/s)$, $\{v_1, v_2, v_3\}_{[\mathbb{C}]} = v(T_{z,i})$ and $v_{k[\mathbb{R}]} = s$ if either $v_{k[\mathbb{C}]} = \mathbf{0}$ or $v_{k[\mathbb{C}]} = w_{0;z}$, otherwise $v_{k[\mathbb{R}]} = t$ for $k = 1, 2, 3$. We say that the set $\mathcal{H}_{z,t}^{(s,m)} = \{\Delta_{z,i,t}^{(s)} \,|\, i = 1, 2, \ldots, 9^m\}$ is the *simplicial honeycomb* in $\mathbb{H}^3$ of type (z, m, s, t). See Figure 4.2, where $l_{\mathbf{0}}$ is the geodesic line in $\mathbb{H}^3$ connecting $\mathbf{0}$ with ∞. We set

$$\varepsilon_m(s) = \sup\{\boldsymbol{v}_3 - \mathrm{Vol}(\Delta)\,;\, \Delta \in \mathcal{H}_{z,t}^{(s,m)}, 0 < t \le s, z \in B_1(4J)\},$$

where the radius 1 of $B_1(4J)$ is taken as a positive constant sufficiently smaller than $4J$ and independent of δ. Since any $\Delta \in \mathcal{H}_{z,t}^{(s,m)}$ geometrically converges to an regular ideal simplex uniformly on any compact subsets in $\mathbb{H}^3$ as $0 < t \le s \to 0$,

$$\lim_{s \to 0} \varepsilon_m(s) = 0. \tag{4.1}$$

The next lemma follows immediately from the definition of $\mathcal{H}_{z,t}^{(s,m)}$. Here we recall that $x_0 = (\mathbf{0}, 1)$ is the base point of $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$.

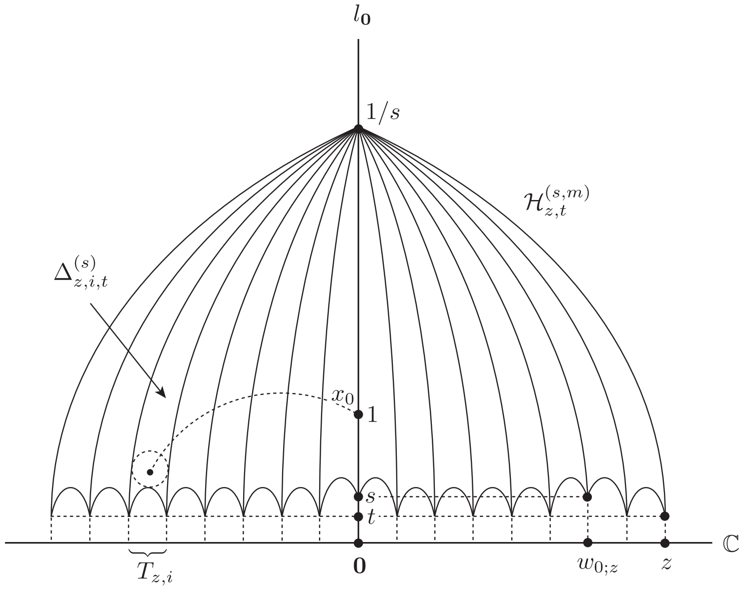


Figure 4.2

Lemma 4.1. *There exists a uniform constant* $d_0(m) > 0$ *independent of* $0 < s < 1$ *such that, for any element* Δ *of* $\mathcal{H}^{(s,m)}_{z,t}$, $\mathrm{dist}_{\mathbb{H}^3}(x_0, o(\Delta)) \le d_0(m)$.

Note that $\lim_{m\to\infty} d_0(m) = \infty$.

Let $\widetilde{\psi} : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ be the lift of the continuous map $\psi : M \longrightarrow M'$ given in Subsection 3.2. For $\alpha \in \mathrm{PSL}_2(\mathbb{C})$, $\widetilde{\psi} \circ \alpha$ is denoted by $\widetilde{\psi}_\alpha$. If necessary deforming $\widetilde{\psi}_\alpha$ slightly by homotopy, one can suppose that $\widetilde{\psi}_\alpha(\mathbf{0}, 1/s) \neq \widetilde{\psi}_\alpha(\mathbf{0}, s)$. Then the composition $\widetilde{\psi}_{\alpha,\beta} = \beta\circ\widetilde{\psi}\circ\alpha : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ with $\beta \in \mathrm{PSL}_2(\mathbb{C})$ is called a *normalization* of $\widetilde{\psi}_\alpha$ if it satisfies

$$\{\widetilde{\psi}_{\alpha,\beta}(\mathbf{0}, 1/s), \widetilde{\psi}_{\alpha,\beta}(\mathbf{0}, s)\} \subset l_{\mathbf{0}} \quad \text{and} \quad \widetilde{\psi}_{\alpha,\beta}(\mathbf{0}, s)_{[\mathbb{R}]} < \widetilde{\psi}_{\alpha,\beta}(\mathbf{0}, 1/s)_{[\mathbb{R}]}. \tag{4.2}$$

See Figure 4.6 for the normalization $\widetilde{\psi}_{\alpha,\beta}$ with $s = u_n$.

Definition 4.2. For any non-degenerate straight 3-simplex Δ in $\mathbb{H}^3$, we denote a positive straight 3-simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ with $\sigma(\Delta^3) = \Delta$ by σ_Δ. We say that Δ is *η-efficient* with respect to $\widetilde{\psi}_\alpha$ if σ_Δ is η-efficient, that is, $\mathrm{Vol}(\mathrm{straight}(\widetilde{\psi}_\alpha \circ \sigma_\Delta)) > \boldsymbol{v}_3 - \eta$. A finite set $\{\Delta_1, \dots, \Delta_n\}$ of positive straight 3-simplices in $\mathbb{H}^3$ satisfies the *property* $\mathbf{P}^\eta_{\mathrm{effi}}(\widetilde{\psi}_\alpha)$ if each Δ_i $(i = 1, \dots, n)$ is η-efficient with respect to $\widetilde{\psi}_\alpha$.

Now we present two technical lemmas, which are proved by arguments quite similar to those in [So2]. Here $\mathrm{dist}_{\mathbb{C}\times\mathbb{R}_+}$ is the distance function and $\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}$ is the Lebesgue measure on $\mathbb{C} \times \mathbb{R}_+$ with respect to the standard Euclidean metric on $\mathbb{C} \times \mathbb{R}_+ \subset \mathbb{E}^2 \times \mathbb{E} = \mathbb{E}^3$. Let $\mathcal{V}^{(s,m)}_{z,t}$ be the union of all vertices of $\Delta^{(s)}_{z,i,t} \in \mathcal{H}^{(s,m)}_{z,t}$ other than the top vertex $(\mathbf{0}, 1/s)$. Then we have ${\mathcal{V}^{(s,m)}_{z,t}}_{[\mathbb{C}]} = V^{(m)}(\widehat{T}_z)$. Note that $\mathcal{V}^{(s,m)}_{z,t}$ contains $(\mathbf{0}, s)$ and $(w_{0;z}, s)$, any other elements of which are of height t.

Lemma 4.3 (cf. [So2, Lemma 3]). *For any $\delta > 0$ sufficiently smaller than r_0, there exist constants $s_1 = s_1(\delta, m) > 0$ and $\eta = \eta(\delta, m) > 0$ satisfying the following (*).*

(*) If $\mathcal{H}_{z,t}^{(s,m)}$ has the property $\mathbf{P}_{\mathrm{effi}}^{\eta}(\widetilde{\psi}_\alpha)$ for some $\alpha \in \mathrm{PSL}_2(\mathbb{C})$, $0 < s \le s_1$ and $(z,t) \in B_\delta(4J) \times (0, s]$ and $\widetilde{\psi}_\alpha$ has a normalization β with $\widetilde{\psi}_{\alpha,\beta}(w_{0;z,s})_{[\mathbb{C}]} \in B_{2\delta}(w_{0;z})$, then there exists a constant $c_0 > 0$ independent of δ such that

$$\mathrm{dist}_{\mathbb{C}}(v_{[\mathbb{C}]}, \widetilde{\psi}_{\alpha,\beta}(v)_{[\mathbb{C}]}) < c_0\delta \quad \text{and} \quad \widetilde{\psi}_{\alpha,\beta}(v)_{[\mathbb{R}]} < c_0\delta \tag{4.3}$$

for any $v \in \mathcal{V}_{z,t}^{(s,m)}$, see Figure 4 in [So2] (*and also Figure 6 in* [So1]).

We also suppose that any *constant* is independent of n and δ. A function $f(\delta)$ of δ is often denoted by $\langle\delta\rangle$ if $0 \le f(\delta) < R\delta$ holds for some constant $R > 0$. For example, if $f_0(\delta)$, $f_1(\delta)$ are such functions and a, b are non-negative constants, then $af_0(\delta) + bf_1(\delta)$ can be represented as $a\langle\delta\rangle + b\langle\delta\rangle = \langle\delta\rangle$.

Suppose that X is a subset of $\mathbb{C} \times (0, s]$. We say that $\widetilde{\psi}_{\alpha,\beta}|_X$ is a $\langle\delta\rangle$-*almost identity* if $\widetilde{\psi}_{\alpha,\beta}$ satisfies (4.3) for any $v \in X$. Lemma 4.3 asserts that one can choose a normalizing factor β of $\widetilde{\psi}_\alpha$ so that $\widetilde{\psi}_{\alpha,\beta}|_{\mathcal{V}_{z,t}^{(s,m)}}$ is a $\langle\delta\rangle$-almost identity when $\mathcal{H}_{z,t}^{(s,m)}$ has the property $\mathbf{P}_{\mathrm{effi}}^{\eta}(\widetilde{\psi}_\alpha)$. In general, the choice of β depends on (z, t). Lemma 4.10 will show that, in our case, there exists a normalizing factor without depending on (z, t).

For any Borel subset L of $B_\delta(4J) \times (0, s]$, we set

$$W^{(s,m)}(L) = B_{2J}(\mathbf{0}) \times (0, s] \cap \left(\bigcup_{(z,t)\in L} \mathcal{V}_{z,t}^{(s,m)} \right),$$
$$N^{(s,m)}(L) = W^{(s,m)}(L) \cap B_\delta(\mathbf{0}) \times (0, s].$$

The following lemma corresponds to Lemma 5 in [So2]. For the proof, it was crucial that the fineness of $V^{(m)}(\widehat{T}_z)$ converges to zero as $m \to \infty$.

Lemma 4.4 (cf. [So2, Lemma 5]). *Fix a constant $c \ge 1$ and suppose that $\delta > 0$ and $s > 0$ are any sufficiently small numbers. Then there exists $m_0 = m_0(\delta) \in \mathbb{N}$ and a constant $\theta_0 > 0$ independent of c, s such that, for any integer $m \ge m_0$, the followings hold.*

$$\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(W^{(s,m)}(L)) > (1 - \theta_0 c\delta)\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_{2J}(\mathbf{0}) \times (0, s]),$$
$$\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(N^{(s,m)}(L)) > (1 - \theta_0 c\delta)\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(\mathbf{0}) \times (0, s])$$

for any Borel subset L of $B_\delta(4J) \times (0, s]$ with

$$\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L) > (1 - c\delta)\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(4J) \times (0, s]).$$

In fact, the lemma holds if $\theta_0 = 5$. However we just need that θ_0 is a positive constant in our argument.

4.2. **Applications to simply degenerate ends.** Let E be the neighborhood of a simply degenerate end $\mathcal{E}$ with respect to a finite core of M. The submanifold $\bigcup_{n=0}^{\infty} N_n$ of E given in Section 2 is also a neighborhood of $\mathcal{E}$. Throughout the remainder of this section, we suppose that $\psi : M \longrightarrow M'$ satisfies the ω-upper bound condition $(0.1)'$ on E. So one can use results in Subsection 3.2.

Let $p : \mathbb{H}^3 \longrightarrow M$ be the universal covering and $q : \mathrm{PSL}_2(\mathbb{C}) \longrightarrow P(M)$ the quotient map given in Section 3. We may suppose that the base point x_0 of $\mathbb{H}^3$

is taken so that $y_0 = p(x_0)$ is the base point of N_0. For the constant r_0 given in Lemma 2.2 (3), consider the open subset $\mathcal{A}$ of $\mathrm{PSL}_2(\mathbb{C})$ consisting of elements α with $\mathrm{dist}_{\mathbb{H}^3}(\alpha x_0, x_0) < 2r_0/3$. Recall that, for any $n \in \mathbb{N} \cup \{0\}$, R_n is the main part of N_n with the base point y_n and satisfying the conditions (R1) and (R2) in Section 2, see Figure 2.4. Let α_n be an element of $\mathrm{PSL}_2(\mathbb{C})$ with $q(\alpha_n) \bullet x_0 = y_n$. Set $\alpha_n x_0 = x_n$, $\alpha_n \mathcal{A} = \mathcal{A}_n$ and $q(\mathcal{A}_n) = A_n$. For any $\alpha' = \alpha_n \alpha \in \mathcal{A}_n$ with $\alpha \in \mathcal{A}$, $\mathrm{dist}_{\mathbb{H}^3}(\alpha' x_0, \alpha_n x_0) = \mathrm{dist}_{\mathbb{H}^3}(\alpha x_0, x_0) < 2r_0/3$. By Lemma 2.2 (3), the following properties hold.

- For any $n \in \mathbb{N} \cup \{0\}$, the restriction $q|_{\mathcal{A}_n}$ of q is injective and hence
$$\mu_{\mathrm{Haar}}(\mathcal{A}) = \mu_{\mathrm{Haar}}(\mathcal{A}_n) = \widehat{\mu}_{\mathrm{Haar}}(A_n).$$
- For any n_0, $n_1 \in \mathbb{N} \cup \{0\}$ with $n_0 \neq n_1$, $A_{n_0} \cap A_{n_1}$ is empty.

For any non-degenerate straight 3-simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$, let $A^{\eta}_{n,\mathrm{ineffi}}(\sigma)$ be the subset of A_n consisting of elements $a \in A_n$ such that $a \bullet \sigma$ is η-inefficient and set $\mathcal{A}^{\eta}_{n,\mathrm{ineffi}}(\sigma) = (q|_{\mathcal{A}_n})^{-1}(A^{\eta}_{n,\mathrm{ineffi}}(\sigma))$.

For a given $0 < \delta < 1$, we fix a integer $m \geq m_0(\delta)$ for $m_0(\delta)$ in Lemma 4.4 and let $\eta = \eta(\delta, m)$ be the positive number in Lemma 4.3 for an integer m is greater than m_0. Let $\mathcal{E}^{(s,m);\eta}_n$ be the subset of $\mathcal{A}_n \times B_\delta(4J) \times (0, s]$ consisting of elements (α, z, t) such that $\alpha\Delta_i$ is η-efficient for all Δ_i $(i = 1, 2, \dots, 9^m)$, where we denote the elements $\Delta^{(s)}_{z,i,t}$ of $\mathcal{H}^{(s,m)}_{z,t}$ by Δ_i for simplicity. If we set $\mathcal{X}^{\eta}_{n;z,i,t} = \mathcal{A}^{\eta}_{n,\mathrm{ineffi}}(\sigma_i)$ for $\sigma_i = \sigma_{\Delta_i} : \Delta^3 \longrightarrow \mathbb{H}^3$, then
$$\mathcal{E}^{(s,m);\eta}_n = \left\{ (\alpha, z, t) \,\middle|\, (z, t) \in B_\delta(4J) \times (0, s], \alpha \in \mathcal{A}_n \setminus \bigcup_{i=1}^{9^m} \mathcal{X}^{\eta}_{n;z,i,t} \right\}.$$
For any $k \in \mathbb{N}$,
$$\begin{aligned}\sum_{n=1}^{k} \mu_{\mathrm{Haar}}(\mathcal{X}^{\eta}_{n;z,i,t}) &= \sum_{n=1}^{k} \widehat{\mu}_{\mathrm{Haar}}(A^{\eta}_{n,\mathrm{ineffi}}(\sigma_i)) \\ &\leq \widehat{\mu}_{\mathrm{Haar}}(C^{\eta}_{\mathrm{ineffi}}(\sigma_i; 1, k; d_0(m, r_0)))\end{aligned}$$
for $i = 1, 2, \dots, 9^m$, where $d_0(m, r_0) = d_0(m) + r_0$ for the constant $d_0(m)$ given in Lemma 4.1. By Corollary 3.5,
$$\sum_{n=1}^{k} \sum_{i=1}^{9^m} \mu_{\mathrm{Haar}}(\mathcal{X}^{\eta}_{n;z,i,t}) \leq \frac{2 \cdot 9^m \varepsilon_m(s) V_1 k}{\eta} + \frac{2 \cdot 9^m e(d_0(m, r_0))}{\eta}. \tag{4.4}$$

Let $V_n(\lambda)$, $W_n(\lambda)$ $(\lambda > 0, n = 1, 2, \dots)$ be measurable subsets of a measure space (X, μ) with $V_n(\lambda) \subset W_n(\lambda)$. Then $\mu(V_n(\lambda)) \approx_{(\lambda)} \mu(W_n(\lambda))$ means that there exists a constant$c > 0$ independent of λ and n and satisfying
$$\mu(V_n(\lambda)) > (1 - c\lambda)\mu(W_n(\lambda))$$
for any sufficiently small $\lambda > 0$ and any n greater than some $n(\lambda) \in \mathbb{N}$.

The following lemma is an infinite volume version of Lemma 6 in [So2]. Here, we denote by $L^{(s)}_\alpha$ the α-section of $\mathcal{E}^{(s,m);\eta}_n$ in $\mathcal{A}_n \times B_\delta(4J) \times (0, s]$ for $\alpha \in \mathcal{A}_n$, which is a Borel subset of $B_\delta(4J) \times (0, s]$.

Lemma 4.5. *For any sufficiently small $\delta > 0$, there exists $n_0 = n_0(\delta) \in \mathbb{N}$ such that, for any $n \geq n_0$, there are $s_n > 0$ with $\lim_{n\to\infty} s_n = 0$ and such that, for any $0 < s \leq s_n$, there exists a Borel subset $\mathcal{O}^{(s)}_n$ of $\mathcal{A}_n$ satisfying the following conditions.*

(1) *For any* $\alpha \in \mathcal{A}_n \setminus \mathcal{O}_n^{(s)}$ *and any* $0 < s \le s_n$, $\mathcal{H}_{z,t}^{(s,m)}$ *has the property* $\mathbf{P}_{\mathrm{effi}}^{\eta}(\widetilde{\psi}_\alpha)$ *if* $(z,t) \in L_\alpha^{(s)}$.

(2) $\mu_{\mathrm{Haar}}(\mathcal{A}_n \setminus \mathcal{O}_n^{(s)}) \approx_{(\delta)} \mu_{\mathrm{Haar}}(\mathcal{A}_n)$.

(3) *For any* $\alpha \in \mathcal{A}_n \setminus \mathcal{O}_n^{(s)}$, $\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L_\alpha^{(s)}) \approx_{(\delta)} \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(4J) \times (0,s])$.

Proof. Suppose that there would exist infinitely many $n(a) \in \mathbb{N}$ with $n(a) < n(a+1)$ and such that, for any $0 < \widehat{s} < 1/n(a)$ $(a = 1, 2, \dots)$, there exists $0 < s \le \widehat{s}$ such that any Borel subset $\mathcal{O}_{n(a)}$ of $\mathcal{A}_{n(a)}$ does not satisfy at least one of the conditions (1)–(3).

Let a_0 be the smallest integer satisfying

$$a_0 \ge \frac{2 \cdot 4 \cdot 9^m \cdot e(d_0(m, r_0))}{\eta\delta^2\mu_{\mathrm{Haar}}(\mathcal{A})}.$$

Since $\lim_{s\to 0} \varepsilon_m(s) = 0$ by (4.1), $9^m \cdot 2\varepsilon_m(s)V_1\eta^{-1}n(a_0) < a_0\delta^2\mu_{\mathrm{Haar}}(\mathcal{A})/2$ holds for any $0 < s \le \widehat{s}_0$ if we take $0 < \widehat{s}_0 \le 1/n(a_0)$ sufficiently small. For any $(z,t) \in B_\delta(4J) \times (0,s]$, since

$$\sum_{a=1}^{a_0} \mu_{\mathrm{Haar}}(\mathcal{X}^{\eta}_{n(a);z,i,t}) \le \sum_{n=1}^{n(a_0)} \mu_{\mathrm{Haar}}(\mathcal{X}^{\eta}_{n;z,i,t}),$$

the inequality (4.4) with $k = n(a_0)$ implies

$$\sum_{a=1}^{a_0}\sum_{i=1}^{9^m} \mu_{\mathrm{Haar}}(\mathcal{X}^{\eta}_{n(a);z,i,t}) \le \frac{4 \cdot 9^m \varepsilon_m(s)V_1 n(a_0)}{\eta} + \frac{4 \cdot 9^m e(d_0(m,r_0))}{\eta} \le a_0\delta^2\mu_{\mathrm{Haar}}(\mathcal{A}).$$

Then, for some $a \in \{1, \dots, a_0\}$ and $s_n(a) = \widehat{s}_0$, we have

$$\frac{\mu_{\mathrm{Haar}} \times \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(\mathcal{E}^{(s,m);\eta}_{n(a)})}{\mu_{\mathrm{Haar}} \times \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(\mathcal{A}_{n(a)} \times B_\delta(4J) \times (0,s])} > 1 - \delta^2.$$

Hence there exists a Borel subset $\mathcal{O}_{n(a)}$ of $\mathcal{A}_{n(a)}$ with (1)–(3). This contradicts our definition of $n(a)$. So one can have a positive integer n_0 and $0 < s_n < 1/n$ for any $n \ge n_0$ which are desired in Lemma 4.5. This completes the proof. □

Take $n \ge n_0(\delta)$ arbitrarily. For a fixed $0 < s \le s_n$, let α be an element of $\mathcal{A}_n \setminus \mathcal{O}_n^{(s)}$ and $L_\alpha^{(s)}$ the subset of $B_\delta(4J) \times (0,s]$ given in Lemma 4.5 (3). Recall that, for any $\beta \in \mathrm{PSL}_2(\mathbb{C})$ satisfying (4.2), $\widetilde{\psi}_{\alpha,\beta} = \beta \circ \widetilde{\psi}_\alpha$ is called a normalization of $\widetilde{\psi}_\alpha$. Suppose that $\rho_{\alpha,\beta} : B_\delta(4J) \longrightarrow \mathbb{C}$ is a continuous map defined by

$$\rho_{\alpha,\beta}(z) = \widetilde{\psi}_{\alpha,\beta}(w_{0;z}, s)_{[\mathbb{C}]} \cdot (w_{0;z})^{-1}, \tag{4.5}$$

where $w_{0;z}$ is the specified point of $V^{(m)}(\widehat{T}_z)$ given in Subsection 4.1. Then the correspondence $w \longmapsto \rho_{\alpha,\beta}(z) \cdot w$ defines the similar map on $\mathbb{C}$ fixing $\mathbf{0}$ and mapping $w_{0;z}$ to $\widetilde{\psi}_{\alpha,\beta}(w_{0;z}, s)_{[\mathbb{C}]}$. Since any $\mathcal{V}_{z,t} = \mathcal{V}_{z,t}^{(s,m)}$ with $0 < t \le s$ contains $(w_{0;z}, s_n)$ as a common point, it follows from Lemmas 4.3 and 4.5 that, for any $(z,t) \in L_\alpha^{(s)}$,

$$\left| \widetilde{\psi}_{\alpha,\beta}(v)_{[\mathbb{C}]} - \rho_{\alpha,\beta}(z) \cdot v_{[\mathbb{C}]} \right| < |\rho_{\alpha,\beta}(z)|\langle\delta\rangle \quad (v \in \mathcal{V}_{z,t}). \tag{4.6}$$

Remark 4.6. Note that the normalization β of $\widetilde{\psi}_\alpha$ depends on the choice of $z_0 \in B_\delta(4J)$ with $(z_0, t_0) \in L_\alpha^{(s)}$. For any $(z,t) \in L_\alpha^{(s)}$ with $z \ne z_0$, $\widetilde{\psi}_{\alpha,\beta}|_{\mathcal{V}_{z,t}}$ is approximated by either the identity or a conformal map on $\mathbb{C}$ fixing $\mathbf{0}$. We would like to choose a common β so that $\widetilde{\psi}_{\alpha,\beta}|\mathcal{V}_{z,t}$ is $\langle\delta\rangle$-almost identical for 'most' $(z,t) \in L_\alpha^{(s)}$. To accomplish the object, we consider a counter part $\alpha'(\mathcal{V}_{z',t})$ of $\alpha(\mathcal{V}_{z,t})$ for some $\alpha' \in \mathrm{PSL}_2(\mathbb{C})$ and $(z',t) \in L_{\alpha'}^{(s)}$. First we show that $\alpha'(\mathcal{V}_{z',t})$ is stuck on a solid cylinder with the axis $\alpha(l_{\mathbf{0}})$ (see (4.10) below), so that $\widetilde{\psi}_{\alpha,\beta}$ can not move $\alpha^{-1} \circ \alpha'(\mathcal{V}_{z',t})$ essentially. By using this fact, one can prove that $\alpha(\mathcal{V}_{z,t})$ is also stuck on an opposite solid cylinder with the axis $\alpha'(l_{\mathbf{0}})$, and hence $\widetilde{\psi}_{\alpha,\beta}|\mathcal{V}_{z,t}$ is also almost identical.

For any $\alpha \in \mathcal{A}_n$ and any $z \in B_\delta(4J)$, there exists a unique element $\alpha' = \tau_n(\alpha, z) \in \mathrm{PSL}_2(\mathbb{C})$ with $\alpha'(\infty) = \alpha(\infty)$, $\alpha'(w_z^+) = \alpha(w_z^-)$ and $\alpha'(w_z^-) = \alpha(w_z^+)$. Then $\alpha'(\mathbf{0})$ is equal to $\alpha(w_{0;z})$. See Figure 4.3. Let r_z be the elliptic element of

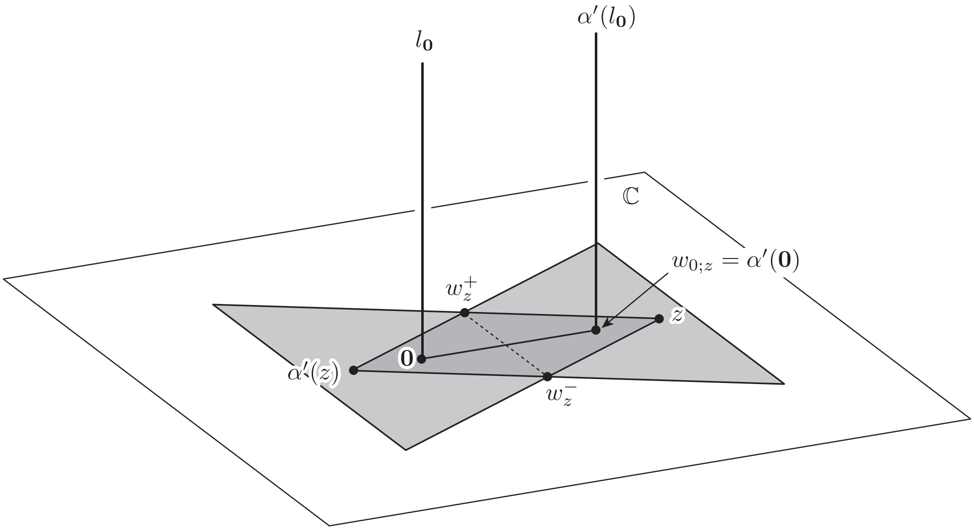


Figure 4.3. The coordinate on Image(α) $= \mathbb{C} \times \mathbb{R}_+$ is taken so that $\alpha(\infty) = \infty$, $\alpha(\mathbf{0}) = \mathbf{0}$ and $\alpha(z) = z$, that is, $\alpha = \mathrm{Id}_{\mathbb{C}\times\mathbb{R}_+}$.

$\mathrm{PSL}_2(\mathbb{C})$ of rotation angle π and fixing $z/3$, ∞. Then $\tau_n(\alpha, z)$ is represented as $\alpha \circ r_z$.

Lemma 4.7. *The map*

$$\zeta_n : \mathcal{A}_n \times B_\delta(4J) \longrightarrow \mathrm{PSL}_2(\mathbb{C}) \times B_\delta(4J)$$

defined by $\zeta_n(\alpha, z) = (\tau_n(\alpha, z), z)$ *is a smooth embedding.*

Proof. We set $\alpha_i' = \tau(\alpha_i, z_i)$ for $i = 0, 1$ and suppose that $(\alpha_0', z_0) = (\alpha_1', z_1)$. Then $\alpha_0 \circ r_{z_0} = \alpha_1 \circ r_{z_0}$ and hence $\alpha_0 = \alpha_1$. This shows ζ_n is injective. For a fixed $z \in B_\delta(4J)$, the correspondence $\alpha' \longmapsto \alpha = \alpha' \circ r_z$ defines a smooth map as well as τ_n. It follows that ζ_n is a local diffeomorphism and hence an smooth embedding. □

We set

$$\mathcal{A}_n' = \{\tau_n(\alpha, z) \in \mathrm{PSL}_2(\mathbb{C})\,;\, z \in B_\delta(4J), \alpha \in \mathcal{A}_n\}.$$

Then $\zeta_n(\mathcal{A}_n \times B_\delta(4J))$ is a subset of $\mathcal{A}_n' \times B_\delta(4J)$. See Figure 4.4. Recall that

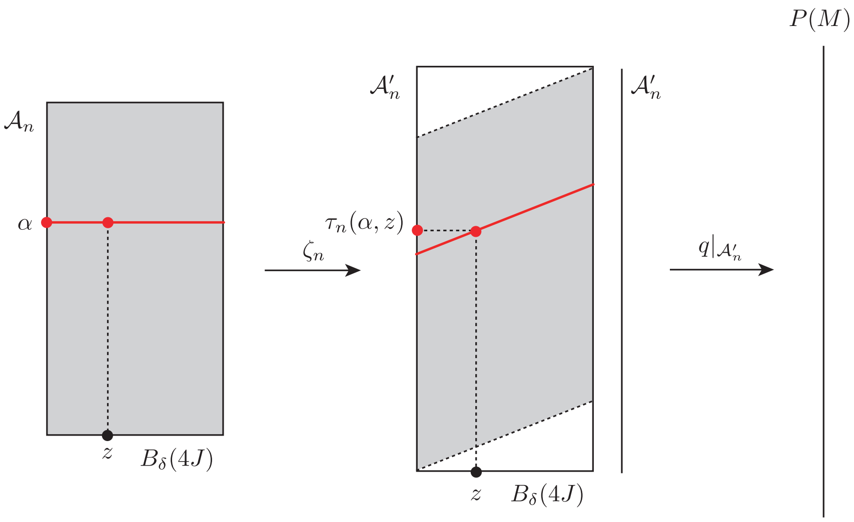


Figure 4.4

$q : \mathrm{PSL}_2(\mathbb{C}) \longrightarrow P(M) = \Gamma\backslash\mathrm{PSL}_2(\mathbb{C})$ is the quotient map given in the paragraph containing the equation (3.1).

Lemma 4.8. *The restriction $q|_{\mathcal{A}_n'} : \mathcal{A}_n' \longrightarrow P(M)$ is injective.*

Proof. If $q|_{\mathcal{A}_n'}$ were not injective, then there would exist $\gamma \in \pi_1(M, x_0) \setminus \{1\}$ and $\alpha_i' = \tau_n(\alpha_i, z_i) \in \mathcal{A}_n'$ $(i = 0, 1)$ with $\alpha_1' = \gamma \circ \alpha_0'$. From the definition of τ_n, this implies $\alpha_1 = \gamma \circ \alpha_0 \circ r_{z_0} \circ r_{z_1}$. Since both z_1, z_2 are contained in $B_\delta(4J)$, $r_{z_0} \circ r_{z_1}$ well approximated by the identify of $\mathbb{H}^3$ in a fixed neighborhood of x_0 in $\mathbb{H}^3$. Since moreover δ is sufficiently smaller than r_0, $\mathrm{dist}_{\mathbb{H}^3}(\alpha_1(x_0), \gamma \circ \alpha_0(x_0)) < 2r_0/3$. It follows that

$$\begin{aligned}\mathrm{dist}_{\mathbb{H}^3}(x_n, \gamma x_n) \le \mathrm{dist}_{\mathbb{H}^3}(x_n, \alpha_1(x_0)) + \mathrm{dist}_{\mathbb{H}^3}(\alpha_1(x_0), \gamma \circ \alpha_0(x_0))\\ + \mathrm{dist}_{\mathbb{H}^3}(\gamma \circ \alpha_0(x_0), \gamma x_n) < \frac{2r_0}{3} + \frac{2r_0}{3} + \frac{2r_0}{3} = 2r_0.\end{aligned}$$

This contradicts $\mathrm{dist}_{\mathbb{H}^3}(x_n, \gamma x_n) \ge 2r_0$. ☐

By using the injectivity of $q|_{\mathcal{A}_n'}$ instead of that of $q|_{\mathcal{A}_n}$, we have the following lemma corresponding to Lemma 4.5.

Lemma 4.9. *For any sufficiently small $\delta > 0$, there exists $n_0' = n_0'(\delta) \in \mathbb{N}$ such that, for any $n \ge n_0'$, there are $s_n' > 0$ with $\lim_{n\to\infty} s_n' = 0$ and such that, for any $0 < s \le s_n'$, there exists a Borel subset $\mathcal{O}_n'^{(s)}$ of $\mathcal{A}_n'$ satisfying the following conditions.*

(1) *For any $\alpha' \in \mathcal{A}_n' \setminus \mathcal{O}_n'^{(s)}$ and any $0 < s \le s_n'$, $\mathcal{H}_{z,t}^{(s,m)}$ has the property $\mathbf{P}^{\eta}_{\mathrm{effi}}(\widetilde{\psi}_{\alpha'})$ if $(z, t) \in L_{\alpha'}^{(s)}$.*
(2) *$\mu_{\mathrm{Haar}}(\mathcal{A}_n' \setminus \mathcal{O}_n'^{(s)}) \approx_{(\delta)} \mu_{\mathrm{Haar}}(\mathcal{A}_n')$.*
(3) *For any $\alpha' \in \mathcal{A}_n' \setminus \mathcal{O}_n'^{(s)}$, $\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L_{\alpha'}^{(s)}) \approx_{(\delta)} \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(4J) \times (0, s])$.*

The following is a key lemma to prove Theorem A. In fact, we will find an element $(z_0, t_0) \in L_\alpha^{(s)}$ such that (i) $|\rho_{\alpha,\beta}(z_0)| \geq |\rho_{\alpha,\beta}(z)|$ for 'most' $(z,t) \in L_\alpha^{(s)}$ and (ii) (z_0, t_0) is also an element of $L_{\alpha'}^{(s)}$ for $\alpha' = \tau_n(\alpha, z_0)$. We use a truncating trick for the proof.

Lemma 4.10. *With the notations as above, for any $n \geq \max\{n_0, n_0'\}$, there exists an element α of $\mathcal{A}_n$ and a Borel subset W_α of $B_{2J}(\mathbf{0})\times(0,u_n]$ satisfying the following conditions, where $u_n = \min\{s_n, s_n'\}$.*

(1) $\widetilde{\psi}_{\alpha,\beta}|_{W_\alpha}$ *is a $\langle\delta\rangle$-almost identity for some normalization β of $\widetilde{\psi}_\alpha$.*

(2) $\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(W_\alpha) \approx_{(\delta)} \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_{2J}(\mathbf{0}) \times (0, u_n])$.

Proof. We set shortly $\mathcal{O}_n^{(u_n)} = \mathcal{O}_n$, $\mathcal{O}_n'^{(u_n)} = \mathcal{O}_n'$, $L_\alpha^{(u_n)} = L_\alpha$, $L_{\alpha'}^{(u_n)} = L_{\alpha'}$ for $\alpha \in \mathcal{A}_n \setminus \mathcal{O}_n$ and $\alpha' \in \mathcal{A}_n' \setminus \mathcal{O}_n'$. For a fixed constant $K > 0$ and any $\alpha \in \mathcal{A}_n \setminus \mathcal{O}_n$, we have a Borel subset $L_{\alpha,K\delta}$ of L_α satisfying the following conditions.

- $\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L_{\alpha,K\delta}) = K\delta \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L_\alpha)$.
- For any $(z,t) \in L_{\alpha,K\delta}$ and $(w,u) \in L_\alpha \setminus L_{\alpha,K\delta}$, $|\rho_{\alpha,\beta}(z)| \geq |\rho_{\alpha,\beta}(w)|$, where β is a normalization of $\widetilde{\psi}_\alpha$.

The existence of such a subset $L_{\alpha,K\delta}$ is guaranteed by the continuity of $\rho_{\alpha,\beta}$. Since ζ_n is an orientation-preserving embedding on the compact space $\mathcal{A}_n \times B_\delta(4J)$ by Lemma 4.7, $\inf_{(\alpha,z)}\{\det(D\zeta_n(\alpha,z))\} = c(J) > 0$, where (α, z) ranges over $\mathcal{A}_n \times B_\delta(4J)$. By this fact together with Lemma 4.9, one can choose the constant K so that

$$\begin{aligned}\widehat{\zeta}_n\Bigg(\mu_{\mathrm{Haar}} \times \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}\bigg(\bigcup_{\alpha\in\mathcal{A}_n\setminus\mathcal{O}_n}\{\alpha\}\times L_{\alpha,K\delta}\bigg)\Bigg)\\ > \mu_{\mathrm{Haar}} \times \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}\bigg((\mathcal{A}_n'\setminus\mathcal{O}_n')\times B_\delta(4J)\times(0,u_n] \setminus \bigcup_{\alpha'\in\mathcal{A}_n'\setminus\mathcal{O}_n'}\{\alpha'\}\times L_{\alpha'}\bigg),\end{aligned}$$

where $\widehat{\zeta}_n$ is the direct product embedding

$$\widehat{\zeta}_n = \zeta_n \times \mathrm{id}_{(0,u_n]} : \mathcal{A}_n \times B_\delta(4J) \times (0, u_n] \longrightarrow \mathcal{A}_n' \times B_\delta(4J) \times (0, u_n].$$

In fact, the left side term of the preceding inequality is greater than $c_1 c(J) K\delta$ and the right smaller than $c_2\delta$ for some constants $c_1, c_2 > 0$. It follows that there exists $(z_0, t_0) \in L_{\alpha,K\delta}$ with $\alpha \in \mathcal{A}_n \setminus \mathcal{O}_n$ such that $(\alpha', z_0, t_0) = \widehat{\zeta}_n(\alpha, z_0, t_0)$ is an element of $\{\alpha'\} \times L_{\alpha'}$ with $\alpha' \in \mathcal{A}_n' \setminus \mathcal{O}_n'$.

We will truncate elements (z,t) with relatively large $|\rho_{\alpha,\beta}(z)|$ in L_α. For simplicity, the coordinate on $\mathrm{Image}(\alpha) = \mathbb{C}\times\mathbb{R}_+$ is taken so that $\alpha(\infty) = \infty$, $\alpha(\mathbf{0}) = \mathbf{0}$ and $\alpha(w_{0;z_0}) = w_{0;z_0}$, or equivalently $\alpha' = r_{z_0}$. Let $L_\alpha^{(1)}$ be the Borel subset of L_α consisting of elements $(z,t) \in L_\alpha$ with $|\rho_{\alpha,\beta}(z)| \leq |\rho_{\alpha,\beta}(z_0)|$. Since $L_\alpha^{(1)}$ contains $L_\alpha \setminus L_{\alpha,K\delta}$, by Lemma 4.5

$$\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L_\alpha^{(1)}) \approx_{(\delta)} \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(4J) \times (0, u_n]). \tag{4.7}$$

Here we choose the normalization β with (4.2) so that $\widetilde{\psi}_{\alpha,\beta}(w_{0;z_0}, u_n)_{[\mathbb{C}]}$ coincides with $w_{0;z_0}$. This implies that $\rho_{\alpha,\beta}(z_0) = 1$ and hence

$$|\rho_{\alpha,\beta}(z)| \leq 1 \tag{4.8}$$

for any $(z,t) \in L_\alpha^{(1)}$. It follows from Lemma 4.4 and (4.7) that

$$\text{meas}_{\mathbb{C}\times\mathbb{R}_+}(N^{(u_n,m)}(L_\alpha^{(1)})) \approx_{(\delta)} \text{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(\mathbf{0}) \times (0,u_n]). \tag{4.9}$$

For any $v \in V^{(m)}(\widehat{T}_{4J})$, let

$$y_v : B_\delta(4J) \longrightarrow \mathbb{C}$$

be the similar map defined by $y_v(z) = \dfrac{z}{4J}v$. Note that $y_v(z)$ is the element of $V^{(m)}(\widehat{T}_z)$ which is the continuation of $v \in V^{(m)}(\widehat{T}_{4J})$. Let $L_{\alpha'}^{(1)}$ be the Borel subset of $L_{\alpha'}$ consisting of elements $(z',t) \in L_{\alpha'}$ such that $(\alpha' \circ y_{v'}(z'), \bar{t})$ belongs to $N^{(u_n,m)}(L_\alpha^{(1)})$ for some $v' \in V^{(m)}(\widehat{T}_{4J})$, where $\bar{t} = u_n$ if $z' = w_{0;z'}$ and otherwise $\bar{t} = t$. See Figure 4.5. Since $\alpha' \circ y_{w_{0;4J}}(z_0) = \alpha'(w_{0;z_0}) = \mathbf{0}$, (z_0,t_0) is an element of

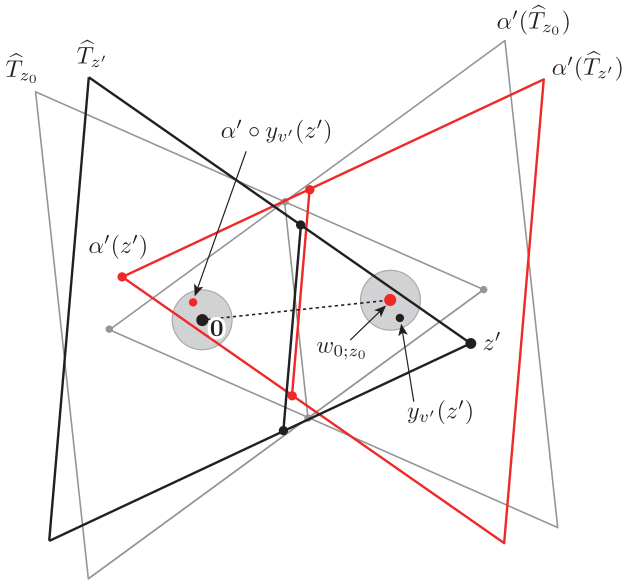


Figure 4.5. The left side shaded disk represents $B_\delta(\mathbf{0})$.

$L_{\alpha'}^{(1)}$ as well as of $L_\alpha^{(1)}$. By (4.8), we have

$$\text{dist}_{\mathbb{C}}(\mathbf{0}, \widetilde{\psi}_{\alpha,\beta}(\alpha' \circ y_{v'}(z'), \bar{t})_{[\mathbb{C}]}) \le \delta. \tag{4.10}$$

Furthermore, by Lemma 4.9 (3) and (4.9),

$$\text{meas}_{\mathbb{C}\times\mathbb{R}_+}(L_{\alpha'}^{(1)}) \approx_{(\delta)} \text{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(4J) \times (0,u_n]).$$

Let $\Delta_{z_0,i_0,t_0}^{(u_n)}$ and $\Delta_{z_0,i_1,t_0}^{(u_n)}$ be elements of $\mathcal{H}_{z_0,t_0}^{(u_n,m)}$ with $v(T_{z_0,i_0}) \ni \mathbf{0}$ and $v(T_{z_0,i_1}) = \alpha'(v(T_{z_0,i_0}))$. Then $v(T_{z_0,i_1})$ contains $w_{0;z_0} = \alpha'(\mathbf{0})$. Since $(z_0,t_0) \in L_\alpha^{(1)} \cap L_{\alpha'}^{(1)}$, $\Delta_{z_0,i_0,t_0}^{(u_n)}$, $\Delta_{z_0,i_1,t_0}^{(u_n)}$ and $\alpha'(\Delta_{z_0,i_0,t_0}^{(u_n)})$ are η-efficient. See Lemma 4.3 for $\eta = \eta(\delta,m)$. In particular, this implies that $\psi_{\alpha,\beta}(v(T_{z_0,i_1}))_{[\mathbb{C}]}$ spans a triangle T'_{z_0,i_1} arbitrarily well approximated by the regular triangle T_{z_0,i_1} if we take η sufficiently small. Since $\widetilde{\psi}_{\alpha,\beta}(v(T_{z_0,i_1})) = \widetilde{\psi}_{\alpha,\beta}(\alpha'(v(T_{z_0,i_0})))$ and $\widetilde{\psi}_{\alpha,\beta}(w_{0;z_0}, u_n)_{[\mathbb{C}]} = w_{0;z_0}$ by our choice of β, the geodesic line in $\mathbb{H}^3$ passing through $\widetilde{\psi}_{\alpha,\beta}(\alpha'(\mathbf{0}), u_n)$ and $\widetilde{\psi}_{\alpha,\beta}(\alpha'(\mathbf{0}), 1/u_n)$ is also well approximated by the Euclidean geodesic ray $\alpha'(l_{\mathbf{0}})$ in $\mathbb{C}\times\mathbb{R}_+$ in a half 3-ball centered at $\alpha'(\mathbf{0})$ and with sufficiently large radius. See Figure 4.6. This means

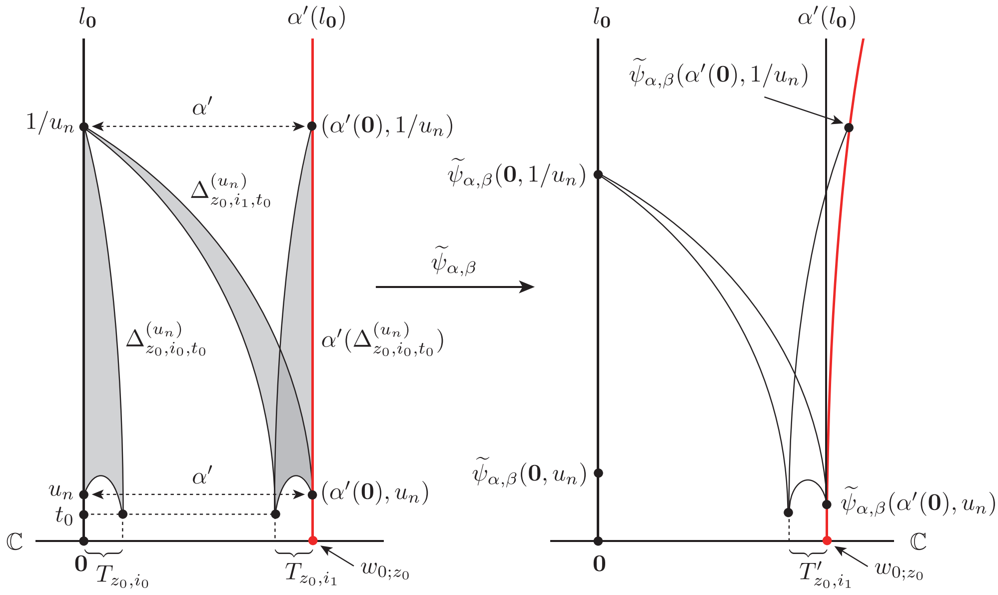


Figure 4.6. The normalization of $\widetilde{\psi}_\alpha$ centered at $\mathbf{0}$.

that β works just like a normalization of $\widetilde{\psi}_\alpha$ centered at $w_{0;z_0} = \alpha'(\mathbf{0})$. So, for any $(z', t) \in L^{(1)}_{\alpha'}$, by relying on (4.10) and Lemma 4.3 with use of $\alpha'(\mathbf{0})$ and $\alpha' \circ y_{v'}(z')$ instead of $\mathbf{0}$ and $w_{0;z}$ respectively, one can prove that $\widetilde{\psi}_{\alpha,\beta}$ is $\langle\delta\rangle$-almost identical on $\alpha'(\mathcal{V}_{z',t})$ and hence in particular on $\alpha'(W^{(u_n,m)}(L^{(1)}_{\alpha'})) \supset \alpha'(N^{(u_n,m)}(L^{(1)}_{\alpha'}))$. See Figure 4.7 (a).

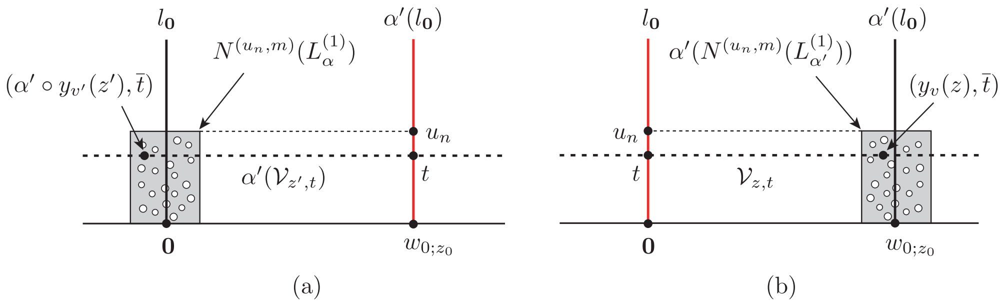


Figure 4.7. (a) For any $(z', t) \in L^{(1)}_{\alpha'}$, $\widetilde{\psi}_{\alpha,\beta}|_{\alpha'(\mathcal{V}_{z',t})}$ is a 'rotation' with the shaft $\alpha'(l_{\mathbf{0}})$, but $(\alpha' \circ y_{v'}(z'), \bar{t})$ can not go out of $B_\delta(\mathbf{0}) \times (0, u_n]$ due to (4.10). (b) A similar situation occurs for any $(z, t) \in L^{(2)}_\alpha$.

Next, by using $\alpha'(N^{(u_n,m)}(L^{(1)}_{\alpha'}))$ instead of $N^{(u_n,m)}(L^{(1)}_{\alpha})$, one can show in turn that there exists a Borel subset $L^{(2)}_\alpha$ of $L^{(1)}_\alpha$ with

$$\mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(L^{(2)}_\alpha) \approx_{(\delta)} \mathrm{meas}_{\mathbb{C}\times\mathbb{R}_+}(B_\delta(4J)\times(0,u_n])$$

and such that, for any $(z,t)\in L^{(2)}_\alpha$ and some $v\in\widehat{V}^{(m)}(\widehat{T}_{4J})$, $(y_v(z),\bar{t})$ belongs to $\alpha'(N^{(u_n,m)}(L^{(1)}_{\alpha'}))$. See Figure 4.7 (b). Since $\widetilde{\psi}_{\alpha,\beta}|_{\alpha'(N^{(u_n,m)}(L^{(1)}_{\alpha'}))}$ is $\langle\delta\rangle$-almost identical as seen above, this implies

$$\mathrm{dist}_{\mathbb{C}}(w_{0;z_0},\widetilde{\psi}_{\alpha,\beta}(y_v(z),\bar{t})_{[\mathbb{C}]})\le\delta(1+\langle\delta\rangle),$$

which corresponds to (4.10) in the first case. It follows from this fact together with Lemma 4.3 that $\widetilde{\psi}_{\alpha,\beta}$ is $\langle\delta\rangle$-almost identical also on $W_\alpha = W^{(u_n,m)}(L^{(2)}_\alpha)$, which satisfies the condition (2) by Lemma 4.4. This completes the proof. □

5. Proof of Theorem A

Throughout this section, we work under the definitions and notations given in Section 2 and prove Theorem A.

5.1. **Construction of locally bi-Lipschitz maps.** A continuous map $f: X\longrightarrow Y$ between metric spaces is called a *locally K-bi-Lipschitz* if, for any $x\in X$, the restriction of f on the r-ball $\mathcal{B}_r(x)$ for some $r>0$ is a K-bi-Lipschitz map onto a closed neighborhood of $f(x)$ in Y. The aim of this subsection is to show that, for the neighborhood E of any simply degenerate end $\mathcal{E}$ of M with respect to a finite core, the restriction $\varphi|_{E_{\mathrm{thick}}}: E_{\mathrm{thick}}\longrightarrow E'=\varphi(E)$ is properly homotopic to a locally bi-Lipschitz map if φ and hence ψ satisfy the ω-upper bound condition on E.

By Lemmas 1.4 and 2.2, there exists a generator system $\gamma^{(n)}_1,\dots,\gamma^{(n)}_u$ of $\pi_1(R_n,y_n)$ with $u\le u_0$ and

$$2\mu_0<\mathrm{tl}(\rho_n(\gamma^{(n)}_j))\le\mathrm{tl}(\rho_n(\gamma^{(n)}_j),x_n)\le\lambda_0 \tag{5.1}$$

for any $j=1,\dots,u$ and some constant $\lambda_0>0$ independent of j and n, where $\rho_n:\pi_1(R_n,y_n)\longrightarrow \mathrm{PSL}_2(\mathbb{C})$ is the holonomy associated the covering transformation on $\mathbb{H}^3$ based at x_n. We set $\widehat{\gamma}_j=\rho_n(\gamma^{(n)}_j)$ for short and denote by $l(\widehat{\gamma}_j)$ the axis of the loxodromic element $\widehat{\gamma}_j$. For any point $x'_n\in\mathbb{H}^3$ with $\mathrm{dist}_{\mathbb{H}^3}(x_n,x'_n)\le 2r_0/3$, by (5.1), there exists a constant $d(r_0)>0$ with $\mathrm{dist}_{\mathbb{H}^3}(x'_n,l(\gamma^{(n)}_j))\le d$. So one can have a uniform constant $J=J(r_0)>4$ such that, for any $\alpha\in\mathcal{A}_n\setminus\mathcal{O}_n$ and any coordinate $\mathbb{C}\times\mathbb{R}_+$ on $\mathbb{H}^3$ with $\alpha x_n=(\mathbf{0},1)$, at least one of the end points of $\widehat{\gamma}_j$ is contained in $B_J(\mathbf{0})$. If necessary replacing $\widehat{\gamma}_j$ by $\widehat{\gamma}_j^{-1}$, we may assume that the attracting fixed point of $\widehat{\gamma}_j$ is contained in $B_J(\mathbf{0})$. For a $\tau>0$, two representations $\rho_0,\rho_1:\pi_1(R_n,y_n)\longrightarrow\mathrm{PSL}_2(\mathbb{C})$ are said to be *τ-close* to each other with respect to $\gamma^{(n)}_j$ $(j=1,\dots,u)$ if $\rho_0(\gamma^{(n)}_j)\rho_1(\gamma^{(n)}_j)^{-1}=\pm\begin{pmatrix}1+\tau_1&\tau_2\\ \tau_3&1+\tau_4\end{pmatrix}$ satisfies $|\tau_i|\le\tau$ for $i=1,2,3,4$ under a suitable coordinate $\mathbb{C}\times\mathbb{R}_+$ on $\mathbb{H}^3$ with $x_n=(\mathbf{0},1)$. Let $\rho':\pi_1(E',y'_0)\longrightarrow\mathrm{PSL}_2(\mathbb{C})$ be the holonomy of E'. For a vertical core F_n of R_n, the inclusion $F_n\longrightarrow E$ is π_1-injective. Since F_n is a deformation retract of R_n, $(\varphi|_{R_n})_*=(\psi|_{R_n})_*:\pi_1(R_n,y_n)\longrightarrow\pi_1(E',\varphi(y_n))$ is also injective, where $\psi: M\longrightarrow M'$ is the continuous map defined in Subsection 3.2.

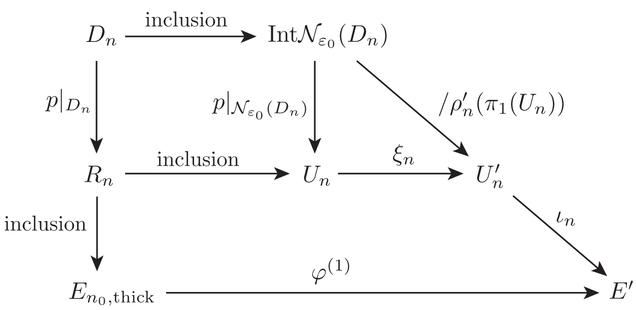
Lemma 5.1. *Let τ be any positive number. Then there exists $n_0 \in \mathbb{N}$ such, for any $n \geq n_0$, the following condition (*) holds.*
(*) $\rho_n : \pi_1(R_n, y_n) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ *is τ-close to* $\rho'_n : \pi_1(R_n, y_n) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ *with respect to $\gamma_1^{(n)}, \dots, \gamma_u^{(n)}$, where ρ'_n is the representation defined by* $\rho'_n(\cdot) = \beta_n(\rho' \circ (\varphi|_{R_n})_*(\cdot))\beta_n^{-1}$ *for some $\beta_n \in \mathrm{PSL}_2(\mathbb{C})$.*

Proof. By using an argument quite similar to that in the proof of the assertion (3.7) in [So2, page 2767], one can show that ρ'_n satisfies (*) for a $\tau(\delta) > 0$ with $\lim_{\delta \to 0} \tau(\delta) = 0$ for all sufficiently large n. Here we use Lemma 4.10 instead of [So2, Lemma 6]. See Figure 5 in [So2] for the situation. To complete the proof, it suffices to take $\delta > 0$ with $\tau(\delta) < \tau$. □

For the integer $n_0 > 0$ given in Lemma 5.1, let $E_{n_0} = \bigcup_{n=n_0}^{\infty} N_n$, $E_{n_0,\mathrm{thick}} = E_{n_0} \cap E_{\mathrm{thick}}$ and $\partial_1 E_{n_0,\mathrm{thick}} = E_{n_0,\mathrm{thick}} \cap E_{\mathrm{thin}}$. Then we have $E_{n_0,\mathrm{thick}} = \bigcup_{n \geq n_0} R_n$ and $\partial_1 E_{n_0,\mathrm{thick}} = \bigcup_{n \geq n_0} \partial_1 R_n$. See Figures 2.3 and 2.4 in Section 2. For any $n \geq n_0$, let D_n be a Dirichlet fundamental domain of R_n in $\mathbb{H}^3$ centered at x_n. By an argument used in the proof of Proposition 5.1 in [Th1, Chapter 5] (see also [CEG, Theorem I.1.7.1]), one can show that there exists an $\varepsilon_0 > 0$ independent of n which satisfies the following conditions.

- For the open ε_0-neighborhood $\mathrm{Int}\mathcal{N}_{\varepsilon_0}(D_n)$ of D_n in $\mathbb{H}^3$, the image $U_n = p(\mathrm{Int}\mathcal{N}_{\varepsilon_0}(D_n))$ is a deformation retract of R_n.
- There exists an (abstract) incomplete hyperbolic 3-manifold U'_n and a $(1+\kappa)$-bi-Lipschitz map $\xi_n : U_n \longrightarrow U'_n$ such that the holonomy of the hyperbolic structure on U'_n with the marking ξ_n is equal to the representation $\rho'_n : \pi_1(R_n, y_n) = \pi_1(U_n, y_n) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ in Lemma 5.1, where $\kappa = \kappa(\tau) > 0$ is a constant with $\lim_{\tau \to 0} \kappa(\tau) = 0$.

Here the fact of ε_0 being independent of n is derived from the boundedness of geometry on R_n ($n \geq n_0$). If $U_m \cap U_n \neq \emptyset$ for $m \neq n$, then $U_m \cap U_n$ is a slim open neighborhood of the compact surface $R_m \cap R_n$. By Lemma 5.1, $\rho'_m = (\beta_m \beta_n^{-1})\rho'_n(\beta_m \beta_n^{-1})^{-1}$ on $\pi_1(U_n \cap U_m)$. Hence one can choose ξ_m and ξ_n so that there exists a marking-preserving isometry $\zeta_{m,n} : \xi_m(U_m \cap U_n) \longrightarrow \xi_n(U_m \cap U_n)$ with $\zeta_{m,n} \circ \xi_m|_{U_m \cap U_n} = \xi_n|_{U_m \cap U_n}$. Note that ρ'_n is the restriction of the holonomy $\rho' : \pi_1(E', y'_0) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ of $E' = \varphi(E)$. Thus there exists a locally isometric marking-preserving immersion $\iota_n : U'_n \longrightarrow E'$. By using ξ_n's and ι_n's, we have a locally $(1+\kappa)$-bi-Lipschitz immersion $\varphi^{(1)} : E_{n_0,\mathrm{thick}} \longrightarrow E'$ properly homotopic to $\varphi|_{E_{n_0,\mathrm{thick}}}$. The following diagram presents the connection of the above maps.



By applying our arguments with $\mu_0/2$ instead of μ_0, we may extend $\varphi^{(1)}$ to a locally $(1+\kappa)$-bi-Lipschitz map from $E_{n_0,\mathrm{thick}(\mu_0/2)}$ to E', which is still denoted by $\varphi^{(1)}$. For any $x \in E_{n_0,\mathrm{thick}(\mu_0/2)}$, let $i_* : \pi_1(E_{n_0,\mathrm{thick}(\mu_0/2)}, x) \longrightarrow \pi_1(E, x)$ be the homomorphism induced from the inclusion. We denote by $\rho_x : \pi_1(E, x) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ and $\rho'_{\varphi^{(1)}(x)} : \pi_1(E', \varphi^{(1)}(x)) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ the holonomies of (E, x) and $(E', \varphi^{(1)}(x))$ respectively. By Lemma 5.1 together with the construction of $\varphi^{(1)}$, one can suppose that, for any $\gamma \in i_*(\pi_1(E_{n_0,\mathrm{thick}(\mu_0/2)}, x)) \subset \pi_1(E, x)$ with $\mu_0 \leq \mathrm{tl}(\rho_x(\gamma), x) \leq 4\mu_0$,

$$\frac{1}{1+\kappa} \leq \frac{\mathrm{tl}\big(\rho_x(\gamma), x\big)}{\mathrm{tl}\big(\rho_{\varphi^{(1)}(x)}(\varphi^{(1)}_*(\gamma)), \varphi^{(1)}(x)\big)} \leq 1+\kappa \tag{5.2}$$

holds if necessary replacing τ by a smaller positive number.

5.2. **Proof of Theorem A.** We will extend the locally $(1+\kappa)$-bi-Lipschitz map $\varphi^{(1)} : E_{n_0,\mathrm{thick}} \longrightarrow E'$ given in the previous subsection to a bi-Lipschitz map $\Phi_E : E \longrightarrow E'$ required in Theorem A.

Proof of Theorem A. First we show that $\varphi^{(1)} : E_{n_0,\mathrm{thick}} \longrightarrow E'$ is a proper map. There exist closed non-contractible loops l_n in R_n with $\sup_n\{\mathrm{length}_M(l_n)\} < \infty$ and not homotopic to a loop in $\partial_1 R_n$. If $\varphi^{(1)}$ were not proper, then there would exist infinitely many R_n $(n \geq n_0)$ the $\varphi^{(1)}$-images of which stay a bounded region of E'. If necessary passing to a subsequence, we may assume that l_{n_i} are not mutually homotopic in E. Then $\varphi^{(1)}(l_{n_i})$ are non-contractible loops in E' which are not mutually homotopic. On the other hand, since $\sup_i\{\mathrm{length}_{E'}(\varphi^{(1)}(l_{n_i}))\} < \infty$, $\bigcup_i \varphi^{(1)}(l_{n_i})$ would not be in a bounded region of E', a contradiction. This shows that $\varphi^{(1)}$ is a proper map. Moreover this implies that E' is the neighborhood of a simply degenerated end of M' with respect to the finite core $C' = \varphi(C)$ of M'.

Let T be any component of $\partial_1 E_{n_0,\mathrm{thick}}$ homeomorphic to either a torus or a half-open annulus. Since T excises from E_{n_0} a connected submanifold containing a component of the pure $\mu_0/2$-thin part $E_{n_0,\text{p-thin}(\mu_0/2)}$ (see Definition 1.1), it follows from (5.2) that $\varphi^{(1)}(T)$ is contained in $E'_{\mathrm{thin}((1+\kappa)\mu_0)}$. Consider the union $E'^{(1)}_{\mathrm{thin}((1+\kappa)\mu_0)}$ of components of $E'_{\mathrm{thin}((1+\kappa)\mu_0)}$ meeting $\varphi^{(1)}(\partial_1 E_{n_0,\mathrm{thick}})$ non-trivially and denote the closure $\overline{E' \setminus E'^{(1)}_{\mathrm{thin}((1+\kappa)\mu_0)}}$ by $E'^{(1)}_{\mathrm{thick}((1+\kappa)\mu_0)}$. Since φ is cusp-preserving, $E'^{(1)}_{\mathrm{thin}((1+\kappa)\mu_0)}$ contains $E'_{\mathrm{cusp}((1+\kappa)\mu_0)}$. Let H' be a properly embedded surface in E' satisfying the following conditions.

- The inclusion $H' \longrightarrow E'$ is a homotopy equivalence. Moreover H' is disjoint from $E'_{\mathrm{tube}((1+\kappa)\mu_0)}$ and meets $\partial E'_{\mathrm{cusp}((1+\kappa)\mu_0)}$ transversely.
- The inclusion $H'_0 = H' \cap E'_{\mathrm{main}((1+\kappa)\mu_0)} \longrightarrow E'_{\mathrm{main}((1+\kappa)\mu_0)}$ is a homotopy equivalence, and H'_0 is contained in $\varphi^{(1)}(\mathrm{Int}E_{n_0,\mathrm{thick}})$. See Figure 5.1.

Let $E'_{H'}$ be the closure of the component of $E'^{(1)}_{\mathrm{thick}((1+\kappa)\mu_0)} \setminus H'$ adjacent to $\mathcal{E}'$ and $E^\flat_H = (\varphi^{(1)})^{-1}(E'_{H'})$. Since the restriction $\varphi^{(1)}|_{E^\flat_H} : E^\flat_H \longrightarrow E'_{H'}$ is a proper surjective immersion, $\varphi^{(1)}|_{E^\flat_H}$ is a locally $(1+\kappa)$-bi-Lipschitz covering. Since E has only one end, $E^\flat_H$ is connected. We set $(\varphi^{(1)})^{-1}(H') = H_0$ and $H^\flat_0 = H_0 \cap E^\flat_H$. Consider the restriction $f_F : F(\sigma) \longrightarrow E_{n_0,\mathrm{thick}}$ of any pleated map

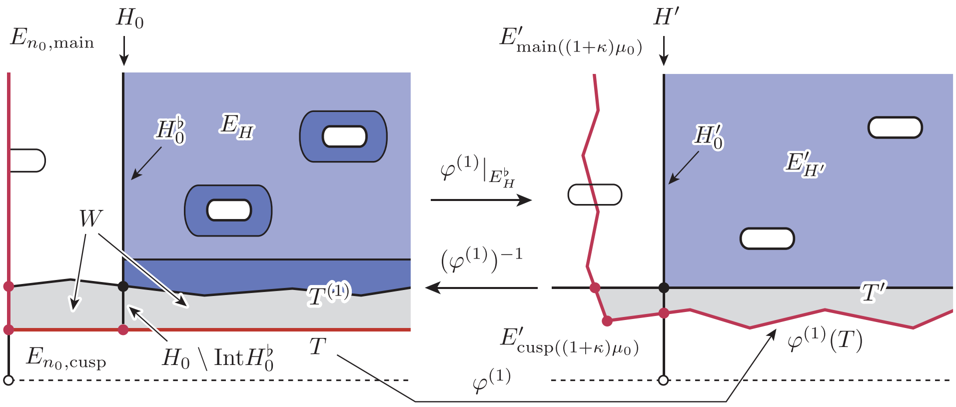


Figure 5.1. The union of light and dark blue regions in E_{n_0} (on the left hand side) represents $E_H^\flat$.

$f : \Sigma(\sigma) \longrightarrow E_{n_0}$ satisfying the conditions (Y1) and (Y2) in Subsection 1.1. By applying an argument in the proof of Lemma 1.3 to the composition $f_F^{(1)} = \varphi^{(1)} \circ f_F : F(\sigma) \longrightarrow E'$, one can prove that any component of $(f_F^{(1)})^{-1}(E'^{(1)}_{\mathrm{thin}((1+\kappa)\mu_0)})$ is a peripheral annulus in $F(\sigma)$. In particular, this implies that, for each component T' of $E'^{(1)}_{\mathrm{thick}((1+\kappa)\mu_0)} \cap E'_{\mathrm{cusp}((1+\kappa)\mu_0)}$, any component $T^{(1)}$ of $(\varphi^{(1)})^{-1}(T')$ excises from $E_{n_0,\mathrm{thick}}$ a manifold W such that $(W, T^{(1)}, W \cap E_{n_0,\mathrm{cusp}})$ is homeomorphic to $(A\times[0,\infty), l_0\times[0,\infty), l_1\times[0,\infty))$, where A is an annulus the boundary ∂A of which is a disjoint union of two loops l_0 and l_1. Deforming $\varphi^{(1)}$ by a homotopy supported on W, we may assume that each component of $H_0 \setminus \mathrm{Int}H_0^\flat$ is an annulus. Such a deformation can be accomplished by a standard argument of 3-manifold topology. For example, see Lemma 6.5 in Hempel [He]. Note that $H_0^\flat = H_0$ if $E_{n_0,\mathrm{cusp}} = \emptyset$.

Since $i \circ \varphi^{(1)}|_{E_H^\flat}$ is homotopic to $\varphi|_{E_H^\flat} : E_H^\flat \longrightarrow E'$, $\varphi^{(1)}|_{E_H^\flat}$ is extended to a (not necessarily locally bi-Lipschitz) continuous map from $E_{n_0,\mathrm{main}}$ to E', where $i : E'_{H'} \longrightarrow E'$ is the inclusion. If H_0 were compressible in $E_{n_0,\mathrm{main}}$, then $H_0^\flat$ would not be π_1-injective in $E_{n_0,\mathrm{main}}$. Since the covering $\varphi^{(1)}|_{H_0^\flat} : H_0^\flat \longrightarrow H'_0$ is π_1-injective, it follows that H'_0 is not π_1-injective in E'. This contradicts that H'_0 is incompressible in $E'_{\mathrm{main}((1+\kappa)\mu_0)}$. So H_0 is incompressible in $E_{n_0,\mathrm{main}}$. Since $H_0^\flat$ is not an annulus, any component of H_0 is not so. Note that ∂H_0 is contained in $\partial_1 E_{n_0,\mathrm{main}} = E_{n_0,\mathrm{main}} \cap E_{n_0,\mathrm{cusp}}$. Each component of $\partial_1 E_{n_0,\mathrm{main}}$ is a half-open annulus. If a component F_0 of H_0 were boundary-compressible in $(E_{n_0,\mathrm{main}}, \partial_1 E_{n_0,\mathrm{main}})$, then the boundary $\partial F_0^\vee$ of the boundary-compressed surface $F_0^\vee$ would have a component which is contractible in $\partial_1 E_{n_0,\mathrm{main}}$. Since F_0 is incompressible in $E_{n_0,\mathrm{main}}$, $F_0^\vee$ is a disk and hence F_0 is an annulus, a contradiction. It follows that H_0 is not only incompressible but also boundary-incompressible in $(E_{n_0,\mathrm{main}}, \partial_1 E_{n_0,\mathrm{main}})$. Since moreover $E_{n_0,\mathrm{main}}$ is homeomorphic to $\Sigma_{\mathrm{main}} \times [0,\infty)$, H_0 is a disjoint union of mutually parallel surfaces in $E_{n_0,\mathrm{main}}$, which are homeomorphic to Σ_{main}. Since $E_H^\flat$ is connected and adjacent to $\mathcal{E}$, $H_0^\flat$ and hence H_0 are connected. So, as well as H_0, $H_0^\flat$ is homeomorphic to Σ_{main}. This proves that the covering $\varphi^{(1)}|_{H_0^\flat} : H_0^\flat \longrightarrow H'_0$ is a homeomorphism. Thus $\varphi^{(1)}|_{E_H^\flat} : E_H^\flat \longrightarrow E'_{H'}$ is a $(1+\kappa)$-bi-Lipschitz map. Each component of $E_{n_0,\mathrm{thin}((1+\kappa)^2\mu_0)}$ contains a component of $E_{n_0,\text{p-thin}((1+\kappa)^2\mu_0/2)}$ and hence that of $E_{n_0,\text{p-thin}}$. Consider the union G_H of components of $E_{n_0,\mathrm{thin}((1+\kappa)^2\mu_0)} \setminus \mathrm{Int}E_{n_0,\text{p-thin}}$ meeting $\partial E_H^\flat \setminus \mathrm{Int}H_0^\flat$ non-trivially. By (5.2), $\partial G_H \cap \partial E_H^\flat = \emptyset$ and hence G_H contains $\partial E_H^\flat \setminus \mathrm{Int}H_0^\flat$ as a core. The union of dark blue regions in Figure 5.1 represents $G_H \cap E_H^\flat$. Let E_H be the closure of $E_H^\flat \setminus G_H$. Composing $(\varphi^{(1)}|_{E_H^\flat})^{-1}$ with an ambient isotopy in E_{n_0}, we have a $K^{(2)}$-bi-Lipschitz map $(\varphi^{(2)})^{-1} : E'_{H'} \longrightarrow E_H$ such that $i \circ \varphi^{(2)}$ is homotopic to $\varphi|_{E_H} : E_H \longrightarrow E'$ for some constant $K^{(2)} > 1 + \kappa$.

We denote by E_H^+ (resp. $E'^+_{H'}$) the closure of the component of $E_{\mathrm{main}((1+\kappa)^2\mu_0)} \setminus H_0$ (resp. $E'_{\mathrm{main}((1+\kappa)\mu_0)} \setminus H'_0$) containing E_H (resp. $E'_{H'}$). Then any component V of the closure $\overline{E_H^+ \setminus E_H}$ is a solid torus. Let m be a meridian of V. Since $\varphi(m)$ is contractible in E', $\varphi^{(2)}(m)$ is so. It follows that $\varphi^{(2)}(m)$ is a meridian of the component V' of $E'^{(1)}_{\mathrm{thin}((1+\kappa)\mu_0)}$ with $\partial V' = \varphi^{(2)}(\partial V)$. By using Lemma 3.4 in Minsky [Mi2, Subsection 3.4], one can extend $\varphi^{(2)}$ to a $K^{(3)}$-bi-Lipschitz map $\varphi^{(3)} : E_H^+ \longrightarrow E'^+_{H'}$ for some constant $K^{(3)} > K^{(2)}$. Since both $\overline{E_{\mathrm{main}((1+\kappa)^2\mu_0)} \setminus E_H^+}$ and $\overline{E'_{\mathrm{main}((1+\kappa)\mu_0)} \setminus E'^+_{H'}}$ are compact, $\varphi^{(3)}$ is also extended to a K_E-bi-Lipschitz map $\Phi_E : E \longrightarrow E'$ for some $K_E > K^{(3)}$. Since the original $\varphi|_E$ and Φ_E are marking preserving homeomorphisms from E to E', they are properly homotopic to each other. This completes the proof of Theorem A. □

Here we note that the above result by Minsky is proved by using standard arguments of hyperbolic and differential geometry and has no connection with the theory of curve complex.

6. Geometric limits of limits

Ending laminations are geometric limits of geodesic loops tending toward ends of hyperbolic 3-manifolds. Earthquakes are limit operations of Finchel-Nielsen twists. We study here geometric limits of ending laminations and earthquakes.

Throughout this section, we suppose that $\mathcal{E}$ is a simply degenerate end of M with ending lamination ν, E is the neighborhood of $\mathcal{E}$ with respect to a finite core C of M, and $f_n : \Sigma_n = \Sigma(f_n) \longrightarrow E$ are pleated maps tending toward $\mathcal{E}$.

6.1. Geometric limits of pleated maps and supervising markings.

Convention 6.1. Let $\{x_n\}$ be a sequence in a metric space X. If $\{x_n\}$ has a subsequence converging to x_0 in X, then we usually say that $\{x_n\}$ converges to x_0 *if necessary passing to a subsequence*. However, for short, we may omit the phrase 'if necessary ...' if it does not cause any confusions. In particular, for a sequence $\{t_n\}$ of real numbers, $\limsup_{n\to\infty} t_n$ (or $\liminf_{n\to\infty} t_n$) is often considered as $\lim_{n\to\infty} t_n$.

Definition 6.2 (Geometric limits of pleated maps)**.** Consider a maximal union $J(f_n)$ of simple geodesic loops in $\Sigma_{n,\mathrm{thin}}$ such that $\mathrm{length}_{\Sigma_n}(J(f_n))$ converges to zero. The union $J(f_n)$ is called the *joint* of f_n. Set $\Sigma_n^\vee = \Sigma_n \setminus J(f_n)$. Let $F_{n,1}, \dots, F_{n,k_n}$ be the components of $\Sigma_n^\vee$. Fix a base point $x_{n,i}$ of $F_{n,i}$ with $x_{n,i} \in \Sigma_{n,\mathrm{thick}}$ and set $y_{n,i} = f_n(x_{n,i})$. Let $E_{n,i}$ be the manifold E with $y_{n,i}$ as its base point. If necessary renumbering 'i' of $f_{n,i}$, one can assume that the sequence $\{f_{n,i}\}$ with $f_{n,i} = f|_{F_{n,i}} : F_{n,i} \longrightarrow E_{n,i}$ geometrically converges to a pleated map $f_{\infty,i} : F_{\infty,i} \longrightarrow E_{\infty,i}$, where all k_n $(n = 1, 2, \dots)$ have the same value k_0 and $E_{\infty,i}$ is a geometric limit hyperbolic 3-manifold of $\{E_{n,i}\}_n$. If $\sup_n\{\mathrm{dist}_{E_{\mathrm{thick}}}(y_{n,i}, y_{n,j})\} < \infty$ for fixed $i, j \in \{1, \dots, k_0\}$, then one can suppose that $E_{\infty,i} = E_{\infty,j}$, and otherwise $E_{\infty,i} \cap E_{\infty,j} = \emptyset$. Let E_∞ be a maximal union of mutually disjoint $E_{\infty,i}$'s. By matching up them, we have an locally pathwise isometric map $f_\infty : \Sigma_\infty \longrightarrow E_\infty$ satisfying the following conditions.

- Σ_∞ is a disjoint union of connected complete hyperbolic surfaces $F_{\infty,j}$ ($j = 1, \dots, k_0$) of finite area homeomorphic to $F_{n,j}$ such that the restriction $f_\infty|_{F_{\infty,j}} : F_{\infty,j} \longrightarrow E_\infty$ is a pleated map. In particular, $\Sigma_n^\vee$ is homeomorphic to Σ_∞ for all sufficiently large n.
- There exists the R_n-neighborhood $\mathcal{N}_{\infty,j;n}$ of $f_\infty(F_{\infty,j,\mathrm{main}})$ in E_∞ and a locally K_n-bi-Lipschitz embedding $\zeta_{n,j}$ from $\mathcal{N}_{\infty,j;n}$ to E with $\lim_{n\to\infty} R_n = \infty$ and $\lim_{n\to\infty} K_n = 1$. Moreover, $\zeta_{n,j} \circ f_\infty|_{F_{\infty,j,\mathrm{main}}}$ is homotopic to $f|_{F_{n,j,\mathrm{main}}}$ up to marking by a homotopy with an arbitrarily small translation distance for all sufficiently large.

Note that $\mathcal{N}_{\infty,i;n} \cap \mathcal{N}_{\infty,j;n} \neq \emptyset$ for all sufficiently large n if $E_{\infty,i} = E_{\infty,j}$. Then one can choose $\zeta_{n,i}$, $\zeta_{n,j}$ so that $\zeta_{n,i}|_{\mathcal{N}_{\infty,i;n}\cap\mathcal{N}_{\infty,j;n}} = \zeta_{n,j}|_{\mathcal{N}_{\infty,i;n}\cap\mathcal{N}_{\infty,j;n}}$. In general, the topological type of E_∞ is very complicated. It is possible that E_∞ has infinitely many simply degenerate ends and infinitely many wild (i.e. geometrically infinite but not simply degenerate) ends simultaneously. For example, see Ohshika-Soma [OS, Theorem C]. However, since we are mainly concerned with a bounded neighborhood of $f_\infty(\Sigma_{\infty,\mathrm{main}})$ in E_∞, the complexity does not influence our arguments essentially.

Now we consider the locally bi-Lipschitz embedding

$$\zeta_n : \mathcal{N}_{\infty;n} = \mathcal{N}_{\infty,1;n} \cup \cdots \cup \mathcal{N}_{\infty,k_0;n} \longrightarrow E \tag{6.1}$$

defined by $\zeta_n|_{\mathcal{N}_{\infty,j;n}} = \zeta_{n,j}$. We denote by $E_{n(\mathrm{cusp})}$ (resp. $\Sigma_{n(\mathrm{cusp})}$) the union of the components of E_{thin} (resp. $\Sigma_{n,\mathrm{thin}}$) corresponding to cusps of E_∞ (resp. Σ_∞) via ζ_n^{-1}. We define $\Sigma_{n(\mathrm{main})} = \Sigma_n \setminus \mathrm{Int}\Sigma_{n(\mathrm{cusp})}$. Here '(cusp)' and '(main)' in parenthesis mean that the eventually cuspidal and permanently main parts of Σ_n, respectively. We say that f_∞ is a *limit pleated map* of $\{f_n\}$. Then there exists a K_n-bi-Lipschitz map

$$\xi_n : \Sigma_{\infty,\mathrm{main}} \longrightarrow \Sigma_{n(\mathrm{main})} \tag{6.2}$$

with $\lim_{n\to\infty} K_n = 1$ and such that $\{\zeta_n^{-1} \circ f_n \circ \xi_n\}$ converges to $f_\infty|_{\Sigma_{\infty,\mathrm{main}}}$ uniformly as $n \to \infty$. We denote by $\mathcal{E}_\infty$ the union of all ends $\mathcal{E}_\alpha$ of E_∞ which are not $\mathbb{Z}\times\mathbb{Z}$-cusps and have neighborhoods N_α in E_∞ such that $\zeta_n(N_\alpha \cap \mathcal{N}_{\infty;n})$ is contained in the component of $E \setminus f_n(\Sigma_n)$ adjacent to $\mathcal{E}$ for all sufficiently large n.

Now we give the definition of geometric limits of geodesic laminations.

Definition 6.3 (Geometric limits of laminations)**.** A geodesic segment α in a hyperbolic surface is called *unit* if the length of α is one. We say that a sequence of laminations μ_n on Σ_n with compact support *geometrically converges* to a lamination μ_∞ on Σ_∞ if the following (1) and (2) hold.

(1) For any unit geodesic segment α_∞ in $\mu_\infty \cap \Sigma_{\infty,\mathrm{main}}$, there exist unit geodesic segments α_n in $\mu_n \cap \Sigma_{n(\mathrm{main})}$ such that $\xi_n^{-1}(\alpha_n)$ uniformly converges to α_∞.
(2) Consider any subsequence of unit geodesic segments α_{n_j} in $\mu_{n_j} \cap F_{n_j(\mathrm{main})}$ such that $\xi_{n_j}^{-1}(\alpha_{n_j})$ is geometrically convergent. Then the limit of $\xi_{n_j}^{-1}(\alpha_{n_j})$ is a unit geodesic segment in $\mu_\infty \cap \Sigma_{\infty,\mathrm{main}}$.

Note that a geodesic lamination μ_∞ on Σ_∞ extending $\mu_\infty \cap \Sigma_{\infty,\mathrm{main}}$ is uniquely determined, which is called a *limit lamination* of $\{\mu_n\}$.

To compare structures of limit hyperbolic surfaces, we introduce the notion of supervising markings. We study the deformation of such structures by using limits of left earthquakes on the supervising surface. We know that the hyperbolic structures on Σ_n are not in a bounded region of the Teichmüller space Teich(Σ). Let $\Sigma^\natural$ be the surface Σ with a fixed hyperbolic structure of finite area and $\mathcal{H}^\natural$ a fixed hoop family of $\Sigma^\natural$. Then the next lemma follows immediately from standard facts on hyperbolic geometry. Recall here that we work under Convention 6.1.

Lemma 6.4. *There exists a constant $K = K(\{f_n\}) > 1$ and a sufficiently small $\delta > 0$ satisfying the following condition.*

- *For some unions J_n $(n = 1, 2, \dots)$ and J_∞ of components of $\mathcal{H}^\natural$, there exist K-bi-Lipschitz maps $h_n : \Sigma^\natural_{\mathrm{main}} \setminus \mathrm{Int}\mathcal{N}_\delta(J_n) \longrightarrow \Sigma_{n(\mathrm{main})}$ and $h_\infty : \Sigma^\natural_{\mathrm{main}} \setminus \mathrm{Int}\mathcal{N}_\delta(J_\infty) \longrightarrow \Sigma_{\infty,\mathrm{main}}$ such that $\xi_n^{-1} \circ h_n$ converges uniformly to h_∞, where $\xi_n : \Sigma_{\infty,\mathrm{main}} \longrightarrow \Sigma_{n(\mathrm{main})}$ is a K_n-bi-Lipschitz map with $\lim_{n\to\infty} K_n = 1$ given in* (6.2).

Let $\mathcal{V}(f_\infty)$ be the union of cusps in E_∞ meeting $f_\infty(\Sigma_{\infty,\mathrm{cusp}})$ non-trivially and corresponding to components of E_{tube} via ζ_n. Note that the components of J_∞ bijectively correspond to the components of $\mathcal{V}(f_\infty)$. We may assume that all J_n and J_∞ are the same union J_0 of components of $\mathcal{H}^\natural$. Set $\Sigma^{\natural(\delta)}_{\mathrm{main}} = \Sigma^\natural_{\mathrm{main}} \setminus \mathrm{Int}\mathcal{N}_\delta(J_0)$ for short. We say that h_∞ is a *supervising marking* of $\Sigma^\natural$ for Σ_∞. Let $\widehat{h}_n : \Sigma^\natural \longrightarrow \Sigma_n$ be a homeomorphism extending h_n and such that $\widehat{h}_n(J_0)$ is equal to the joint $J(f_n)$ of f_n (see Definition 6.2) and the following diagram is eventually commutative as $n \to \infty$ in the sense of (6.2) and Lemma 6.4, where ζ_n, ξ_n are the maps defined respectively as (6.1), (6.2) and $i_n : \Sigma_{\infty,\mathrm{main}} \longrightarrow \Sigma_\infty$ and $j_n : \Sigma_{n(\mathrm{main})} \longrightarrow \Sigma_n$ are the inclusions.

$$
\begin{array}{ccccc}
\Sigma^{\natural(\delta)}_{\mathrm{main}} & \xrightarrow{h_\infty} & \Sigma_{\infty,\mathrm{main}} & \xrightarrow{f_\infty \circ i_n} & E_\infty \\
{\scriptstyle \mathrm{inclusion}}\downarrow & & \downarrow{\scriptstyle j_n \circ \xi_n} & & \downarrow{\scriptstyle \zeta_n \ (\mathrm{locally})} \\
\Sigma^\natural & \xrightarrow[\widehat{h}_n]{} & \Sigma_n & \xrightarrow[f_n]{} & E
\end{array}
\tag{6.3}
$$

We say that J_0 is the *joint* for $\{\widehat{h}_n\}$.

From now on, if the supervising marking $\widehat{h}_n$ is fixed, then the lamination supervised by $\mu_n^\natural$ is denoted by μ_n and vice versa. Let $\mu_\infty^\natural$ be a geometric limit of $\mu_n^\natural$ and μ_∞ the lamination in Σ_∞ supervised by $\mu_\infty^\natural$. We note that, if $\mu_\infty^\natural \subset J_0$, then μ_∞ is empty. If $f_n(\mu_n)$ is realizable as a geometric lamination in E, then the realized lamination in E is denoted by μ_n^*.

Note that our choice of $\widehat{h}_n$ has some ambiguity. For a simple closed geodesic $l^\natural$ in $\Sigma^\natural$ meeting a component j_0 of J_0 transversely, one can choose supervising markings $\widehat{h}_n : \Sigma^\natural \longrightarrow \Sigma_n$ and $\widehat{h}'_n : \Sigma^\natural \longrightarrow \Sigma_n$ so that $\widehat{h}_n(j_0) = \widehat{h}'_n(j_0)$ and the intersection number of l_n and l'_n in Σ_n diverges to infinity, where l_n, l'_n are the realizations of $\widehat{h}_n(l^\natural)$ and $\widehat{h}'_n(l^\natural)$ in Σ_n respectively. In the next subsection, we will take $\widehat{h}_n$ satisfying Assumption 6.6 so as to avoid such a difficulty.

Let $f_n^{(i)} : \Sigma_n^{(i)} \longrightarrow E$ $(i = 1, 2)$ be pleated maps geometrically converging to $f_\infty^{(i)} : \Sigma_\infty^{(i)} \longrightarrow E_\infty^{(i)}$. Suppose that there exist components $F_\infty^{(i)}$ of $\Sigma_\infty^{(i)}$ such that

$f_\infty^{(1)}|_{F_\infty^{(1)}}$ is properly homotopic to $f_\infty^{(2)}|_{F_\infty^{(2)}}$ in $\widehat{E}_\infty = E_\infty^{(1)} \cap E_\infty^{(2)}$. Then it is not hard to see that there exist subsurfaces $F_n^{(i)}$ of $\Sigma_n^{(i)}$ with geodesic boundary and a marking-preserving smooth K-bi-Lipschitz map $\iota_\infty : F_\infty^{(1)} \longrightarrow F_\infty^{(2)}$ for some $K > 1$ such that $\iota_\infty|_{F^{(1)}_{\infty,\mathrm{cusp}}} : F^{(1)}_{\infty,\mathrm{cusp}} \longrightarrow F^{(2)}_{\infty,\mathrm{cusp}}$ is isometric.

Lemma 6.5. *With the assumptions as above, suppose that $\lambda_n^{(i)}$ $(i = 1, 2)$ are laminations in $\Sigma_n^{(i)}$ realizing the same lamination in Σ and $\lambda_\infty^{(i)}$ are geometric limits of $\lambda_n^{(i)}$ in $\Sigma_\infty^{(i)}$. Then $\lambda_\infty^{(2)}|_{F_\infty^{(2)}}$ coincides with the realization of $\iota_\infty(\lambda_\infty^{(1)}|_{F_\infty^{(1)}})$ as a geodesic lamination in $F_\infty^{(2)}$. In particular, if μ_∞ is a compact sub-lamination of $\lambda_\infty^{(1)}$ contained in $F_\infty^{(1)}$, then the geodesic lamination $\mu_\infty^{(2)}$ in $F_\infty^{(2)}$ realizing $\iota_\infty(\mu_\infty)$ is a sub-lamination of $\lambda_\infty^{(2)}$.*

Proof. Let $\widehat{\lambda}_n^{(i)}$ be laminations in $\Sigma_n^{(i)}$ obtained by winding $\lambda_n^{(i)}$ around $\partial F_n^{(i)}$ so that each component of $\partial F_n^{(i)}$ is either a leaf of $\widehat{\lambda}_n^{(i)}$ or disjoint from $\widehat{\lambda}_n^{(i)}$. Intuitively, for any component l of $\partial F_n^{(i)}$ meeting $\lambda_n^{(i)}$ transversely, we reduce $l \cap \lambda_n^{(i)}$ to a single point on l and then spin it around l (see for example Figure 2.2 in [Th3]). Since each component of $\partial F_n^{(i)}$ geometrically converges to a cusp of $F_\infty^{(i)}$, $\widehat{\lambda}_n^{(i)}|_{F_n^{(i)}}$ as well as $\lambda_n^{(i)}|_{F_n^{(i)}}$ geometrically converges to $\lambda_\infty^{(i)}|_{F_\infty^{(i)}}$. It follows from the property of ι_∞ that there exists a monotone decreasing sequence $\{\varepsilon_n\}$ of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and a marking-preserving homeomorphism $\iota_n : F_n^{(1)} \longrightarrow F_n^{(2)}$ satisfying the following conditions.

- ι_n geometrically converges to ι_∞.
- For any leaf $l_n^{(i)}$ of $\widehat{\lambda}_n^{(i)}$ with $l_n^{(i)} \cap F^{(i)}_{n,\mathrm{thin}(\varepsilon_n)} \neq \emptyset$, the angle of $l_n^{(i)}$ and $\partial F^{(i)}_{n,\mathrm{thin}(\varepsilon_n)}$ at any point of $l_n^{(i)} \cap \partial F^{(i)}_{n,\mathrm{thin}(\varepsilon_n)}$ uniformly converges to $\pi/2$ as $n \to \infty$.
- $\iota_n(F^{(1)}_{n,\mathrm{thick}(\varepsilon_n)}) = F^{(2)}_{n,\mathrm{thick}(\varepsilon_n)}$ and the restriction $\iota_n|_{F^{(1)}_{n,\mathrm{thick}(\varepsilon_n)}} : F^{(1)}_{n,\mathrm{thick}(\varepsilon_n)} \longrightarrow F^{(2)}_{n,\mathrm{thick}(\varepsilon_n)}$ is $2K$-bi-Lipschitz, where $2K$ is just taken as a constant greater than K.

Since ι_n is marking-preserving, for any leaf l_n of $\widehat{\lambda}_n|_{F_n^{(1)}}$, $\iota_n(l_n)$ is an arc in $F_n^{(2)}$ properly homotopic to a leaf of $\widehat{\lambda}_n^{(2)}|_{F_n^{(2)}}$. One can suppose that such a proper homotopy has uniformly bounded translation distance depending only on $2K$, which is a standard fact in hyperbolic geometry. For example, see [BP, Lemma C.1.6], [Th1, Proposition 5.9.2] and so on. A geometric limit argument shows that, for any leaf l_∞ of $\lambda_\infty|_{F_\infty^{(1)}}$, $\iota_\infty(l_\infty)$ is an arc in $F_\infty^{(2)}$ properly homotopic to a leaf of $\lambda_\infty^{(2)}|_{F_\infty^{(2)}}$ by a homotopy with uniformly bounded translation distance. It follows that $\lambda_\infty^{(2)}|_{F_\infty^{(2)}}$ is equal to the realization of $\iota_\infty(\lambda_\infty|_{F_\infty^{(1)}})$ in $F_\infty^{(2)}$. □

6.2. **Geometric limits of ending laminations.** Suppose that ν_n is the realization in Σ_n of the ending lamination ν of $\mathcal{E}$. By Proposition 9.3.9 in [Th1], ν_n has no compact leaves and $\Sigma_n \setminus \nu_n$ contains no simple closed geodesic. In particular, ν_n meets each components of $\widehat{h}_n(J_0)$ non-trivially and transversely. Hence one can retake the supervising markings $\widehat{h}_n$ if necessary so that a geometric limit of $\nu_\infty^\natural$ of $\nu_n^\natural$ satisfies the following assumption.

Assumption 6.6. Any component of J_0 is not a leaf of $\nu_\infty^\natural$.

We may also assume that $\nu_n^\natural$ and $\nu_\infty^\natural$ are full laminations if necessary adding finitely many non-compact isolated leaves. See Subsection 1.1 for full laminations.

Under these assumptions, we prove the following lemma.

Lemma 6.7. *Suppose that Σ_n contains a disjoint union of simple geodesic loops η_n realized by $f_n : \Sigma_n \longrightarrow E$ such that $\eta_n^\natural$ geometrically converges to a lamination $\eta_\infty^\natural$ in $\Sigma^\natural$. If $\eta_\infty^\natural$ contains a connected sub-lamination $\mu_\infty^\natural$ which is also a sub-lamination of $\nu_\infty^\natural$, then μ_∞ is not realizable in E_∞.*

Proof. If μ_∞ were empty, then $\mu_\infty^\natural$ would consist of a single compact leaf corresponding to a parabolic cups of E_∞. This contradicts Assumption 6.6 and hence $\mu_\infty \neq \emptyset$. We suppose that μ_∞ is realizable in E_∞ and will introduce a contradiction.

When μ_∞ is not a closed geodesic in Σ_∞, we denote by J_0' the union of components of J_0 meeting μ_∞ non-trivially and by $F^\natural$ the smallest complete subsurface of $\Sigma^\natural$ with geodesic boundary and containing $\mu_\infty^\natural \cup J_0'$. Then any component of $\mathrm{Int}F^\natural \setminus (\mu_\infty^\natural \cup J_0')$ contains at most one simple closed geodesic of $\Sigma^\natural$. Let $\partial_+ F^\natural$ be the union of $\partial F^\natural$ and all such closed geodesics. Note that, if $\mu_\infty^\natural$ is a simple closed geodesic but μ_∞ is not so, then μ_∞ consists of finitely many simple geodesic lines in Σ_∞. When μ_∞ is a closed geodesic in Σ_∞, we set $\mu_\infty^\natural = F^\natural$. Let F_n and F_∞ be the subsurfaces of Σ_n and Σ_∞ respectively supervised by $F^\natural$. We denote by $C_n^{(1)}$ the union of simple closed geodesics in $\mathrm{Int}F_n$ supervised by $\mathrm{Int}F^\natural \cap J_0'$. The components of $\partial_+ F_n$ are divided into the two unions $C_n^{(2)}$ and $C_n^{(3)}$ such that $\inf_n\{\mathrm{length}_E(\widehat{b}_n^{(2)})\} > 0$ if $c_n^{(2)}$ is a component of $C_n^{(2)}$, and $\lim_{n\to\infty} \mathrm{length}_E(\widehat{b}_n^{(3)}) = 0$ if $c_n^{(3)}$ is a component of $C_n^{(3)}$, where $\widehat{b}_n^{(i)}$ is the closed geodesic in E freely homotopic to $f_n(c_n^{(i)})$ for $i = 2, 3$. Let $C_n = C_n^{(1)} \cup C_n^{(2)} \cup C_n^{(3)}$ and let $\widehat{B}_n = \widehat{B}_n^{(1)} \cup \widehat{B}_n^{(2)} \cup \widehat{B}_n^{(3)}$ be the union of closed geodesics in E freely homotopic to $f_n(C_n)$. We define a continuous map $\widehat{f}_n : \widehat{\Sigma}_n \longrightarrow E$ properly homotopic to $f_n : \Sigma_n \longrightarrow E$ and satisfying the following conditions, where the subsurface of $\widehat{\Sigma}_n$ corresponding to F_n is still denoted by F_n for simplicity.

- For any component c_n of C_n, $\widehat{f}_n|_{c_n}$ is a submersion onto $\widehat{B}_n$.
- For the closure Y_n of any component of $\widehat{\Sigma}_n \setminus C_n$, the restriction $\widehat{f}_n|_{Y_n}$ is a partial pleated map. Moreover, $\widehat{f}_n|_{\widehat{\Sigma}_n \setminus \mathrm{Int}F_n}$ realizes $\nu_n|_{\Sigma_n \setminus \mathrm{Int}F_n}$ as a geodesic lamination in E. Strictly this means that, for any leaf l of $\nu_n|_{\Sigma_n \setminus \mathrm{Int}F_n}$, $\widehat{f}_n(l)$ is either a geodesic line or a geodesic arc connecting points of $\widehat{B}_n$ in E.
- $\widehat{f}_n|_{F_{n(\mathrm{main})}}$ is homotopic to $f_n|_{F_{n(\mathrm{main})}}$ by a homotopy with uniformly bounded translation distance.

For any subsurface Y_n of F_n, we do not require at this point that $\widehat{f}_n|_{Y_n}$ realizes $\nu_n|_{Y_n}$ as a geodesic lamination in E, because it may not be compatible with the third condition. Let $\widehat{C}_n = \widehat{C}_n^{(1)} \cup \widehat{C}_n^{(2)} \cup \widehat{C}_n^{(3)}$, $\widehat{F}_n$, $\widehat{\nu}_n$ be the realizations of C_n, F_n and ν in $\widehat{\Sigma}_n$ respectively. Let $\widehat{f}_\infty : \widehat{\Sigma}_\infty \longrightarrow \widehat{E}_\infty$, $\widehat{F}_\infty$, $\widehat{C}_\infty^{(2)}$, $\widehat{B}_\infty^{(2)}$ and $\widehat{\nu}_\infty$ be geometric limits of $\widehat{f}_n$, $\widehat{F}_n$, $\widehat{C}_n^{(2)}$, $\widehat{B}_n^{(2)}$ and $\widehat{\nu}_n$ respectively. See Figure 6.1. These definitions imply that $\widehat{f}_n(\widehat{C}_n^{(i)}) = \widehat{B}_n^{(i)}$ for $i = 1, 2, 3$, $\widehat{f}_\infty(\widehat{C}_\infty^{(2)}) = \widehat{B}_\infty^{(2)}$ and $\widehat{f}_\infty(\widehat{F}_\infty) \subset E_\infty \cap \widehat{E}_\infty$. We say that $\widehat{f}_n$ and $\widehat{f}_\infty$ are *pseudo-pleated maps* bound by $\widehat{C}_n$ and $\widehat{C}_\infty^{(2)}$ respectively. By Lemma 6.5, there exists a sub-lamination $\widehat{\mu}_\infty$ of $\widehat{\nu}_\infty$ corresponding to μ_∞ via the bi-Lipschitz map $\iota_\infty : F_\infty \longrightarrow \widehat{F}_\infty$ given in the paragraph preceding Lemma 6.5. Since we supposed that μ_∞ is realizable in E_∞, we may assume that $f_\infty|_{F_\infty}$ itself realizes μ_∞. Moreover, by the condition (1) on $F^\natural$, the f_∞-image of any simple closed

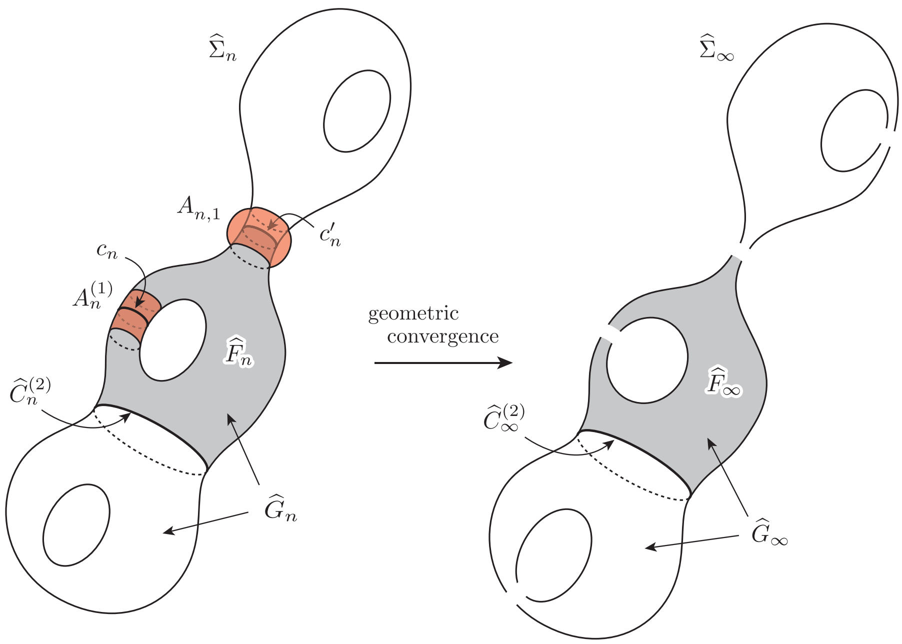


Figure 6.1. The loops c_n and c'_n represent components of $\widehat{C}_n^{(1)}$ and $\widehat{C}_n^{(3)}$ respectively.

geodesic in $F_\infty \setminus \mu_\infty$ is freely homotopic to a closed geodesic in E_∞. So one can suppose that f_∞ realizes $\nu_\infty|_{F_\infty}$ and hence the limit $\widehat{f}_\infty$ of $\widehat{f}_n$ also realizes $\widehat{\nu}_\infty|_{\widehat{F}_\infty}$ as a piecewise geodesic lamination with respect to $\widehat{B}_\infty^{(2)}$ if necessary modifying $\widehat{f}_n|_{\widehat{F}_n}$ by a proper homotopy with uniformly bounded translation distance. Here $\widehat{f}_\infty(\widehat{\nu}_\infty|_{\widehat{F}_\infty})$ being *piecewise geodesic with respect to* $\widehat{B}_\infty^{(2)}$ means that it consists of geodesic lines and geodesic arcs α^* in E with $\partial\alpha^* \subset \widehat{B}_\infty^{(2)}$.

Since $\widehat{f}_n$ is a pseudo-pleated map bound by $\widehat{C}_n$, $\widehat{f}_n(\widehat{\nu}_n)$ is a geodesic lamination in E 'bent' along $\widehat{B}_n$. We will smooth the bending in the following three steps, where $\widehat{\mathcal{V}}_n$ is the union of components of E_{tube} containing $\widehat{B}_n^{(1)} \cup \widehat{B}_n^{(3)}$ as a core.

Step 1. For each component c_n of $\widehat{C}_n^{(1)}$, let $V(c_n)$ be the component of $\widehat{\mathcal{V}}_n$ with $\widehat{f}_n(c_n)$ as a geodesic core and $A_n^{(1)} = \widehat{f}_n^{-1}(V(c_n))$. For any leaf $\widehat{\alpha}_n$ of $\widehat{\nu}_n|_{A_n^{(1)}}$, let $\widehat{\alpha}_n^*$ is a geodesic arc in $V(c_n)$ homotopic to $\widehat{f}_n(\widehat{\alpha}_n)$ rel. $\partial\widehat{\alpha}_n$. Let $\widehat{l}_n$ be the realization of a component l_n of λ_n in $\widehat{\Sigma}_n$ with $\widehat{l}_n \cap A_n^{(1)} \neq \emptyset$. From the assumption on f_n, for any component β_n of $l_n \cap f_n^{-1}(V(c_n))$, $f_n(\beta_n)$ is a geodesic arc in $V(c_n)$. See Figure 6.2. Since the annulus $f_n^{-1}(V(c_n))$ geometrically converges to a parabolic cusp of Σ_∞, $\lim_{n\to\infty} \mathrm{length}_{\Sigma_n}(\beta_n) = \lim_{n\to\infty} \mathrm{length}_E(f_n(\beta_n)) = \infty$. Let $\widehat{\beta}_n$ be the component of $\widehat{\lambda}_n \cap A_n^{(1)}$ corresponding to β_n and $\widehat{\beta}_n^*$ the geodesic arc in $V(c_n)$ properly homotopic to $\widehat{f}_n(\widehat{\beta}_n)$ rel. $\partial\widehat{\beta}_n$, where $\widehat{\lambda}_n$ is the realization of λ_n in $\widehat{\Sigma}_n$. By Assumption 6.6, either $\alpha_n^\natural = \beta_n^\natural$ or the cardinality of $\alpha_n^\natural \cap \beta_n^\natural$ is less than a constant $m_0 \in \mathbb{N}$. This implies that $|\,\mathrm{length}_E(\widehat{\alpha}_n^*) - \mathrm{length}_E(\widehat{\beta}_n^*)|$ is uniformly bounded. Since $f(\beta_n)$

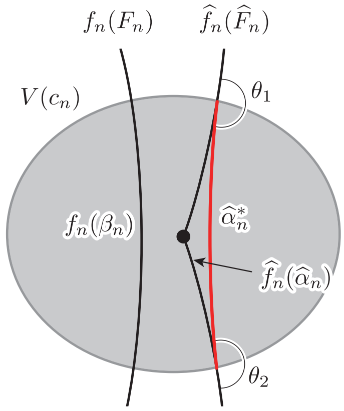


Figure 6.2. The face angles θ_1, θ_2 are nearly equal to π.

is properly homotopic to $\widehat{\beta}_n^*$ by a homotopy with uniformly bounded translation distance. It follows that $\lim_{n\to\infty} \mathrm{length}_E(\widehat{\beta}_n^*) = \infty$ and hence $\lim_{n\to\infty} \mathrm{length}_E(\widehat{\alpha}_n^*) = \infty$. Since the radius of any meridian disk of $V(c_n)$ diverges, the face angle between $\partial\widetilde{V}(c_n)$ and the boundary of $\mathbb{H}^3$ at the cone points of $\widetilde{V}(c_n)$ is arbitrarily small for all sufficiently large n, where $\widetilde{V}(c_n)$ is a component of the inverse image of $V(c_n)$ by the universal covering $p : \mathbb{H}^3 \longrightarrow M$. This implies that $\widehat{\alpha}_n^*$ meets $\partial V(c_n)$ almost orthogonally. Thus the angle of $\widehat{\alpha}_n^*$ and $\widehat{f}_n(\widehat{\nu}_n \setminus \mathrm{Int}\widehat{\alpha}_n)$ at any point of $\partial\widehat{\alpha}_n^*$ is arbitrarily close to π. From this fact, we know that there exists a pseudo-pleated map $\widehat{f}_n^{(1)} : \widehat{\Sigma}_n^{(1)} \longrightarrow E$ bound by $\widehat{C}_n^{(2)} \cup \widehat{C}_n^{(3)}$ and such that $\widehat{f}_n^{(1)}(\widehat{\nu}_n^{(1)})$ is a piecewise geodesic lamination in E with respect to $\widehat{B}_n^{(2)} \cup \widehat{B}_n^{(3)}$, where $\widehat{\nu}_n^{(1)}$ is the realization of ν in $\widehat{\Sigma}^{(1)}$. Moreover, we may take $\widehat{f}_n^{(1)}$ so that it has a geometric limit $\widehat{f}_\infty^{(1)} : \widehat{\Sigma}_\infty^{(1)} \longrightarrow \widehat{E}_\infty$ properly homotopic to $\widehat{f}_\infty$.

Step 2. For short, we set $\widehat{f}_n^{(1)} = \widehat{f}_n$ and $\widehat{f}_\infty^{(1)} = \widehat{f}_\infty$. Let $\widehat{G}_\infty$ be a component of $\widehat{\Sigma}_\infty$ containing a component of $\widehat{F}_\infty$ and $\widehat{G}_n$ the connected subsurface of $\widehat{\Sigma}_n$ geometrically converging to $\widehat{G}_\infty$ and with geodesic boundary. Note that $\widehat{G}_n \neq \widehat{F}_n$ if and only if $\mathrm{Int}\widehat{G}_n \cap \widehat{C}_n^{(2)} \cap \partial\widehat{F}_n$ is non-empty. See Figure 6.1 again. Since $\widehat{f}_n|_{\widehat{G}_n} : \widehat{G}_n \longrightarrow E$ is π_1-injective, $\widehat{f}_\infty|_{\widehat{G}_\infty} : \widehat{G}_\infty \longrightarrow E_\infty$ is also π_1-injective. Hence $\Gamma_\infty = \pi_1(\widehat{f}_\infty)_*(\pi_1(\widehat{G}_\infty))$ is a surface sub-group of a Kleinian group $\pi_1(E_\infty)$. Since both $\widehat{\nu}_\infty|_{\widehat{F}_\infty}$ and $\widehat{\nu}_\infty|_{\widehat{\Sigma}_\infty \setminus \mathrm{Int}\widehat{F}_\infty}$ are realized by $\widehat{f}_n$, if there existed a non-realizable compact leaf l_∞ of $\widehat{\nu}_\infty|_{\widehat{G}_\infty}$, then l_∞ would meet $\widehat{C}_\infty^{(2)}$ transversely and non-trivially. In particular, $(l_\infty \cap \widehat{F}_\infty) \setminus \widehat{C}_\infty^{(2)}$ consists of proper geodesic arcs disjoint from $\widehat{\mu}_\infty \cup \mathcal{A}_\infty^{(1)}$, where $\mathcal{A}_\infty^{(1)}$ is the union of parabolic cusps of $\widehat{\Sigma}_\infty$ corresponding to J_0'. This contradicts that $F^\natural$ is the smallest surface in the sense as above. It follows that any element of Γ_∞ represented by a compact leaf of $\widehat{\nu}_\infty|_{\widehat{G}_\infty}$ is not parabolic. By applying [Th1, Proposition 9.3.7] to the covering of E_∞ with respect to Γ_∞, one can prove that there exists a pleated map $\widehat{f}_{\infty,G} : \widehat{G}_\infty \longrightarrow E_\infty$ properly homotopic to $\widehat{f}_\infty|_{\widehat{G}_\infty}$ rel. $\widehat{\mu}_\infty$ and realizing $\widehat{\nu}_\infty|_{\widehat{G}_\infty}$. Thus there exists a pseudo-pleated map

$\widehat{f}_n^{(2)} : \widehat{\Sigma}_n^{(2)} \longrightarrow E$ bound by $\widehat{C}_n^{(3)}$ and such that $\widehat{f}_n^{(2)}(\widehat{\nu}_n^{(2)})$ is a piecewise geodesic laminations in E with respect to $\widehat{B}_n^{(3)}$, where $\widehat{\nu}_n^{(2)}$ is the realization of ν in $\widehat{\Sigma}_n^{(2)}$. Also in this case, one can suppose that $\widehat{f}_n^{(2)}$ has a geometric limit $\widehat{f}_\infty^{(2)} : \widehat{\Sigma}_\infty^{(2)} \longrightarrow \widehat{E}_\infty$ properly homotopic to $\widehat{f}_\infty$.

Step 3. Again we set simply $\widehat{f}_n^{(2)} = \widehat{f}_n$ and $\widehat{f}_\infty^{(2)} = \widehat{f}_\infty$. For any component c_n' of $\widehat{C}_n^{(3)}$, the boundary $\partial V(c_n')$ is a Euclidean torus which is the union of annuli $A_{n,1}$ and $A_{n,2}$ with $\partial A_{n,1} = \partial A_{n,2} = \partial V(c_n') \cap \widehat{f}_n(\widehat{\Sigma}_n)$. Let $\alpha_{n,i}'$ $(i = 1, 2)$ be any geodesic arc in $A_{n,i}$ connecting the components of $\partial A_{n,i}$ and homotopic rel. $\partial\alpha_{n,i}'$ to a component α_n' of $\widehat{f}_n(\widehat{\nu}_n) \cap V(c_n')$. If $\lim_{n\to\infty} \mathrm{length}_{A_{n,1}}(\alpha_{n,1}') = \lim_{n\to\infty} \mathrm{length}_{A_{n,2}}(\alpha_{n,2}') = \infty$, then we have $\lim_{n\to\infty} \mathrm{length}_E(\alpha_n'^*) = \infty$ by elementary hyperbolic geometry, where $\alpha_n'^*$ is the geodesic arc in E homotopic to $\widehat{f}_n(\alpha_n')$ rel. $\widehat{f}_n(\partial\alpha_n')$. Then one can apply an argument similar to Step 1. Next we consider the case that $\mathrm{length}_{A_{n,i}}(\alpha_{n,i})$ is uniformly bounded for one of $i = 1, 2$, say $i = 1$. Suppose that $\widehat{\Sigma}_{n,A}$ is the surface obtained from $\widehat{\Sigma}_n$ by cutting off $(\widehat{f}_n)^{-1}(V(c_n'))$ and attaching $A_{n,1}$. Consider the map $\widehat{f}_{n,A} : \widehat{\Sigma}_{n,A} \longrightarrow E$ with $\widehat{f}_{n,A}|_{\widehat{\Sigma}_{n,A} \setminus \mathrm{Int} A_{n,1}} = \widehat{f}_n|_{\widehat{\Sigma}_n \setminus \mathrm{Int}(\widehat{f}_n)^{-1}(V(c_n'))}$ and such that $\widehat{f}_{n,A}|_{A_{n,1}} : A_{n,1} \longrightarrow E$ is the inclusion. We say that $\widehat{f}_{n,A}$ is the *bulged map* of $\widehat{f}_n$ along $A_{n,1}$. See Figure 6.3. Then $\widehat{f}_{n,A}$ has a geometric limit $\widehat{f}_{\infty,A} : \widehat{\Sigma}_{\infty,A} \longrightarrow \widehat{E}_\infty$

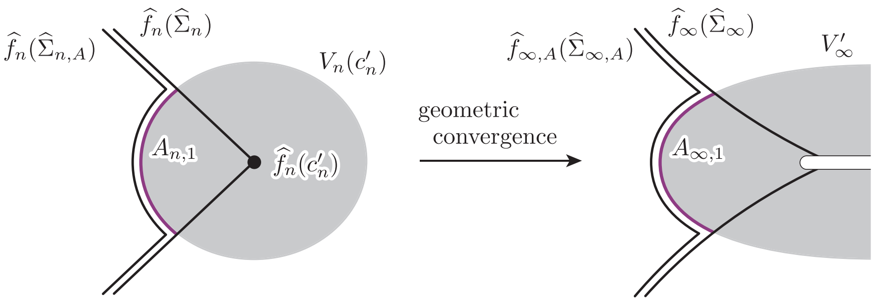


Figure 6.3

with $\widehat{f}_{\infty,A}|_{\widehat{\Sigma}_{\infty,A} \setminus \mathrm{Int} A_{\infty,1}} = \widehat{f}_\infty|_{\widehat{\Sigma}_\infty \setminus \mathrm{Int}\widehat{f}_\infty^{-1}(V_\infty')}$, where $A_{\infty,1}$ and V_∞' are geometric limits of $A_{n,1}$ and $V(c_n')$ respectively. Since

$$\sup_n\{\mathrm{width}(A_{n,1})\} \le \sup_n\{\mathrm{length}_{A_{n,1}}(\alpha_{n,1}')\} < \infty,$$

$A_{\infty,1}$ is an annulus, where $\mathrm{width}(A_{n,1})$ denotes the length of a shortest arc in $A_{n,1}$ connecting the components of $\partial A_{n,1}$. So one can regard that $\widehat{f}_{\infty,A}$ is the bulged map of $\widehat{f}_\infty$ along $A_{\infty,1}$. Then one can apply an argument similar to Step 2. In either case, $\widehat{f}_n$ is properly homotopic to a (real) pleated map $\widehat{f}_n^{(3)} : \widehat{\Sigma}_n^{(3)} \longrightarrow E$ such that, for the realization $\widehat{\nu}_n^{(3)}$ of ν in $\widehat{\Sigma}_n^{(3)}$, $\widehat{f}_n^{(3)}(\widehat{\nu}_n^{(3)})$ is a geodesic lamination in E, which contradicts that ν is the ending lamination of $\mathcal{E}$. Thus μ_∞ is not realizable in E_∞. □

Remark 6.8. Arguments used in the proof of Lemma 6.7 work for certain hyperbolic-like 3-manifolds for which ending laminations of simply degenerate ends are well defined, for example locally CAT(-1)-spaces defined in the next subsection.

6.3. **Irreversibility Lemma and** CAT(-1)**-ruled maps.** Let $f_n : \Sigma_n \longrightarrow E$ be a pleated map realizing a hoop family $\mathcal{H}(f_n)$ of Σ_n and $f_\infty : \Sigma_\infty \longrightarrow E_\infty$ a geometric limit of f_n. If necessary replacing the hoop families $\mathcal{H}(f_n)$ of Σ_n, we may assume that Σ_∞ has no simple closed geodesics the f_∞-images of which are freely homotopic into parabolic cusps in E_∞. Then f_∞ is properly homotopic to a normalized map $f_\infty^\flat : \Sigma_\infty^\flat \longrightarrow E_\infty$ satisfying the properties given in Lemma 1.7. Then f_n is also properly homotopic to a normalized map $f_n^\flat : \Sigma_n^\flat \longrightarrow E$ rel. $f_n(\widehat{h}_n(J_0))$ which geometrically converges to $f_\infty^\flat$. Let $E_{n(a)}$ be the closure of the (a)-side component of $E \setminus f_n^\flat(\Sigma_n^\flat)$ for $a = \pm$ and $\mathcal{V}_n$ the union of components of $E_{n(\mathrm{cusp})}$ meeting $f_n^\flat(\widehat{h}_n(J_0))$ non-trivially. Note that f_n itself may not be an embedding and $f_n(\Sigma_n)$ may wrap around $\mathcal{V}_n$. Then it would be difficult to distinguish the $(+)$ and $(-)$-sides of E with respect to $f_n(\Sigma_n)$ strictly. Since normalized maps have the bounded geometry as in Subsection 1.2, one can define supervising maps $\widehat{h}_n : \Sigma^\natural \longrightarrow \Sigma_n^\flat$ and their limit $h_\infty : \Sigma_{\mathrm{main}}^{\natural(\delta)} \longrightarrow \Sigma_{\infty,\mathrm{main}}^\flat$ just as for pleated maps in Subsection 6.1, see the diagram (6.3). A *lamination* (resp. *geodesic*) μ_n in $\Sigma_n^\flat$ is the $\widehat{h}_n$-image of a geodesic lamination (resp. geodesic) $\mu_n^\natural$ in the hyperbolic surface $\Sigma^\natural$.

Let $\mathcal{V}_\infty$ be a geometric limit of $\mathcal{V}_n$. A component $V_{\infty,i}$ of $\mathcal{V}_\infty$ is *of type I* with respect to $f_\infty^\flat$ if $A_i = \partial V_{\infty,i} \cap E_{\infty(+)}$ is an annulus and *of type II* if it is not of type I and $A_i = \partial V_{\infty,i} \cap E_{\infty(-)}$ is an annulus. Any other component of $\mathcal{V}_\infty$ is *of type III*. See Figure 6.4. We say that a component V_n of $\mathcal{V}_n$ is of *type I, II* or *III*

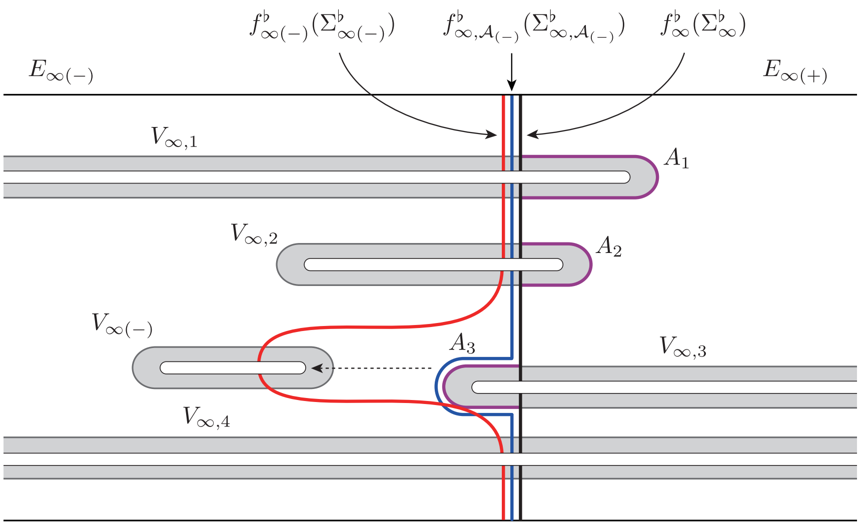


Figure 6.4. Both $V_{\infty,1}$ and $V_{\infty,2}$ are of type I, $V_{\infty,3}$ is of type II, and $V_{\infty,4}$ is of type III. $V_{\infty(-)}$ is a parabolic cusp of $E_{\infty(-)}$ into which $f^\flat_{\infty,\mathcal{A}_{(-)}}(l)$ is freely homotopic for some component l of $L_{(-)}$.

if V_n geometrically converges respectively to a component of $\mathcal{V}_\infty$ of type I, II or

III. Let $f^\flat_{\infty,\mathcal{A}_{(-)}} : \Sigma^\flat_{\infty,\mathcal{A}_{(-)}} \longrightarrow E_\infty$ be the bulged map of $f^\flat_\infty$ along the union $\mathcal{A}_{(-)}$ of annuli $\partial V_{\infty,i} \cap E_{\infty(-)}$ for type II components $V_{\infty,i}$ of $\mathcal{V}_\infty$. Consider a maximal union $L_{(-)}$ of mutually disjoint simple closed geodesics in $\Sigma^\flat_{\infty,\mathcal{A}_{(-)}}$ such that, for any component l of $L_{(-)}$, $f^\flat_{\infty,\mathcal{A}_{(-)}}(l)$ is freely homotopic into a parabolic cusp of $E_{\infty(-)}$. Let $f^\flat_{\infty(-)} : \Sigma^\flat_{\infty(-)} \longrightarrow E_\infty$ be a normalized map obtained by reducing $f^\flat_{\infty,\mathcal{A}_{(-)}}$ along a homotopy from $f^\flat_{\infty,\mathcal{A}_{(-)}}(L_{(-)})$ to parabolic cusps in $E_{\infty(-)}$. In particular, $\Sigma^\flat_{\infty(-)}$ is homeomorphic to $\Sigma^\flat_{\infty,\mathcal{A}_{(-)}} \setminus L_{(-)}$. See Figure 6.4 again. Let $f^\flat_{n(-)} : \Sigma^\flat_{n(-)} \longrightarrow E$ be normalized maps geometrically converging to $f^\flat_{\infty(-)}$. We say that $f^\flat_{n(-)}$ and $f^\flat_{\infty(-)}$ are $(-)$-*reduced normalized maps* of $f^\flat_n$ and $f^\flat_\infty$ respectively. Let $\mathcal{V}_{n(-)}$ be the union of components of E_{tube} meeting $f^\flat_{n(-)}\big(\Sigma^\flat_{n(-),\,(\mathrm{cusp})}\big)$ non-trivially and $\mathcal{V}_{\infty(-)}$ a geometric limit of $\mathcal{V}_{n(-)}$. Note that $\mathcal{V}_{\infty(-)}$ has no components of type II with respect to $f^\flat_{\infty(-)}$. This fact will be used in Case 3 of the proof of Lemma 6.9.

The following lemma is a main result in this section.

Lemma 6.9 (Irreversibility Lemma)**.** *Under the assumptions as above, let η_n be a disjoint union of simple closed geodesics in $\Sigma^\flat_{n(-)}$ supervised by a lamination $\eta^\natural_n$ in $\Sigma^\natural$ which geometrically converges to a sub-lamination $\eta^\natural_\infty$ of $\nu^\natural_\infty$. Then there exists a constant $R > 0$ such that the realization η^*_n of $f^\flat_{n(-)}(\eta_n)$ in E is disjoint from $E_{n(-)} \setminus \mathcal{N}_R\big(f^\flat_{n(-)}\big(\Sigma^\flat_{n(-),\,(\mathrm{main})}\big)\big)$ for any n.*

Intuitively this lemma means that pleated maps realizing η_n as geodesic laminations in E do not diverge to any $(-)$-end of E_∞. Since we do not assume that $f^\flat_{n(-)}$ itself realizes η_n in E in contrast to Lemma 6.7, one can not use any argument similar to that in Step 1 in the proof of the lemma. To overcome the defect, we employ the notion of CAT(-1)-ruled maps introduced in [So3], which were called ruled wrappings there.

A simply connected geodesic metric space X is called a CAT(-1)-*space* if any geodesic triangle Δ in X is *not thicker* than a comparison triangle $\overline{\Delta}$ in $\mathbb{H}^2$, that is, for any two points s and t in the edges of Δ and their comparison points $\overline{s}$ and $\overline{t}$ in $\overline{\Delta}$, $\mathrm{dist}_X(s,t) \le \mathrm{dist}_{\mathbb{H}^2}(\overline{s},\overline{t})$. A metric space whose universal covering is a CAT(-1)-space is called a *locally* CAT(-1)-*space*. See Bridson and Haefliger [BH] for fundamental properties of such spaces.

Definition 6.10 (CAT(-1)-rulded maps)**.** Let δ be a union of simple closed geodesics in E and let $f : \Sigma \longrightarrow E$ be a homotopy equivalence embedding with $\delta \cap f(\Sigma) = \emptyset$ and such that $f(\Sigma)$ is closer to the end $\mathcal{E}$ of E compared with δ. Suppose that $p : Z \longrightarrow M \setminus \delta$ is the covering associated to $f_*(\pi_1(\Sigma)) \subset \pi_1(M \setminus \delta)$ and $\overline{Z}$ is the metric completion of Z. By [So3], $\overline{Z}_n$ is a locally CAT(-1)-space. Then p is uniquely extended to a branched covering $\overline{p}_n : \overline{Z} \longrightarrow M$ branched over δ. A proper homotopy equivalence $\rho : \Sigma \longrightarrow \overline{Z}$ is called a CAT(-1)-*ruled map* (for short a *ruled map*) realizing a lamination μ in Σ if, for any leaf l of μ, $\rho(l)$ is a geodesic in $\overline{Z}$ and, for any component Δ of $\Sigma \setminus \mu$, the restriction $\rho|_\Delta : \Delta \longrightarrow \overline{Z}$ is a ruled map. Note that Σ is a locally CAT(-1)-space with respect to the metric induced from that on $\overline{Z}$ via ρ. Let $\{l_n\}$ be a sequence of simple closed geodesics in Σ geometrically converging to μ. By the Ascoli-Arzelà Theorem, ruled maps $\rho_n : \Sigma \longrightarrow \overline{Z}$ realizing l_n uniformly converge to a ruled map ρ realizing μ as $n \to \infty$ if μ is not the ending lamination of an end of $\overline{Z}$. Strictly, since $\overline{Z}$ is not locally compact at any point of $\overline{Z} \setminus Z$, one can not apply the Ascoli-Arzelà Theorem directly. We consider a uniformly convergence limit $r : \Sigma \longrightarrow M$ of $r_n = \overline{p} \circ \rho_n : \Sigma \longrightarrow M$. Since any r_n are liftable to ρ_n in $\overline{Z}$, r is also liftable to the limit ρ of ρ_n in $\overline{Z}$. In the case when $r(\Sigma)$ is contained in E, we may regard that r is a map to E. Then $r : \Sigma \longrightarrow E$ is called a *ruled map* realizing μ in (E, δ) *with respect to* f and δ is the *branching locus* of r. We also say that the image $\overline{p}(l)$ of a closed geodesic l in $\overline{Z}$ is a closed geodesic in (E, δ). Note that $\overline{p}(l)$ is a piecewise geodesic loop with respect to the original hyperbolic metric on E all vertices of which are contained in δ.

Now we are ready to prove Irreversibility Lemma.

Proof of Lemma 6.9. For simplicity, we suppose that $f_\infty^\flat : \Sigma_\infty^\flat \longrightarrow E_\infty$ itself is a $(-)$-reduced normalized map and set $\mathcal{V}_{n(-)} = \mathcal{V}_n$ and $\mathcal{V}_{\infty(-)} = \mathcal{V}_\infty$. Then $\mathcal{V}_\infty$ has no components of type II with respect to $f_\infty^\flat$. If μ_∞ is a compact sub-lamination of η_∞ which is an ending lamination of some $(-)$-end $\mathcal{E}_{\infty,i}$ of E_∞, then by [Th1, Proposition 9.3.8] there exists a component $F_{\infty,i}^\flat$ of $\Sigma_\infty^\flat$ such that μ_∞ is a full lamination of $F_{\infty,i}^\flat$ and $f_\infty^\flat(F_{\infty,i}^\flat)$ excises from E_∞ a neighborhood $E_{\infty,i}$ of $\mathcal{E}_{\infty,i}$ which is homeomorphic to $F_{\infty,i}^\flat \times (-\infty, 0]$. Since $\mathcal{E}_{\infty,i}$ is a simply degenerate end, there exists a simple closed geodesic $\delta_{\infty,i}$ in $E_{\infty,i}$ such that $\mathrm{dist}_{E_\infty}(\delta_{\infty,i}, f_\infty^\flat(\Sigma_\infty^\flat))$ is sufficiently large. Let L_∞ be the union of all such $\delta_{\infty,i}$ and L_n the union of closed geodesics in E geometrically converging to L_∞. If $f_\infty^\flat(\Sigma_\infty^\flat)$ has other components $F'^\flat_{\infty,j}$ such that $f_\infty^\flat(F'^\flat_{\infty,j})$ excises from $\widehat{E}_\infty$ $(-)$-side submanifolds $E'_{\infty,j}$ homeomorphic to $F'^\flat_{\infty,j} \times (-\infty, 0]$. Let $\widehat{f}_{\infty,j} : \widehat{F}'_{\infty,j} \longrightarrow E_\infty$ be a pleated map realizing $\eta_\infty|_{F'_{\infty,j}}$. Since the sequence $\{f_n^\flat(\Sigma_{n,\mathrm{main}}^\flat)\}$ escapes from any bounded neighborhood of the boundary ∂E in E, the end $\mathcal{E}'_{\infty,j}$ of $E'_{\infty,j}$ is not geometrically finite. Thus there exists a simple closed geodesic $\delta'_{\infty,j}$ in $E'_{\infty,j}$ with $\delta'_{\infty,j} \cap \widehat{f}_{\infty,j}(\widehat{F}'_{\infty,j}) = \emptyset$ which is closer to $\mathcal{E}'_{\infty,j}$ compared with $\widehat{f}_{\infty,j}(\widehat{F}'_{\infty,j})$ and $\mathrm{dist}_{E_\infty}(\delta'_{\infty,j}, f_\infty^\flat(\Sigma_\infty^\flat))$ is sufficiently large. Let L'_∞ be the union of all such $\delta'_{\infty,j}$ and L'_n the union of closed geodesics in E geometrically converging to L'_∞. We set $\Delta_\infty = L_\infty \cup L'_\infty$. One can suppose that Δ_∞ is a disjoint union of simple closed geodesics in E_∞ if necessary slightly modifying the Riemannian metric on E_∞ in a small neighborhood of Δ_∞.

Let Δ_n be the union of $L_n \cup L'_n$ and the geodesic cores of all components of $\mathcal{V}_n$. We denote by $f_n^+ : \Sigma_n^+ \longrightarrow E$ an embedded proper homotopy equivalence such that Δ_n is contained in the $(-)$-component of $E \setminus f_n^+(\Sigma_n^+)$. Let $r_n : \overline{\Sigma}_n \longrightarrow E$ be a ruled map realizing η_n as a union of geodesics in the locally CAT(-1)-space (E, Δ_n) with respect to f_n^+. Let $\overline{\eta}_n$ be the realization of η_n in $\overline{\Sigma}_n$ and $\overline{\eta}_n^* = r_n(\overline{\eta}_n)$. If $\overline{\eta}_n^*$ is disjoint from Δ_n for all sufficiently large n, then each component of $\overline{\eta}_n^*$ is a closed geodesic of the hyperbolic manifold E rather than that of (E, Δ_n). Then it is not hard to have a constant $R > 0$ satisfying the conditions of this lemma.

We next suppose that $\overline{\eta}_n^*$ intersects only the geodesic cores of components V_n of $\mathcal{V}_n$ of type III. For such V_n, let $A_{n,1}$, $A_{n,2}$ be the annuli in ∂V_n with $A_{n,1} \cap A_{n,2} = \partial V_n \cap f_n^\flat(\Sigma_n^\flat)$. Since $\lim_{n\to\infty} \mathrm{width}(A_{n,1}) = \lim_{n\to\infty} \mathrm{width}(A_{n,2}) = \infty$, one can show as Step 3 in the proof of Lemma 6.7 that, for the realization η_n^* of η_n in E, the restriction $\eta_n^* \cap E_{n(\mathrm{main})}$ is contained in the r-neighborhood of $\overline{\eta}_n^*$ in E for some constant $r > 0$ independent of n. Thus there exists our requiring constant $R > 0$.

So it suffices to get a contradiction under the assumption that $\overline{\eta}_n^*$ meets a components of Δ_n other than the geodesic cores of components of $\mathcal{V}_n$ of type III. Let $r_\infty : \overline{\Sigma}_\infty \longrightarrow \overline{E}_\infty$ be a geometric limit of r_n, which realizes a geometric limit $\overline{\eta}_\infty^*$ of $\overline{\eta}_n^*$ in $\overline{E}_\infty$. We need to consider the following three cases, where we set $\widehat{E}_\infty = E_\infty \cap \overline{E}_\infty$.

Case 1. $\overline{\eta}_n^* \cap L_n \neq \emptyset$ for infinitely many n. Let $\overline{F}_{\infty,i}$ be a component of $\overline{\Sigma}_\infty$ such that $r_\infty(\overline{\eta}_\infty|_{\overline{F}_{\infty,i}})$ meets a component $\delta_{\infty,i}$ of L_∞ non-trivially. Then there exists a subsurface $\overline{F}_{n,i}$ of $\overline{\Sigma}_n$ with geodesic boundary such that $r_n|_{\mathrm{Int}\overline{F}_{n,i}}$ geometrically converges to $r_\infty|_{\overline{F}_{\infty,i}}$. Since $\mathrm{dist}_{\widehat{E}_\infty}(\delta_{\infty,i}, f_\infty^\flat(\Sigma_\infty^\flat))$ is sufficiently large, there exists a component $F_{\infty,i}^\flat$ of $\Sigma_\infty^\flat$ such that $f_\infty^\flat|_{F_{\infty,i}^\flat}$ is properly and freely homotopic to $r_\infty|_{\overline{F}_{\infty,i}}$ in $\widehat{E}_\infty$. See Figure 6.5. Since η_∞ has a compact sub-lamination μ_∞

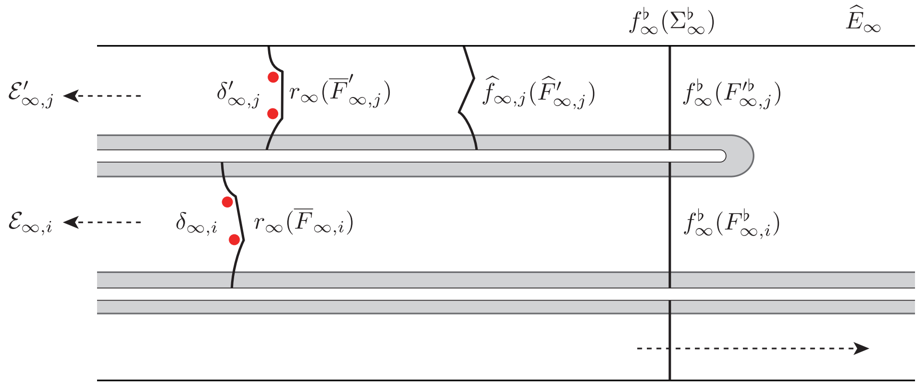


Figure 6.5. The lower '- - →' means that the corresponding part of $r_\infty(\overline{\Sigma}_\infty)$ does not remain in $\widehat{E}_\infty$.

contained in $F_{\infty,i}^\flat$ as a maximal lamination, which is also a sub-lamination of ν_∞. Here a lamination λ in a hyperbolic surface S is called *maximal* if $S \setminus \lambda$ contains no simple closed geodesics. Since $\overline{\eta}_\infty$ is realizable in the locally CAT(-1)-space $(\overline{E}_\infty, \Delta_\infty)$, the sub-lamination $\overline{\mu}_\infty$ of $\overline{\eta}_\infty$ corresponding to μ_∞ is also realizable. By applying Lemma 6.5 to r_n and $f_n^\flat$ instead of f_n, f'_n, one can show that $\overline{\mu}_\infty$ is a sub-lamination of $\overline{\nu}_\infty$. Since (E, Δ_n) has an end which has a neighborhood isometric to a neighborhood of $\mathcal{E}$ in E, $\overline{\nu}_\infty$ is a geometric limit of ending laminations $\overline{\nu}_n$ in (E_n, Δ_n). Then we have a contradiction by applying the locally CAT(-1)-space version of Lemma 6.7 to (E, Δ_n). See Remark 6.8.

Case 2. $\overline{\eta}_n^* \cap L'_n \neq \emptyset$ for infinitely many n. Let $\overline{F}'_{\infty,j}$ be a component of $\overline{\Sigma}_\infty$ such that $r_\infty(\overline{\eta}_\infty|_{\overline{F}'_{\infty,j}})$ meets a component $\delta'_{\infty,j}$ of L'_∞. Note that $r_\infty|_{\overline{F}'_{\infty,j}}$ realizes $\eta_\infty|_{F_{\infty,j}}$ as a geodesic lamination $\overline{\lambda}_\infty^*$ in $(E_\infty, \Delta_\infty)$. On the other hand, $\widehat{f}_{\infty,j}$ realizes $\eta_\infty|_{F'_{\infty,j}}$ as a geodesic lamination $\widehat{\lambda}_\infty^*$ in $\widehat{E}_\infty$. Since $\widehat{f}_{\infty,j}(\widehat{F}'_{\infty,j}) \cap \Delta_\infty = \emptyset$, one can regard $\widehat{\lambda}_\infty^*$ as a geodesic lamination in $(\widehat{E}_\infty, \Delta_\infty)$. However, since $r_\infty(\overline{F}'_{\infty,j}) \cap \widehat{f}_{\infty,j}(\widehat{F}'_{\infty,j}) = \emptyset$, $\overline{\lambda}_\infty^* \neq \widehat{\lambda}_\infty^*$. This contradicts the fact that two geodesic laminations in the same proper homotopy class coincide with each other in the locally CAT(-1)-space $(\widehat{E}_\infty, \Delta_\infty)$. See Figure 6.5 again.

Case 3. Suppose that $\overline{\eta}_n^*$ meets the geodesic core c_n of some component V_n of $\mathcal{V}_n$ other than of type III, which geometrically converges to a component V_∞ of $\mathcal{V}_\infty$. Since $\mathcal{V}_\infty$ has no components of type II with respect to $f_\infty^\flat$, the $(+)$-side annulus $A_{n(+)}$ in ∂V_n with respect to $r_n(\overline{\Sigma}_n)$ geometrically converges to the $(+)$-side annulus $A_{\infty(+)}$ in ∂V_∞ with respect to $r_\infty(\overline{\Sigma}_\infty)$. Note that $A_{\infty(+)}$ is contained in $\widehat{E}_\infty$. Consider the bulged maps $r_{n,A} : \overline{\Sigma}_{n,A} \longrightarrow \overline{E}_n$ of r_n along $A_{n(+)}$, which geometrically converge to the bulged map $r_{\infty,A} : \overline{\Sigma}_{\infty,A} \longrightarrow \overline{E}_\infty$ of r_∞ along $A_{\infty(+)}$. The metrics on $A_{n(+)}$ and $A_{\infty(+)}$ induced respectively from E and $\overline{E}_\infty$ via $r_{n,A}$ and $r_{\infty,A}$ are Euclidean, see Step 3 in the proof of Lemma 6.7 for bulged maps. So the induced metrics on $\overline{\Sigma}_{n,A}$ and $\overline{\Sigma}_{\infty,A}$ are neither hyperbolic nor locally CAT(-1). Since $\overline{\Sigma}_{A,n}$ geometrically converges to $\overline{\Sigma}_{A,\infty}$, there exist hyperbolic metrics on $\overline{\Sigma}_{n,A}$ and $\overline{\Sigma}_{\infty,A}$ K-bi-Lipschitz to their induced metrics respectively for some $K > 1$. Let $\overline{\eta}_{n,A}$, $\overline{\nu}_{n,A}$ be the realizations of η_n and ν_n in $\overline{\Sigma}_{n,A}$ with respect to the hyperbolic metrics, which geometrically converge to laminations $\overline{\eta}_{\infty,A}$ and $\overline{\nu}_{\infty,A}$ in $\overline{\Sigma}_{\infty,A}$ respectively. Since $\eta_\infty^\natural$ is a sub-laminations of $\nu_\infty^\natural$, by Assumption 6.6 $\overline{\eta}_n$ goes across the annulus $r_n^{-1}(V_n)$. This implies that the length of any component of $\overline{\eta}_n \cap r_n^{-1}(V_n)$ diverges and hence the length of any component of $\overline{\eta}_{n,A} \cap r_{n,A}^{-1}(V_n)$ also diverges. It follows from this fact that both $\overline{\eta}_{\infty,A}$ and $\overline{\nu}_{\infty,A}$ contain the closed geodesic $\overline{c}_\infty$ in $\overline{\Sigma}_{\infty,A}$ corresponding to the parabolic cusp of V_∞ as a common compact leaf. Let $\delta_{\infty,A}$ be a simple loop of $\overline{\Sigma}_{\infty,A}$ meeting $A_{\infty(+)}$ homotopically essentially and such that the $r_{\infty,A}$-image of $\delta_{\infty,A}$ is freely homotopic in $\widehat{E}_\infty$ to a closed geodesic δ_∞. Let δ_n be the closed geodesic in E geometrically converging to δ_∞ and let $\widehat{r}_n : \widehat{\Sigma}_n \longrightarrow E$ be a ruled map realizing η_n as a union of closed geodesics in the locally CAT(-1)-space $(E, \widehat{\Delta}_n)$, where $\widehat{\Delta}_n = (\Delta_n \setminus c_n) \cup \delta_n$. Intuitively, $\widehat{r}_n(\widehat{\Sigma}_n)$ is obtained by pushing out the surface $\overline{r}_{n,A}(\overline{\Sigma}_{n,A})$ with the ring δ_n. See Figure 6.6. Since ν is the ending lamination of the end $\mathcal{E}$ of $(E, \widehat{\Delta}_n)$, $\widehat{\nu}_n$ is not

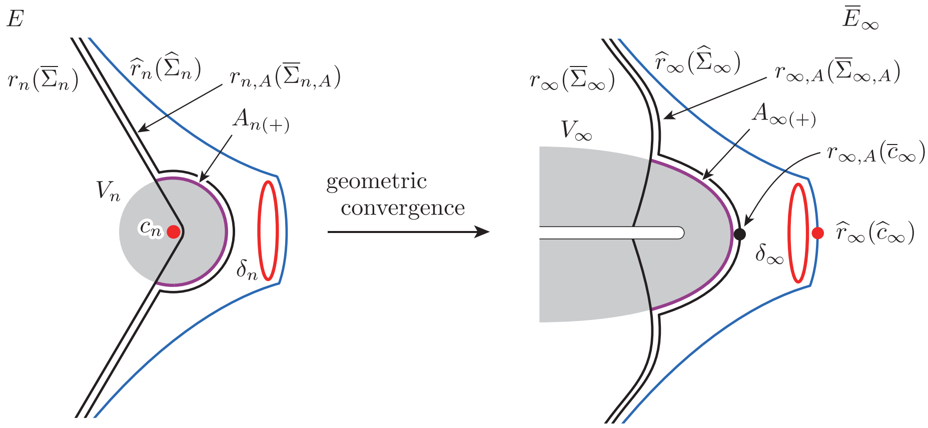


Figure 6.6

realized by $\widehat{r}_n$ as a geodesic lamination in $(E, \widehat{\Delta}_n)$ in contrast to η_n. The ruled map $\widehat{r}_n$ geometrically converges to a limit ruled map $\widehat{r}_\infty : \widehat{\Sigma}_\infty \longrightarrow \overline{E}_\infty$ with the branching locus $\widehat{\Delta}_\infty = \Delta_\infty \cup \delta_\infty$ and the realizations $\widehat{\eta}_n$ of η_n and $\widehat{\nu}_n$ of ν in $\widehat{\Sigma}_n$ also geometrically converge to laminations $\widehat{\eta}_\infty$ and $\widehat{\nu}_\infty$ in $\widehat{\Sigma}_\infty$ respectively. From our construction, we know that $r_{\infty,A}$ is properly homotopic to $\widehat{r}_\infty$ in $\overline{E}_\infty$. The ruled map $\widehat{r}_\infty$ realizes $\widehat{\eta}_\infty$ as a geodesic lamination in $(\overline{E}_\infty, \widehat{\Delta}_\infty)$. Let $\widehat{c}_\infty$ be the closed geodesic in $\widehat{\Sigma}_\infty$ corresponding to $\overline{c}_\infty$. Here we note that $\widehat{r}_\infty(\widehat{c}_\infty)$ meets δ_∞ non-trivially. Otherwise $\widehat{r}_\infty(\widehat{c}_\infty)$ would be a closed geodesic in E_∞ rather than that in $(\overline{E}_\infty, \widehat{\Delta}_\infty)$ and freely homotopic into the parabolic cusp V_∞. By applying an argument similar to that in the proof of Lemma 6.5 to $r_{n,A}$ and $\widehat{r}_n$, one can show that both $\widehat{\eta}_\infty$ and $\widehat{\nu}_\infty$ contain $\widehat{c}_\infty$ as a compact leaf. Then one can get a contradiction by using the locally CAT(-1)-space version of Lemma 6.7. The loop $\widehat{c}_\infty$ here corresponds to μ_∞ of Lemma 6.7 in the case where μ_∞ is a closed geodesic in Σ_∞.

By Cases 1–3, we have our requiring contradiction, which completes the proof. □

6.4. **Geometric limits of earthquakes.** In this subsection, we present the notion and fundamental properties of earthquakes introduced by Thurston, see [Ker, Th2] for details.

Let $\Sigma^\natural$ be the supervising hyperbolic surface given in Subsection 6.1. For a given simple closed geodesic l in $\Sigma^\natural$, let Σ' be the hyperbolic surface obtained from $\Sigma^\natural \setminus l$ by the path-metric completion. The boundary of Σ' consists of two copies of l. For any $t \geq 0$, let Σ_{tl} be the hyperbolic surface obtained by gluing the boundary components of Σ' with left twist of distance t. Then the identity of $\Sigma^\natural \setminus l$ induces a locally isometric map $Q_{tl} : \Sigma^\natural \setminus l \longrightarrow \Sigma_{tl}$. Let l_t be the closed geodesic in Σ_{tl} corresponding to the boundary components of Σ'. Consider a simple geodesic arc α in $\Sigma^\natural$ meeting l transversely. Let $\alpha_{tl}^\vee$ be the piecewise geodesic path in Σ_{tl} obtained by connecting the components of $Q_{tl}(\alpha \setminus l)$ with left directed immersed arcs in l_t of length t. Suppose that α is either a closed geodesic or a geodesic line. Then we denote by α_{tl} the geodesic in Σ_{tl} which is covered by a geodesic line in the universal covering space $\mathbb{H}^3$ with end points the same as those of a lift of $\alpha_{tl}^\vee$. See Figure 3 in [Ker]. We say that α_{tl} is the *straightened geodesic arc* in Σ_{tl} obtained from α. When β is a sub-segment of α, β_{tl} is the sub-segment of α_{tl} in Σ_{tl} obtained by straightening $\beta_{tl}^\vee$. A *marking* $q_{tl} : \Sigma^\natural \longrightarrow \Sigma_{tl}$ *associated with* Q_{tl} is a homeomorphism such that, for any simple closed geodesic α meeting l transversely, $q_{tl}(\alpha)$ is freely homotopic to the straightened geodesic loop α_{tl} in Σ_{tl}. Such a homeomorphism is determined uniquely up to homotopy. Thus the pair (Σ_{tl}, q_{tl}) of the hyperbolic surface Σ_{tl} with the marking q_{tl} uniquely determines an element of the Teichmüller space Teich(Σ). We say that Q_{tl} is the *left Finchel-Nielsen twist* along tl.

Definition 6.11 (Left earthquakes)**.** For any measured lamination ω in $\Sigma^\natural$ with compact support, consider a sequence of weighted simple closed geodesics $t_n l_n$ in $\Sigma^\natural$ converging to ω as measured laminations. Then the sequence of the left Finchel-Nielsen twists $Q_{t_n l_n}$ converges to a locally isometric map $Q_\omega : \Sigma^\natural \setminus \omega \longrightarrow \Sigma^\natural_\omega$ uniformly on any compact subset of $\Sigma^\natural \setminus \omega$ for some hyperbolic surface $\Sigma^\natural_\omega$, see [Ker, Section II] and [Th2] for details. We say that Q_ω is the *left earthquake* associated with ω.

The map Q_ω satisfies the following properties.

- Q_ω does not depend on the choice of the sequence $t_n l_n$ converging to ω.
- Let $\sigma(\omega)$ be the union of compact leaves of ω. Then Q_ω is uniquely extended to a continuous map on $\Sigma^\natural \setminus \sigma(\omega)$, which is still denoted by Q_ω.
- For any strongly simple geodesic arc α meeting ω transversely, the sequence of the piecewise geodesic arc $\alpha^\vee_{t_n l_n}$ converges uniformly to a piecewise geodesic arc $\alpha^\vee_\omega$ in $\Sigma^\natural_\omega$. Here we say that α is *strongly simple* if α is contained in a simple geodesic line in $\Sigma^\natural$. The straightened geodesic arc in $\Sigma^\natural_\omega$ obtained from $\alpha^\vee_\omega$ is denoted by α_ω.
- A marking $q_\omega : \Sigma^\natural \longrightarrow \Sigma^\natural_\omega$ associated with Q_ω is defined by the manner as in the case of q_{tl}. Then $(\Sigma^\natural_{t_n l_n}, q_{t_n l_n})$ converges to $(\Sigma^\natural_\omega, q_\omega)$ in $\mathrm{Teich}(\Sigma^\natural)$.

Theorem 6.12 ([Ker, Theorem 2], [Th2, Sections III.1.5−7])**.** *For any element (Σ, q) in $\mathrm{Teich}(\Sigma^\natural)$, there exists a unique measured lamination ω on $\Sigma^\natural$ with compact support and satisfying $(\Sigma, q) = (\Sigma^\natural_\omega, q_\omega)$ in $\mathrm{Teich}(\Sigma^\natural)$.*

Suppose that $E' = \varphi(E)$ is a neighborhood of a simply degenerate end $\mathcal{E}'$ of M' whose ending lamination ν' is the same as ν via φ. Let λ_n be a maximal lamination in Σ_n realized by f_n and let $g'_n : \Sigma(g'_n) \longrightarrow E'$ be a pleated map realizing the lamination λ'_n in $\Sigma(g'_n)$ corresponding to λ_n via φ.

There exists a homeomorphism $\varphi_n : \Sigma_n \longrightarrow \Sigma(g'_n)$ such that $g'_n \circ \varphi_n$ is properly homotopic to $\varphi \circ f_n$. Let $\widehat{h}_n : \Sigma^\natural \longrightarrow \Sigma_n$ and $\widehat{h}'_n : \Sigma^\natural \longrightarrow \Sigma(g'_n)$ be homeomorphisms as in (6.3). Denote the domains of $\widehat{h}_n$ and $\widehat{h}'_n$ by $\Sigma^{\natural\mathrm{high}}$ and $\Sigma^{\natural\mathrm{low}}$ respectively if we need to distinguish them. Let $q_n : \Sigma^{\natural\mathrm{high}} \longrightarrow \Sigma^{\natural\mathrm{low}}$ be the homeomorphism defined by $q_n = \widehat{h}'^{-1}_n \circ \varphi_n \circ \widehat{h}_n$. Then we have the following diagram which is commutative up to proper homotopy.

$$\begin{array}{ccccc} \Sigma^{\natural\mathrm{high}} & \xrightarrow{\widehat{h}_n} & \Sigma_n & \xrightarrow{f_n} & E \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi} \\ \Sigma^{\natural\mathrm{low}} & \xrightarrow[\widehat{h}'_n]{} & \Sigma(g'_n) & \xrightarrow[g'_n]{} & E' \end{array} \tag{6.4}$$

By Theorem 6.12, there exists a unique measured lamination ω_n on $\Sigma^{\natural\mathrm{high}}$ such that $(\Sigma^{\natural\mathrm{low}}, q_n) = (\Sigma^\natural_{\omega_n}, q_{\omega_n})$. Let ω_∞ be a geometric limit of ω_n in $\Sigma^{\natural\mathrm{high}}$ with limit transverse measure and $\widehat{\omega}_\infty$ the sub-lamination of ω_∞ consisting of leaves l such that, for any open geodesic segment α in $\Sigma^{\natural\mathrm{high}}$ meeting l transversely and non-trivially, the transverse measure of $\widehat{\omega}_\infty$ on α is infinite. Possibly $\widehat{\omega}_\infty$ is empty. For any lamination λ in $\Sigma^{\natural\mathrm{high}}$, the geodesic lamination isotopic to $q_n(\lambda)$ in $\Sigma^{\natural\mathrm{low}}$ is denoted by $q_n(\lambda)^*$. Then $\omega^{\mathrm{low}}_n = q_n(\omega_n)^*$ is the measured laminations in $\Sigma^{\natural\mathrm{low}}$ with the measure induced from that on ω_n via q_n.

Lemma 6.13. *Let α, β and $\beta^{(n)}$ $(n = 1, 2, \dots)$ are strongly simple geodesic arcs in $\Sigma^{\natural\mathrm{high}}$ such that both $\mathrm{Int}\,\alpha$ and $\mathrm{Int}\,\beta$ meet the same leaf l of $\widehat{\omega}_\infty$ transversely and non-trivially and $\beta^{(n)}$ geometrically converges to β. Then the straightened geodesic arcs $\alpha_n = q_n(\alpha)^*$ and $\beta^{(n)}_n = q_n(\beta^{(n)})^*$ contain sub-arcs geometrically converging to the same connected lamination α_∞ in $\Sigma^{\natural\mathrm{low}}$.*

Proof. By the fourth property of earthquakes preceding Theorem 6.12, one can suppose that ω_n consists of a single geodesic loop. First we consider the case that, for any sub-arc α' of α with $\mathrm{Int}\,\alpha' \cap l \neq \emptyset$ and any sufficiently large n, $\alpha' \cap \omega_n$ has a point $x^{(n)}$ such that the transverse measure of ω_n on $\{x^{(n)}\}$ diverges to ∞ but that on $\alpha' \setminus \{x^{(n)}\}$ is uniformly bounded. Then we have an arc $\gamma^{(n)}$ in ω_n connecting $x^{(n)}$ with a point $y_n^{(n)}$ of $\beta^{(n)} \cap \omega_n$. In the other case, there exists a point $x^{(n)}$ in $\mathrm{Int}\,\alpha \setminus \omega_n$ satisfying the following conditions.

- $x^{(n)}$ converges to a point of $\alpha \cap l$.
- For the components $\alpha^{(n)+}$, $\alpha^{(n)-}$ of $\alpha \setminus \{x^{(n)}\}$, the transverse measure of ω_n on $\alpha^{(n)\pm}$ diverges to ∞.

See Figure 6.7 (a). Note that, since ω_n is a geodesic loop, $\alpha_n^\vee$ and $\beta_n^{(n)\vee}$ are piecewise

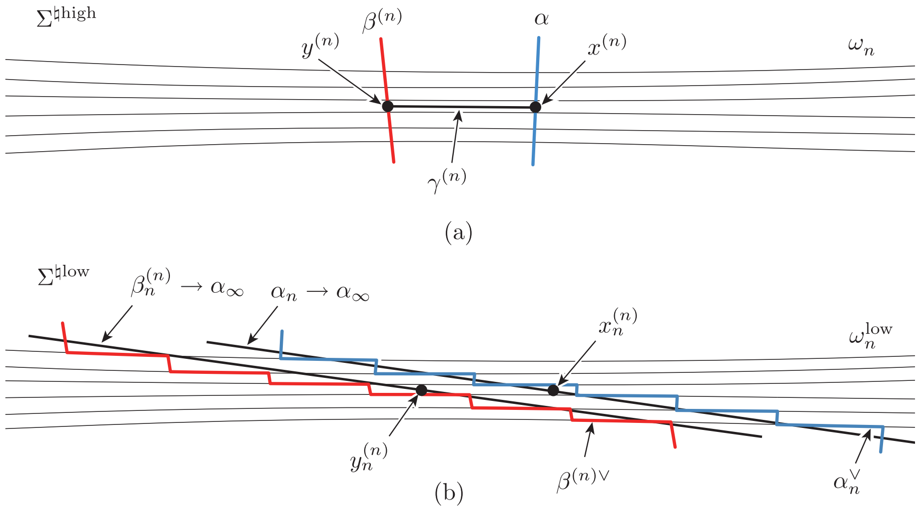


Figure 6.7

geodesic arcs with only finitely many vertices. In either case, we have a geodesic arc $\gamma_n^{(n)}$ in $\Sigma^{\natural\mathrm{low}}$ corresponding to $\gamma^{(n)}$ and connecting points $x_n^{(n)}$ of α_n and $y_n^{(n)}$ of $\beta_n^{(n)}$. We denote by $\alpha_n^\pm$, $\beta_n^{(n)\pm}$ the components of $\alpha_n \setminus \{x_n^{(n)}\}$ and $\beta_n^{(n)} \setminus \{y_n^{(n)}\}$ respectively. From standard facts on earthquakes (for example see Corollary 3.4, Proposition 3.5 and Lemma 3.6 in [Ker]), we have $\mathrm{length}_{\Sigma^{\natural\mathrm{low}}}(\alpha_n^{(n)\pm}) \to \infty$, $\mathrm{length}_{\Sigma^{\natural\mathrm{low}}}(\beta_n^\pm) \to \infty$ and $\sup_n\{\mathrm{length}_{\Sigma^{\natural\mathrm{low}}}(\gamma_n^{(n)})\} < \infty$. Moreover, both α_n and $\beta_n^{(n)}$ contain subarcs centered at $x_n^{(n)}$, $y_n^{(n)}$ respectively which geometrically converge to the same connected lamination α_∞ in $\Sigma^{\natural\mathrm{low}}$. See Figure 6.7 (b).

Intuitively, this fact is explained as follows. Let $p : \mathbb{H}^2 \longrightarrow \Sigma^{\natural\mathrm{low}}$ be the universal covering and $\widetilde{\alpha}_n$, $\widetilde{\beta}_n^{(n)}$ geodesic lines in $\mathbb{H}^2$ with $p(\widetilde{\alpha}_n) \supset \alpha_n$ and $p(\widetilde{\beta}_n^{(n)}) \supset \beta_n^{(n)}$. One can choose these geodesic lines so that their end point sets $\partial\widetilde{\alpha}_n$ and $\partial\widehat{\beta}_n^{(n)}$ converge to the end point set $\partial\widetilde{\alpha}_\infty$ of the same leaf $\widetilde{\alpha}_\infty$ of $p^{-1}(\widehat{\omega}_\infty)$. See Figure 3 in [Ker] again. □

Here we note that that α_∞ is possibly a single closed geodesic in $\Sigma^{\natural\mathrm{low}}$.

Lemma 6.14. *Let λ be a component of $\widehat{\omega}_\infty$ consisting of a single compact leaf. Then $\lambda_n = q_n(\lambda)^*$ geometrically converges to a simple closed geodesic in $\Sigma^{\natural\mathrm{low}}$.*

Proof. Again one can suppose that ω_n consists of a single geodesic loop. For a small $\varepsilon > 0$, let $\mathcal{N}(\lambda)$ be the ε-neighborhood of λ in $\Sigma^{\natural\mathrm{high}}$. Since λ is isolated in

$\widehat{\omega}_\infty$, one can choose ε so that $\widehat{\omega}_\infty \cap \partial\mathcal{N}(\lambda) = \emptyset$ and any leaf of $\omega_n \cap \mathcal{N}(\lambda)$ connects a component b of $\partial\mathcal{N}(\lambda)$ with the other component. See Figure 6.8. So the invariant

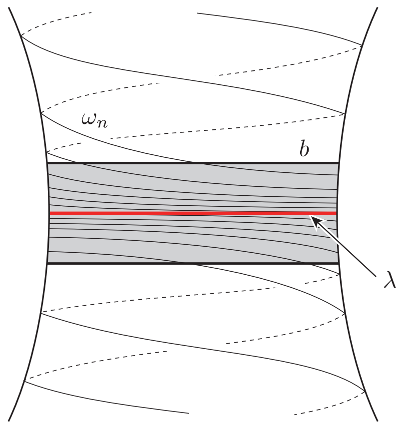


Figure 6.8. View from the side. The shaded region represents $\mathcal{N}(\lambda)$.

transverse measure $\omega_n(b)$ is equal to $\omega_n(\lambda)$, where we suppose that $\omega_n(\lambda) = 0$ if $\omega_n = \lambda$. Let $\lambda_n^\vee$ be the piecewise geodesic loop in $\Sigma^{\natural\mathrm{low}}$ obtained from λ by the earthquake Q_{ω_n}. From the definition of $\lambda_n^\vee$,

$$\mathrm{length}_{\Sigma^{\natural\mathrm{low}}}(\lambda_n^\vee) = \mathrm{length}_{\Sigma^{\natural\mathrm{high}}}(\lambda) + \omega_n(\lambda).$$

Since $b \cap \widehat{\omega}_\infty = \emptyset$, $\omega_\infty(b) < \infty$. This shows that $\sup_n\{\omega_n(\lambda)\} = \sup_n\{\omega_n(b)\} < \infty$. Since λ_n is a closed geodesic freely homotopic to $\lambda_n^\vee$ in $\Sigma^{\natural\mathrm{low}}$, the length of λ_n is uniformly bounded and hence λ_n geometrically converges to a simple closed geodesic in $\Sigma^{\natural\mathrm{low}}$. □

7. Proof of Theorem B

In this section, we will prove Theorem B under the notations and conditions as in Section 6. Then $\varphi : M \longrightarrow M'$ is an orientation and cusp-preserving homeomorphism such that $E' = \varphi(E)$ is a neighborhood of a simply degenerate end of M' with the ending lamination ν' corresponding to ν via φ.

7.1. **Boundedness of volume difference.** For any non-contractible and non-peripheral simple loop l of Σ, we denote by l^* the closed geodesic in Σ freely homotopic to l. Let $f_n : \Sigma_n \longrightarrow E$ be pleated maps tending toward $\mathcal{E}$ and realizing the hoop families $\mathcal{H}(f_n) = \widehat{h}_n(\mathcal{H}^{\natural\mathrm{high}})^*$ of Σ_n supervised by a fixed hoop family $\mathcal{H}^{\natural\mathrm{high}}$ of $\Sigma^{\natural\mathrm{high}}$. Suppose that $g_n' : \Sigma(g_n') \longrightarrow E'$ is the pleated map realizing the union η_n' of closed geodesics corresponding to $\varphi_n(\mathcal{H}(f_n))^*$ in Σ_n'. Then η_n' is supervised by $\eta_n'^\natural = q_n(\mathcal{H}^{\natural\mathrm{high}})^*$ in $\Sigma^{\natural\mathrm{low}}$. See (6.4) for the homeomorphisms q_n and φ_n. Here we use $\mathcal{H}(f_n)$ and η_n' respectively instead of λ_n and λ_n' there. In a similar manner, for pleated maps $f_n' : \Sigma_n' \longrightarrow E'$ tending toward $\mathcal{E}'$ and realizing the hoop families $\mathcal{H}(f_n') = \widehat{h}_n'(\mathcal{H}^{\natural\mathrm{low}})^*$ of Σ_n', one can define pleated maps $g_n : \Sigma(g_n) \longrightarrow E$ realizing the union η_n of closed geodesics corresponding to $\varphi_n^{-1}(\mathcal{H}(f_n'))^*$ in Σ_n.

The following lemma plays a crucial role in the proof of Theorem B.

Lemma 7.1 (Volume Difference Boundedness Lemma)**.** *There exist hoop-realizing pleated maps $f_n : \Sigma_n \longrightarrow E$ and $f'_n : \Sigma'_n \longrightarrow E$ as above with respect to which at least one of the following* (V1) *and* (V2) *holds, where $g'_i : \Sigma(g'_i) \longrightarrow E'$ and $g_i : \Sigma(g_i) \longrightarrow E$ $(i = 0, n)$ are pleated maps realizing η'_i and η_i respectively.*

$$\text{(V1)} \qquad \sup_n \big\{\mathrm{Vol}^{\mathrm{bd}}(f_0, f_n) - \mathrm{Vol}^{\mathrm{bd}}(g'_0, g'_n)\big\} < \infty.$$

$$\text{(V2)} \qquad \sup_n \big\{\mathrm{Vol}^{\mathrm{bd}}(f'_0, f'_n) - \mathrm{Vol}^{\mathrm{bd}}(g_0, g_n)\big\} < \infty.$$

Here we suppose that (V1) does not hold for any such f_n's. Then one can assume that

$$\text{(7.1)} \qquad \lim_{n\to\infty} \big(\mathrm{Vol}^{\mathrm{bd}}(f_0, f_n) - \mathrm{Vol}^{\mathrm{bd}}(g'_0, g'_n)\big) = \infty$$

if necessary passing to a subsequence. We will define pleated maps $f'_n : \Sigma'_n \longrightarrow E'$ and $g_n : \Sigma(g_n) \longrightarrow E$ satisfying (V2) by using the maps f_n, g'_n with (7.1).

Proof. Let $f_\infty : \Sigma_\infty \longrightarrow E_\infty$ be a geometric limit of f_n. One can retake f_n so that f_∞ is properly homotopic in E_∞ to a $(-)$-reduced normalized map, see the paragraph preceding Lemma 6.9 for such maps.

Recall that $\widehat{h}_n : \Sigma^{\natural\mathrm{high}} \longrightarrow \Sigma_n$, $\widehat{h}'_n : \Sigma^{\natural\mathrm{low}} \longrightarrow \Sigma(g'_n)$ are supervising markings satisfying Assumption 6.6 and ω_n is the measured lamination on $\Sigma^{\natural\mathrm{high}}$ with $(\Sigma^{\natural\mathrm{low}}, q_n) = (\Sigma^{\natural\mathrm{high}}_{\omega_n}, q_{\omega_n})$ for $q_n = \widehat{h}'^{-1}_n \circ \varphi_n \circ \widehat{h}_n : \Sigma^{\natural\mathrm{high}} \longrightarrow \Sigma^{\natural\mathrm{low}}$ and $q_{\omega_n} : \Sigma^{\natural\mathrm{high}} \longrightarrow \Sigma^{\natural\mathrm{high}}_{\omega_n}$, see (6.4). Let ω_∞ be a geometric limit of ω_n with limit transverse measure and $\widehat{\omega}_\infty$ the sub-lamination of ω_∞ with infinite transverse measure. First we consider the case when $\widehat{\omega}_\infty$ is contained in $\mathcal{H}^{\natural\mathrm{high}}$ (possibly $\widehat{\omega}_\infty = \emptyset$) and hence each component of $\widehat{\omega}_\infty$ is an isolated closed geodesic. Then, by Lemma 6.14, the length of each component of η'_n is uniformly bounded. Thus, by setting $g'_j = f'_j$, $f_j = g_j$ $(j = 0, n)$ and supposing $\mathcal{H}(f'_j) = \eta'_j$, one can prove that (7.1) implies (V2). So we may assume that $\widehat{\omega}_\infty$ is not a subset of $\mathcal{H}^{\natural\mathrm{high}}$. We denote by $\ell(\widehat{\omega}_\infty)$ the union of loop components of $\widehat{\omega}_\infty$ and by $\eta'^{\natural}_\infty$ a geometric limit of $\eta'^{\natural}_n$ in $\Sigma^{\natural\mathrm{low}}$.

Now we will show that $\widehat{\omega}_\infty \setminus \ell(\widehat{\omega}_\infty)$ is a sub-lamination of $\nu^{\natural}_\infty$. Since $\nu^{\natural}_\infty$ is a full lamination, if it did not hold, then there would exist a non-compact leaf $l^\natural$ of $\widehat{\omega}_\infty \setminus \ell(\widehat{\omega}_\infty)$ meeting a leaf of $\nu^{\natural}_\infty$ transversely and non-trivially. If $l^\natural \cap \mathcal{H}^{\natural\mathrm{high}} \neq \emptyset$, then $l^\natural$ meets a component $l^\natural_H$ of $\mathcal{H}^{\natural\mathrm{high}}$ transversely and non-trivially. By applying Lemma 6.13 with $\alpha \subset l^\natural_H$, $\beta^{(n)} \subset \nu^\natural_n$, $l^\natural \subset \widehat{\omega}_\infty$, one can prove that $\nu'^{\natural}_\infty$ and $\eta'^{\natural}_\infty$ have a common connected lamination $\tau'^{\natural}_\infty$. Since η'_∞ is realizable in E'_∞, so is τ'_∞. This contradicts Lemma 6.7 and hence $l^\natural \cap \mathcal{H}^{\natural\mathrm{high}} = \emptyset$. Thus the closure $\overline{l}^\natural$ of $l^\natural$ in $\Sigma^{\natural\mathrm{high}}$ contains a component $m^\natural_H$ of $\mathcal{H}^{\natural\mathrm{high}}$ as a compact leaf, which is also a leaf of $\widehat{\omega}_\infty$. Take a simple geodesic loop $\gamma^\natural$ in $\Sigma^{\natural\mathrm{high}}$ meeting $m^\natural_H$ with either one or two points and disjoint from $\mathcal{H}^{\natural\mathrm{high}} \setminus m^\natural_H$. Since $\gamma^\natural$ meets $l^\natural$ transversely and non-trivially, again by Lemma 6.13 the geometric limit γ'_∞ of $\gamma'^{\natural}_n = q_n(\gamma^\natural)^*$ and $\nu'^{\natural}_\infty$ have a common connected sub-lamination $\mu'^{\natural}_\infty$. We may assume that $\mu'^{\natural}_\infty$ is *minimal*, that is, $\mu'^{\natural}_\infty$ contains no proper sub-lamination. Suppose that $\mu'^{\natural}_\infty$ did meet $\eta'^{\natural}_\infty$ transversely. If $\mu'^{\natural}_\infty$ is a simple geodesic loop, then we know from $\gamma'^{\natural}_\infty \supset \mu'^{\natural}_\infty$ that $\gamma'^{\natural}_n$ has a sub-arc contained in a small regular neighborhood of $\mu'^{\natural}_\infty$ in $\Sigma^{\natural\mathrm{low}}$ and winding around $\mu'^{\natural}_\infty$ arbitrarily many times. See Figure 7.1. This contradicts that $\gamma'^{\natural}_n$ meets $\eta'^{\natural}_n$ at most two points. If $\mu'^{\natural}_\infty$ is not a simple closed geodesic, then it follows from the minimality of $\mu'^{\natural}_\infty$ that any leaf of $\mu'^{\natural}_\infty$ meets $\eta^{\natural}_n$ transversely infinitely may times for all sufficiently large n. As in the previous case, this also

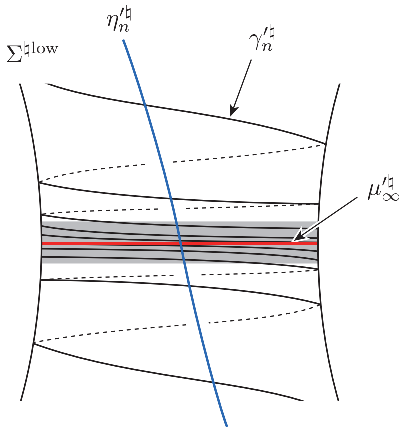


Figure 7.1. Since $\mu_\infty'^\natural$ meets $\eta_\infty'^\natural$ transversely, it also does $\eta_n'^\natural$ with intersection angles bounded away from zero for all sufficiently large n.

gives a contradiction. Thus $\mu_\infty'^\natural \cup \eta_\infty'^\natural$ is a lamination in $\Sigma^{\natural\mathrm{low}}$. Since $\eta_\infty'^\natural$ is maximal in $\Sigma^{\natural\mathrm{low}}$ and $\mu_\infty'^\natural$ has no isolated leaves, $\mu_\infty'^\natural$ is a sub-lamination of $\eta_\infty'^\natural$ as well as of $\nu_\infty'^\natural$. It also contradicts Lemma 6.7. Thus we have shown that $\widehat{\omega}_\infty \setminus \ell(\widehat{\omega}_\infty)$ is a sub-lamination of $\nu_\infty^\natural$.

Suppose that $f_n' : \Sigma_n' \longrightarrow E'$ is a pleated map realizing the hoop family $\mathcal{H}(g_n') = \widehat{h}_n'(\mathcal{H}^{\natural\mathrm{low}})^*$ of $\Sigma(g_n')$. Then one can take a hoop family $\mathcal{H}(f_n')$ of Σ_n' such that $f_n'(\mathcal{H}(f_n'))$ is a union of closed geodesics in E' freely homotopic to $g_n'(\mathcal{H}(g_n'))$. Since $\mathrm{length}_{E'}(g_n'(\mathcal{H}(g_n'))) = \mathrm{length}_{\Sigma(g_n')}(\mathcal{H}(g_n'))$ is uniformly bounded, for any component $F(g_n')$ of $\Sigma(g_n') \setminus \mathcal{H}(g_n')$, the restriction $g_n'|_{F(g_n')}$ geometrically converges to a partial pleated map $g_\infty'|_{F(g_\infty')} : F(g_\infty') \longrightarrow E_\infty'$ such that $F(g_\infty')_{\mathrm{main}}$ is K-bi-Lipschitz to $F(g_n')_{\mathrm{main}}$ for some constant $K > 1$ independent of n. The map $g_\infty'|_{F(g_\infty')}$ is properly homotopic in E_∞' to a continuous map $\iota_\infty' : F_\infty' \longrightarrow E_\infty'$ such that $\iota_\infty'(F_\infty')$ is a union of two totally geodesic ideal triangles in E_∞'. Since $f_n'|_{F_n'}$ also realizes F_n' as a union of two totally geodesic ideal triangles in E' (see for example Figure 2.1 in [Th3]), $f_n'(F_{n(\mathrm{main})}')$ is arbitrarily close to $\zeta_n'(\iota_\infty'(F_{\infty,\mathrm{main}}'))$, where $\zeta_n' : \mathcal{N}_{\infty,n}' \longrightarrow E'$ is a locally bi-Lipschitz embedding defined as ζ in (6.1). See also (6.3). So there exists a constant $C_1 > 0$ with

$$|\mathrm{Vol}^{\mathrm{bd}}(g_n', f_n')| < C_1. \tag{7.2}$$

Let $\eta_\infty^\natural$ be a geometric limit of $\eta_n^\natural = q_n^{-1}(\mathcal{H}^{\natural\mathrm{low}})^*$ in $\Sigma^{\natural\mathrm{high}}$. Then we know that $\eta_\infty^\natural$ does not meet $\widehat{\omega}_\infty$ transversely. Otherwise, there would exist a leaf $l_n^\natural$ of $\eta_n^\natural$ which meets $\widehat{\omega}_\infty$ transversely and non-trivially. Then, for the component $l_n'^\natural = q_n(l_n^\natural)^*$ of $\mathcal{H}^{\natural\mathrm{low}}$, $\lim_{n\to\infty} \mathrm{length}_{\Sigma^{\natural\mathrm{low}}}(l_n'^\natural) = \infty$, a contradiction. Let $\mathcal{H}_{0,n}^{\natural\mathrm{low}}$ be the union of components $l_n'^\natural$ of $\mathcal{H}^{\natural\mathrm{low}}$ such that $q_n^{-1}(l_n'^\natural)^*$ are either disjoint from $\widehat{\omega}_\infty \setminus \ell(\widehat{\omega}_\infty)$ or contained in $\ell(\widehat{\omega}_\infty)$. One can assume that $\mathcal{H}_{0,n}^{\natural\mathrm{low}}$ is independent of n and hence may set $\mathcal{H}_{0,n}^{\natural\mathrm{low}} = \mathcal{H}_0^{\natural\mathrm{low}}$ if necessary passing to a subsequence. Note that the restriction $q_n|_{\Sigma^{\natural\mathrm{high}} \setminus \mathcal{N}(\widehat{\omega}_\infty)}$ is homotopic to a bi-Lipschitz map onto its image, where $\mathcal{N}(\widehat{\omega}_\infty)$ is a small regular neighborhood of $\widehat{\omega}_\infty$ in $\Sigma^{\natural\mathrm{high}}$. Thus, for $\eta^\natural_{0,n} = q_n^{-1}(\mathcal{H}_0^{\natural\mathrm{low}})^*$, length$(\eta^\natural_{0,n})$ is uniformly bounded. This shows that $\eta^\natural_{0,n}$ geometrically converges to a disjoint union $\eta^\natural_{0,\infty}$ of simple closed geodesics in $\Sigma^{\natural\mathrm{high}}$, each component of which is a loop component of $\eta^\natural_\infty$. Any component of $\eta^\natural_\infty \setminus \eta^\natural_{0,\infty}$ is a sub-lamination of $\widehat{\omega}_\infty \setminus \ell(\widehat{\omega}_\infty)$ and hence of $\nu^\natural_\infty$.

Recall that $g_n : \Sigma(g_n) \longrightarrow E$ is a pleated map realizing η_n. Since we supposed in advance that f_∞ is properly homotopic to a $(-)$-reduced normalized map, by Irreversibility Lemma (Lemma 6.9) there exists a constant $C_0 > 0$ with

$$\mathrm{Vol}^{\mathrm{bd}}(f_n, g_n) > -C_0.$$

From this fact together with (7.1) and (7.2),

$$\begin{aligned}
&\mathrm{Vol}^{\mathrm{bd}}(f_0', f_n') - \mathrm{Vol}^{\mathrm{bd}}(g_0, g_n)\\
&\qquad = \mathrm{Vol}^{\mathrm{bd}}(f_0', g_0') + \mathrm{Vol}^{\mathrm{bd}}(g_0', g_n') + \mathrm{Vol}^{\mathrm{bd}}(g_n', f_n')\\
&\qquad\qquad - \mathrm{Vol}^{\mathrm{bd}}(g_0, f_0) - \mathrm{Vol}^{\mathrm{bd}}(f_0, f_n) - \mathrm{Vol}^{\mathrm{bd}}(f_n, g_n)\\
&\qquad < \mathrm{Vol}^{\mathrm{bd}}(f_0', g_0') - \mathrm{Vol}^{\mathrm{bd}}(g_0, f_0) - (\mathrm{Vol}^{\mathrm{bd}}(f_0, f_n) - \mathrm{Vol}^{\mathrm{bd}}(g_0', g_n'))\\
&\qquad\qquad + C_0 + C_1 \to -\infty \quad (n \to \infty).
\end{aligned}$$

This implies (V2). □

7.2. **Proofs of Theorem B and Corollary C.** In this subsection, we suppose that $\tau^\natural$ is a geodesic triangulation on $\Sigma^\natural$ satisfying the conditions (T1)–(T5) in Section 3 and τ_n is the geodesic triangulation on Σ_n supervised by $\tau^\natural$.

The proof of Theorem B is similar to those of Lemmas 3.3 and 3.4.

Proof of Theorem B. By Lemma 7.1, if necessary replacing E with E', we may assume that there exists a sequence $\{f_n\}$ of pleated maps to E satisfying

$$\mathrm{Vol}^{\mathrm{bd}}(f_0, f_n) \le \mathrm{Vol}^{\mathrm{bd}}(g_0', g_n') + C \tag{7.3}$$

for some constant $C > 0$. Let $\widehat{f}_j$ $(j = 0, n)$ be a normalized map whose image is contained in the 1-neighborhood $\mathcal{N}_1(f_j(\Sigma))$ of $f_j(\Sigma)$ in E, see Definition 1.6. If $f_j(\Sigma)$ wraps around a component V of E_{tube}, then by Lemma 1.5 there exists a solid torus V_0 in E with $\partial V_0 \subset \mathcal{N}_1(f_j(\Sigma))$, $V_0 \supset V$ and $\mathrm{Vol}(V_0) < \mathrm{Area}(\Sigma(f_j))$. By this fact together with Proposition 8.12.1 in [Th1] (see also Lemma 1.7 (3)), one can show that $|\mathrm{Vol}^{\mathrm{bd}}(f_j, \widehat{f}_j)|$ is uniformly bounded. It follows from (7.3) that there exists a constant $C' > 0$ satisfying

$$\mathrm{Vol}(E(\widehat{f}_0, \widehat{f}_n)) = \mathrm{Vol}^{\mathrm{bd}}(\widehat{f}_0, \widehat{f}_n) < \mathrm{Vol}^{\mathrm{bd}}(g_0', g_n') + C'. \tag{7.4}$$

Let $\psi : M \longrightarrow M'$ be a continuous map satisfying the conditions (P1) with $\mathrm{Vol}(\mathcal{N}(\widehat{\mathcal{H}}_E)) < \infty$ and (P2) in Subsection 3.2. In particular, ψ is properly homotopic to φ rel. $M \setminus \mathrm{Int}E$. We will show that ψ satisfies the ω-upper bound condition on E.

Recall that the closure of the component of $E \setminus \widehat{f}_0(\Sigma)$ adjacent to $\mathcal{E}$ is denoted by $E^+(\widehat{f}_0)$. For any almost compact 3-dimensional submanifold X of $E^+(\widehat{f}_0)$, there exists $n \in \mathbb{N}$ such that $X \subset E(\widehat{f}_0, \widehat{f}_n) =: \widehat{X}$. By Lemma 3.2, for any straight 3-simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ with $\mathrm{Vol}(\sigma) > 1$, there exists a 3-chain $\widehat{a}_{0,n}$ on $\widehat{X}$ with $\|\widehat{a}_{0,n}\| \le b_0$ and such that $z_{0,n} = z_{\widehat{X}}(\sigma) + \widehat{a}_{0,n}$ is a 3-chain with $\partial_3 z_{0,n} = \mathrm{Vol}(\sigma)(w(\tau_n) - w(\tau_0))$, where $w(\tau_j)$ $(j = 0, n)$ is the fundamental 2-cycle on $\widehat{f}_j(\Sigma)$ given in Lemma 3.2 (2). Let $f_j'^* : \Sigma \longrightarrow E'$ be the piecewise totally geodesic map defined from $\psi \circ \widehat{f}_j$ and satisfying the conditions given in the paragraph preceding Lemma 3.3. Then we have $\omega_{M'}(\psi_*(z_{0,n})) = \mathrm{Vol}(\sigma)\mathrm{Vol}^{\mathrm{bd}}(f_0'^*, f_n'^*)$ as (3.7). Since g_j' realizes $\mathcal{H}(f_j)$ in E', the bending locus of g_j' in $\Sigma(g_j')$ is homeomorphic to a lamination in Σ_j obtained from τ_j by spinning its vertices around $\mathcal{H}(f_j)$. So there exists a 3-chain c_j in E' consisting of ideal straight 3-simplices the number of which is at most $3m_0$ and satisfying $\partial_3 c_j = f_j'^*(\Sigma) - g_j'(\Sigma)$ as 2-cycles. Here '3' means that the triangular prism $\Delta^2 \times [0,1]$ is divided into three 3-simplices. By the property (T4) of τ_j in Section 3, there exists $m_0 \in \mathbb{N}$ independent of j such that the number of elements of $\tau_j^{(2)}$ is not greater than m_0. Since $|\mathrm{Vol}^{\mathrm{bd}}(f_j'^*, g_j')| \le 3m_0\boldsymbol{v}_3$ for $j = 0, n$, it follows from (7.4) that

$$\begin{aligned}\omega_{M'}(\psi_*(z_{0,n})) &= \mathrm{Vol}(\sigma)\mathrm{Vol}^{\mathrm{bd}}(f_0'^*, f_n'^*) \ge \mathrm{Vol}(\sigma)(\mathrm{Vol}^{\mathrm{bd}}(g_0', g_n') - 6m_0\boldsymbol{v}_3)\\ &\ge \mathrm{Vol}(\sigma)(\mathrm{Vol}(\widehat{X}) - 6m_0\boldsymbol{v}_3 - C').\end{aligned}$$

On the other hand,

$$\omega_{M'}(\psi_*(z_{0,n})) = \omega_{M'}(\psi_*(z_{\widehat{X}}(\sigma))) + \omega_{M'}(\psi_*(\widehat{a}_{0,n})) \le \omega_{M'}(\psi_*(z_{\widehat{X}}(\sigma))) + 2b_0\boldsymbol{v}_3.$$

This shows that

$$\omega_{M'}(\psi_*(z_{\widehat{X}}(\sigma))) \ge \mathrm{Vol}(\sigma)\mathrm{Vol}(\widehat{X}) - c_0,$$

where $c_0 = \boldsymbol{v}_3(6m_0\boldsymbol{v}_3 + C' + 2b_0)$. Since moreover

$$\mathrm{Vol}(\sigma)\mathrm{Vol}(\widehat{X}) = \mathrm{Vol}(\sigma)\|z_{\widehat{X}}(\sigma)\| = \omega_M(z_{\widehat{X}}(\sigma)),$$

we have

$$\omega_{M'}(\psi_*(z_{\widehat{X}}(\sigma))) \ge \omega_M(z_{\widehat{X}}(\sigma)) - c_0.$$

Thus ψ satisfies the ω-upper bound condition on $E^+(\widehat{f}_0)$. Since $\mathrm{Vol}(E\backslash \mathrm{Int}E^+(\widehat{f}_0)) < \infty$ and $\mathrm{Vol}(\mathcal{N}(\widehat{\mathcal{H}}_E)) < \infty$, φ as well as ψ satisfies the ω-upper bound condition on E. This completes the proof. □

Proof of Corollary C. Suppose that $\varphi : M \longrightarrow M'$ preserves the end invariants. Let C be a finite core of M and $C' = \varphi(C)$. Then one can suppose that $\varphi|_C : C \longrightarrow C'$ is a bi-Lipschitz map. For any end $\mathcal{E}$ of M, let E be the neighborhood of $\mathcal{E}$ with respect to C and $E' = \varphi(E)$. If $\mathcal{E}$ is simply degenerate, then by Theorems A and B $\varphi|_E : E \longrightarrow E'$ is properly homotopic rel. ∂E to a bi-Lipschitz map $\varphi_E^\flat$. When $\mathcal{E}$ is geometrically finite, consider the domains Ω_Γ, $\Omega_{\Gamma'}$ of discontinuity of Kleinian groups Γ, Γ' with $\mathbb{H}^3/\Gamma = M$ and $\mathbb{H}^3/\Gamma' = M'$ respectively. Since φ preserves the conformal structure on geometrically finite end, $\varphi|_E$ is properly homotopic rel. ∂E to a bi-Lipschitz map $\varphi_E^\flat$ which is extended to a conformal map from O_E to $O_{E'}$, where O_E, $O_{E'}$ are the components of Ω_Γ/Γ and $\Omega_{\Gamma'}/\Gamma'$ adjacent to E and E' respectively. Then the map $\varphi' : M \longrightarrow M'$ defined by $\varphi'|_C = \varphi|_C$ and $\varphi'|_E = \varphi_E^\flat$ is a bi-Lipschitz map the lift $\widetilde{\varphi}' : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ of which is extended to a quasi-conformal map $\varPhi_\infty$ on S^2_∞ such that the restriction $\varPhi_\infty|_{\Omega_\Gamma} : \Omega_\Gamma \longrightarrow \Omega_{\Gamma'}$ is conformal. By Sullivan's Rigidity Theorem [Su], $\varPhi_\infty$ is a conformal map. It follows that φ' and hence φ are properly homotopic to an isometry. □

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