BOUNDED COHOMOLOGY AND VOLUME RIGIDITY OF HYPERBOLIC 3-MANIFOLDS

TERUHIKO SOMA

Abstract. We present a rigidity theorem for hyperbolic 3-manifolds $M = \mathbb{H}^3/\Gamma$ with a Kleinian surface group Γ in terms of the fundamental class $[\omega_M]$ in the bounded cohomology $H_b^3(M;\mathbb{R})$. Under some conditions, we show that a homeomorphism $\varphi : M \longrightarrow M$ between hyperbolic 3-manifolds M, M' are bi-Lipschitz if the pseudo-norm $\|[\omega_M] - \varphi^*([\omega_{M'}])\|$ in $H_b^3(M;\mathbb{R})$ is less than the volume of a regular ideal simplex in the hyperbolic 3-space. We see that the separation constant is best possible.

Let $f : M \longrightarrow M'$ be a proper degree-one map between oriented hyperbolic 3-manifolds of finite volume. Gromov and Thurston [Th1, Chapter 6] proved that f is properly homotopic to an isometry if and only if $\mathrm{Vol}(M) = \mathrm{Vol}(M')$. In the proof, they use the simplicial volume $\|[M]\|$ of M, that is, the simplicial norm of the fundamental homology class $[M]$ of M. In this paper, we consider the case when M is a hyperbolic 3-manifold M with the proper homotopy type equivalent to a hyperbolic surface of finite area. Then, since the volume of M is infinite, we can not use the volume as an invariant. So we use the fundamental class in bounded cohomology instead of simplicial volume. The bounded cohomology $H_b^3(X,\mathbb{R})$ is a homotopy invariant of a topological space X introduced by Gromov [Gr], which has the naturally defined pseudo-norm $\|\cdot\|$, see Section 3. When M is an oriented hyperbolic 3-manifold, we consider the 3-cocycle $\omega_M : C_3(M) \longrightarrow \mathbb{R}$ such that, for any singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$, $\omega_M(\sigma)$ is the oriented volume of the 3-simplex straight(σ) obtained by straightening σ. It is a well know fact in hyperbolic geometry that the supremum norm $\|\omega_M\|$ of ω_M is equal to the volume of a regular ideal simplex $\boldsymbol{v}_3 = 1.01494\ldots$ in $\mathbb{H}^3$. So ω_M represents the fundamental bounded cohomology class $[\omega_M] \in H_b^3(M,\mathbb{R})$ of M with $\|[\omega]\| \le \boldsymbol{v}_3$.

Throughout this paper, we denote by Σ an oriented complete hyperbolic surface of finite area. Possibly Σ has parabolic cusps. We only consider the case that any hyperbolic 3-manifolds M admits a proper homotopy equivalence $\iota : \Sigma \longrightarrow M$, which is called a *marking* of M.

We prove the following theorem by using Connecting Lemma (Lemma 5.1) together with Ending Lamination Theorem [Mi, BCM].

Theorem A. *Let M, M' be hyperbolic 3-manifolds with markings $\iota : \Sigma \longrightarrow M$, $\iota' : \Sigma \longrightarrow M'$ respectively. Suppose that either the $(+)$ or $(-)$-end $\mathcal{E}$ of M with respect to $\iota(\Sigma)$ is totally degenerate. If*

$$\|\iota^*([\omega_M]) - \iota'^*([\omega_{M'}])\| < \boldsymbol{v}_3 \tag{0.1}$$

Date: December 29, 2023.
2020 *Mathematics Subject Classification.* 57K32, 30F40.
Key words and phrases. Hyperbolic 3-manifolds, Kleinian groups, bounded cohomology.
The author was partially supported by JSPS KAKENHI Grant Number 22K03342.

holds in $H^3_b(\Sigma, \mathbb{R})$*, then there exists a marking and orientation-preserving homeomorphism* $\varphi_0 : M \longrightarrow M'$ *and a neighborhood* E *of* $\mathcal{E}$ *such that* $\varphi_0|_E : E \longrightarrow E' = \varphi_0(E)$ *is bi-Lipschitz. In particular,* φ_0 *defines the bijection between the components of* E_{cusp} *and those of* E'_{cusp}*.*

Here we say that the end $\mathcal{E}$ is *totally degenerate* if any genuine sub-end of $\mathcal{E}$ is simply degenerate and set $E^{(\prime)}_{\mathrm{cusp}} = E^{(\prime)} \cap M^{(\prime)}_{\mathrm{cusp}}$. A *genuine end* of M is an end of M with respect to a maximal cusp of M, see Section 1 for the strict definition.

Theorem A says that the fundamental bounded cohomology class keeps the data of the placement of parabolic cusps in a neighborhood of a totally degenerate end. Thus the following corollary is obtained immediately from Theorem A together with Sullivan's Rigidity Theorem [Su].

Corollary B. *Under the assumptions in Theorem A including* (0.1)*, suppose moreover that all genuine ends of* M *are simply degenerate. Then* φ *is properly homotopic to an isometry. In particular,* $\iota^*([\omega_M]) = \iota'^*([\omega_{M'}])$ *in* $H^3_b(\Sigma, \mathbb{R})$*.*

Now we consider the case when the data of the placement of parabolic cusps in M is known in advance. Then φ define the bijection between the genuine ends $\mathcal{E}$ of M and those of M', where $\mathcal{E}$ is possibly geometrically finite. Then, by reforming Theorem A, we have the following result which asserts that the structure of M is uniquely determined by the fundamental bounded cohomology class up to bi-Lipschitz.

Theorem C. *Let* M *be an oriented hyperbolic* 3*-manifold with a marking of* Σ*. Suppose that there exists an orientation-preserving homeomorphism* φ *from* M *to another hyperbolic* 3*-manifold* M' *inducing a bijection between the components of* M_{cusp} *and those of* M'_{cusp}*. If*

$$\|[\omega_M] - \varphi^*([\omega_{M'}])\| < \boldsymbol{v}_3 \tag{0.2}$$

holds in $H^3_b(M, \mathbb{R})$*, then* φ *is properly homotopic to a bi-Lipschitz map.*

Remark 0.1 (Best possibility of separation constant)**.** We refer to Soma [So2, Theorem A], Ohshika-Miyachi [OM, Section 6], Farre [Far1, Corollary 1.5] and so on for precedent results relating to our theorems. In those papers, theorems similar to ours are obtained in suitable settings and under certain conditions with some separation constants as $\boldsymbol{v}_3$ in (0.1) or (0.2). However, the practical values of those constants are not presented there and they depend more or less on either the geometric structure on M or the topological type of Σ. On the other hand, our separation constant is not only concrete but also best possible. In fact, by [So3], if at least one of genuine ends of M is simply degenerate, then $\|[\omega_M]\| = \boldsymbol{v}_3$. In contrast, if M' has no simply degenerate ends, then $[\omega_{M'}] = 0$ in $H^3_b(M', \mathbb{R})$. So we have

$$\|[\omega_M] - \varphi^*([\omega_{M'}])\| \le \|[\omega_M]\| + \|\varphi^*([\omega_{M'}])\| = \boldsymbol{v}_3,$$

but φ is not properly homotopic to a bi-Lipschitz map since any simply degenerate end is not bi-Lipschitz to a geometrically finite end.

Remark 0.2 (Volume rigidity)**.** We use volume arguments in the proof of the above theorems, which show that corresponding ends of E and E' have the same ending lamination. Then Ending Lamination Theorem implies that E and E' are bi-Lipschitz. This means that the latter half of our argument may not be consistent with the title 'volume rigidity' of our paper. It would be possible to prove our theorems only by using volume arguments as in [So4] together the classical (or standard) theory of hyperbolic geometry by Thurston, for example see [Th1], [Th2, Part 1]. However we relied on the established rigidify theorem by Minsky et al. for ensuring completeness of our proofs.

The fundamental group $\pi_1(\Sigma)$ is naturally identified with a Fuchsian group Γ in $\mathrm{PSL}_2(\mathbb{R})$. We denote by $\mathcal{R}^{(\mathrm{p})}(\Gamma)$ the set of representations $\rho : \Gamma \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ which map each parabolic element of Γ to a parabolic element of $\mathrm{PSL}_2(\mathbb{C})$. The holonomy $\rho_M : \Gamma \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ of a hyperbolic 3-manifold M with a marking $\iota : \Sigma \longrightarrow M$ is a discrete and faithful element of $\mathcal{R}^{(\mathrm{p})}(\Gamma)$. We set $\mathcal{R}^{(\mathrm{p})}(\Gamma) = \mathcal{R}(\Gamma)$ if Σ has no parabolic cusps. Farre [Far2] defined the bounded volume class $[\mathrm{Vol}(\rho)]$ of ρ in $H^3_b(\Gamma, \mathbb{R}) = H^3_b(\Sigma, \mathbb{R})$. Then $[\mathrm{Vol}(\rho_M)]$ is equal to $\iota^*([\omega_M])$ if ρ_M is the holonomy as above. In the case when Σ is a closed surface, he presented a rigidity theorem in terms of $[\mathrm{Vol}(\rho)]$ for representations $\rho \in \mathcal{R}(\Gamma)$ such that $\rho(\Gamma)$ contain no parabolic elements. His rigidity theorem also concerns a separating constant but it may depend on the topological type of Σ. Here we propose the following question asking the existence of a concrete separating constant which is valid in volume rigidity theorems of representations.

Question. Does there exist a concrete constant $v > 0$ satisfying the following condition? If it exists, is it best possible?

Let ρ be any element of $\mathcal{R}^{(\mathrm{p})}$ with $\|[\mathrm{Vol}(\rho)] - [\mathrm{Vol}(\rho_M)]\| < v$ in $H^3_b(\Gamma, \mathbb{R})$. If either the $(+)$ or $(-)$-end of M with respect to $\iota(\Sigma)$ is totally degenerate, then ρ is faithful and discrete. Moreover, if the both ends are totally degenerate, then ρ and ρ_M are conjugate in $\mathrm{PSL}_2(\mathbb{C})$.

1. Preliminaries

In this section, we present fundamental definitions and notations in forms suitable to our arguments. Refer to Thurston [Th1], Benedetti and Petronio [BP], Matsuzaki and Taniguchi [MT] and so on for other notations concerning hyperbolic geometry and to Hempel [He] for 3-manifold topology. For a closed subset A of a metric space $X = (X, d)$ and any $r > 0$, the r-neighborhood $\{y \in X \,|\, d(y, A) \le r\}$ of A is denoted by $\mathcal{N}_r(A, X)$ or $\mathcal{N}_r(A)$ for short.

Throughout this paper, we suppose that Γ is a torsion-free finitely generated *Kleinian group*, that is, Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$. Then the quotient map $p : \mathbb{H}^3 \longrightarrow M = \mathbb{H}^3/\Gamma$ is a universal covering and M has a Riemannian metric so that p is locally isometric. Then M is called a *hyperbolic* 3-manifold.

Fundamental notations and definitions. For a $\mu > 0$, the *μ-thin part* $M_{\mathrm{thin}(\mu)}$ of M is the set of points $x \in M$ such that there exists a non-contractible loop l in M of length $\le 2\mu$ and passing through x. The complement $M_{\mathrm{thick}(\mu)} = M \setminus \mathrm{Int} M_{\mathrm{thin}(\mu)}$ is called the *μ-thick part* of M. By the Margulis Lemma [Th1, Corollary 5.10.2], there exists a constant $\mu_* > 0$ independent of M, called a *Margulis constant*, such that, for any $0 < \mu \le \mu_*$, each component of $M_{\mathrm{thin}(\mu)}$ is either an equidistant tubular neighborhood of a simple closed geodesic, called a *Margulis tube*, in M or a parabolic cusp of type $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. In this paper, we fix the constant μ with $0 < \mu < \mu_*$ and set $M_{\mathrm{thick}(\mu)} = M_{\mathrm{thick}}$ and $M_{\mathrm{thin}(\mu)} = M_{\mathrm{thin}}$ for short. Let M_{cusp} be the union of cuspidal components of M_{thin} and $M_{\mathrm{tube}} = M_{\mathrm{thin}} \setminus M_{\mathrm{cusp}}$. In other words, $M_{\rm tube}$ is the union of Margulis tube components of $M_{\rm thin}$. We say that the complement $M_{\rm main} = M \setminus {\rm Int} M_{\rm cusp}$ is the *main part* of M.

As was stated in the introduction, we suppose that Σ is an oriented hyperbolic surface of finite area. Let M be an oriented hyperbolic 3-manifold admitting a marking $\iota : \Sigma \longrightarrow M$, which is supposed to be a proper embedding with $\iota(\Sigma_{\rm cusp}) \subset M_{\rm cusp}$. Then each component of $M_{\rm cusp}$ is a $\mathbb{Z}$-cusp. Let $M_{(\rm cusp)}$ be the union of components of $M_{\rm cusp}$ meeting $\iota(\Sigma_{\rm cusp})$ non-trivially. We consider a union $M_{\rm cusp*}$ of components of $M_{\rm cusp}$ containing $M_{(\rm cusp)}$ and set $M_{\rm main*} = M \setminus M_{\rm cusp*}$. By Scott-McCullough's Core Theorem [Sc, MC], there exists a compact connected submanifold $C_{\rm main*}$ of $M_{\rm main*}$ such that (i) the inclusion $C_{\rm main*} \subset M_{\rm main*}$ is a homotopy equivalence, (ii) $C_{\rm main*} \cap V$ is a non-contractible annulus in ∂V for any component V of $M_{\rm cusp*}$ and (iii) $C_{\rm main*} \cap V' = \emptyset$ for any component V' of $M_{\rm cusp} \setminus M_{\rm cusp*}$. A connected submanifold C_* of M is called a *finite core* of $(M, M_{\rm cusp*})$ if $C_* \cap M_{\rm main*} = C_{\rm main*}$ and $C_* \cap V$ is the union of geodesic rays emanating from the points of $C_{\rm main*} \cap V$ for any component V of $C_{\rm main*}$.

For a finite core C_* of $(M, M_{\rm main*})$, any component E of $M \setminus {\rm Int} C_*$ is considered to be a neighborhood of some end $\mathcal{E}$ of $M_{\rm main*}$. Then $\mathcal{E}$ is called an end of M with respect to the finite core C_* or simply an end of M if it does not cause any confusion. Note that $\Sigma_{\mathcal{E}} = C_* \cap E$ is a properly embedded incompressible surface in M. Any cusp in E of M disjoint from $\Sigma_{\mathcal{E}}$ is called an *accidental parabolic* cusp of $\mathcal{E}$. We say that $\mathcal{E}$ is an *genuine end* of M if $\mathcal{E}$ has no accidental parabolic cusps. A genuine end $\mathcal{E}$ is called *geometrically finite* if the finite core C_* can be taken so that C_* is locally convex in a neighborhood of $\Sigma_{\mathcal{E}}$ in M. According to Bonahon [Bo], if a genuine end $\mathcal{E}$ is not geometrically finite, then there exists a sequence of closed geodesics λ_n^* in E tending toward $\mathcal{E}$ and freely homotopic in E to a simple closed curve λ_n in $\Sigma_{\mathcal{E}}$. Such a genuine end is called *simply degenerate*. Note that E is homeomorphic to $\Sigma_{\mathcal{E}} \times [0, \infty)$ when $\mathcal{E}$ is simply degenerate as well as geometrically finite , see [Th1, Theorem 9.4.1] and [Bo, Corollaire C].

If we suppose that $\iota(\Sigma)$ is a degenerate finite core of M, then M has two ends with respect to $\iota(\Sigma)$. One of them is called the $(+)$-*end* of M if it is adjacent to the closure of the component of $M \setminus \iota(\Sigma)$ which is in the $(+)$-side of $\iota(\Sigma)$ with respect to the orientation of M, and the other is the $(-)$-*end*

A finite core $C_{\rm max}$ of M is *maximal* if $C_{\rm max}$ meets all components of $M_{\rm cusp}$ non-trivially. From the maximality of $C_{\rm max}$, an end $\mathcal{E}$ of M is genuine if and only if it is an end with respect to $C_{\rm max}$. For any end $\mathcal{E}_i$ of M with respect to a finite core C_*, a genuine end $\mathcal{E}_{ij}$ is called a *genuine sub-end* of $\mathcal{E}_i$ if a neighborhood of $\mathcal{E}_i$ in M contains a neighborhood of $\mathcal{E}_{ij}$. See Figure 1.1. A end of M is *totally degenerate* if any genuine sub-end of $\mathcal{E}$ is simply degenerate.

2. Normalized maps tending toward simply degenerate ends

Suppose that M is a hyperbolic 3-manifold admitting a marking embedding $\iota : \Sigma \longrightarrow M$. In this section, we consider the case that the $(+)$-end $\mathcal{E}$ of M is totally degenerate. Then there exists a sequence $\{f_n\}_{n=0}^{\infty}$ of pleated maps $f_n : \Sigma_n \longrightarrow E$ satisfying the following conditions, where E is the neighborhood of $\mathcal{E}$ with respect to $\iota(\Sigma)$. Here we set $E_{\rm main} = E \cap M_{\rm main}$, $E_{\rm thick} = E \cap M_{\rm thick}$ and so on.

- For a sufficiently large $R > 0$, $\mathcal{N}_R(f_n(\Sigma_n)) \cap \mathcal{N}_R(f_{n+1}(\Sigma_{n+1})) \cap E_{\rm main} = \emptyset$ $(n = 0, 1, \dots)$ and $f_{n+1}(\Sigma_{n+1})$ is closer to $\mathcal{E}$ compared with $f_n(\Sigma_n)$.

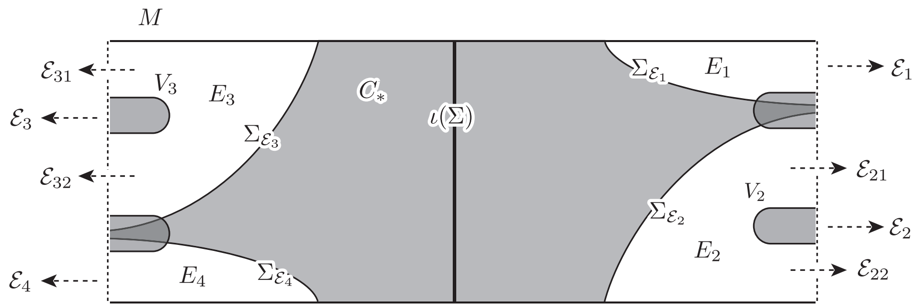

Figure 1.1. $\mathcal{E}_1$, $\mathcal{E}_4$ are genuine ends with respect to the finite core C_*. For $i = 2, 3$, V_i is an accidental parabolic cusp of $\mathcal{E}_i$. $\mathcal{E}_{i1}$, $\mathcal{E}_{i2}$ are genuine sub-ends of $\mathcal{E}_i$.

- If $f_n(\Sigma)$ meets a component V of E_{tube} non-trivially, then Σ_n contains a simple geodesic loop l such that $f_n(l)$ is the geodesic core of V.
- Each f_n is unwrapped with respect to any component V of E_{tube} disjoint from $f_n(\Sigma_n)$, that is, f_n is properly homotopic to an embedding in $E \setminus V$.

These conditions imply that, if $f_n(\Sigma_n) \cap V \neq \emptyset$ for some component V of E_{tube}, then $f_m(\Sigma_m) \cap V = \emptyset$ for any $m \neq n$. If necessary passing to a subsequence of $\{f_n\}$, we may also assume the following.

- For any f_n and any component V of $E_{\text{cusp}} \setminus E_{(\text{cusp})}$, $f_n(\Sigma_n) \cap V$ is an annulus, where $E_{(\text{cusp})} = E \cap M_{(\text{cusp})}$. See Figure 2.1.

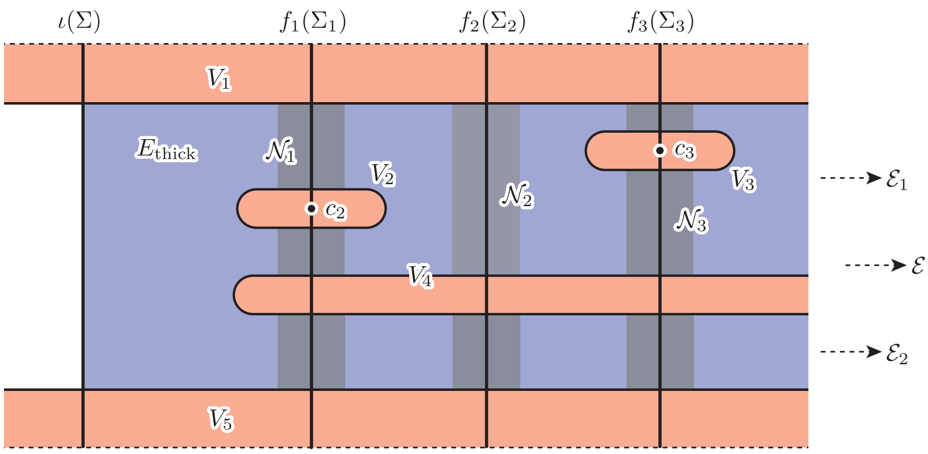

Figure 2.1. V_1, V_5 represent components of $M_{(\text{cusp})}$, V_2, V_3 components of E_{tube} with geodesic cores c_2, c_3 and V_4 an accidental cusp of $\mathcal{E}$. $\mathcal{N}_i = \mathcal{N}_R(f_i(\Sigma_i)) \cap E_{\text{thick}}$ for $i = 1, 2, 3$. $\mathcal{E}_1$, $\mathcal{E}_2$ are simply degenerate sub-ends of $\mathcal{E}$.

The preimage $F_n = f_n^{-1}(E_{\text{thick}})$ is a sub-surface of Σ_n contained in $\Sigma_{n,\text{thick}}$ such that $\Sigma_n \setminus \text{Int}F_n$ is a deformation retract of $\Sigma_{n,\text{thin}}$. Modify the Riemannian metric on E_{thick} in a small neighborhood of $\mathcal{N}_n = \mathcal{N}_R(f_n(\Sigma_n)) \cap E_{\text{thick}}$ so that $\partial\mathcal{N}_n$ is locally convex in $\mathcal{N}_n$. By Freedman-Hass-Scott [FHS], there exists an embedding $h_n : F(h_n) \longrightarrow \mathcal{N}_n$ which has least area among all piecewise smooth maps $h'_F : F \longrightarrow \mathcal{N}_n$ properly homotopic to $f_n|_{F_n}$ *in* $E_{\rm thick}$ and such that each component of $h_{F'}(\partial F')$ is a simple geodesic loop in the Euclidean surface $\partial E_{\rm thick}$. Let $\widehat{f}_n : \widehat{\Sigma}_n \longrightarrow E$ be the embedding satisfying the following conditions.

- The domain $\widehat{\Sigma}_n$ contains $\widehat{F}_n = F(h_n)$ as a sub-surface and $\widehat{f}_n|_{\widehat{F}_n} = h_n|_{F(h_n)}$.
- Let C be a component of $\widehat{\Sigma}_n \setminus {\rm Int}\widehat{F}_n$. If $\widehat{f}_n(C)$ is contained in $M_{({\rm cusp})}$, then $\widehat{f}_n(C)$ is a totally geodesic parabolic cusp. If $\widehat{f}_n(C)$ lies in either an accidental cusp of $\mathcal{E}$ or a component V of $E_{\rm tube}$, then $\widehat{f}_n(C)$ is a smoothly embedded ruled annulus in V consisting of shortest arcs in V connecting the components of ∂V.

We say that $\widehat{f}_n$ is a *normalized map* associated with f_n. Then $\widehat{\Sigma}_n$ has a piecewise smooth Riemannian metric induced from the hyperbolic metric on E via $\widehat{f}_n$. The advantage of normalized maps over pleated maps is that $\widehat{f}_n$ are embeddings.

The following lemma is proved immediately from an argument of bounded geometry together with [Th1, Proposition 8.12.1].

Lemma 2.1. *The following* (1)–(3) *hold, where constants means that they are independent of* n*.*

(1) *There exists a constant* $a_0 > 0$ *with* ${\rm Area}(\widehat{\Sigma}_n) \le a_0$*.*
(2) *There exists a constant* $d_0 > 0$ *with* ${\rm diam}(C) \le d_0$ *for any component* C *of* $\widehat{\Sigma}_n \setminus {\rm Int}\widehat{F}_n$*.*
(3) *For any* $d > 0$*, there exists a constant* $v_0(d) > 0$ *with* ${\rm Vol}(\mathcal{N}_d(\widehat{f}_n(\widehat{\Sigma}_n))) < v_0(d)$*.*

Again by an argument of bounded geometry, there exists a constant $r = r(\Sigma) > 0$ such that, for each n, $\widehat{\Sigma}_n$ contains a disjoint union $\mathcal{H}(\widehat{f}_n) = \lambda_1 \sqcup \cdots \sqcup \lambda_m$ of mutually disjoint simple loops satisfying the following conditions.

- For each component λ_j of $\mathcal{H}(\widehat{f}_n)$, ${\rm length}_{\widehat{\Sigma}_n}(\lambda_j) < r$.
- For each annulus component A of $\widehat{\Sigma}_n \setminus {\rm Int}\widehat{F}_n$, $A \cap \mathcal{H}_n$ is the geodesic core of A.
- The closure G of each component of $\widehat{\Sigma}_n \setminus \mathcal{H}(\widehat{f}_n)$ has bounded geometry and the Euler characteristic -1.

We say that $\mathcal{H}(\widehat{f}_n)$ is an *r-hoop family* (for short *hoop family*) of $\widehat{\Sigma}_n$.

3. Bounded cohomology and smearing chains on 3-manifolds

We denote by Δ^n a regular n-simplex of edge length 1 in the Euclidean n-space. Let $C^*(X)$ be the dual space of the singular chain-complex $C_*(X)$ of a topological space X with real coefficient. Consider the subspace $C_b^*(X)$ of $C^*(X)$ consisting of bounded cochains, that is, $c \in C_b^n(X)$ means that

$$\|c\| = \sup \{|c(\sigma)| \mid \sigma : \Delta^n \longrightarrow X \text{ is a singular } n\text{-simplex}\} < \infty.$$

Since the coboundary operator $\delta^n : C^n(X) \longrightarrow C^{n+1}(X)$ satisfies $\delta^n(C_b^n(X)) \subset C_b^{n+1}(X)$, the bounded cochain complex $(C_b^*(X), \delta_b^*)$ with $\delta_b^* = \delta^*|_{C_b^*(X)}$ defines the *bounded cohomology*

$$H_b^n(X, \mathbb{R}) = Z_b^n(X)/B_b^n(X)$$

with the pseudo-norm

$$\|\alpha\| = \inf \{\|c\| \mid c \text{ is an element of } Z_b^n(X) \text{ with } [c] = \alpha\}$$

for $\alpha \in H_b^n(X, \mathbb{R})$, where $Z_b^n(X) = (\delta_b^n)^{-1}(0)$ and $B_b^n(X) = \delta_b^{n-1}(C_b^{n-1}(X))$.

Suppose that $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold as in Section 1. Then the quotient map $p : \mathbb{H}^3 \longrightarrow M$ is a locally isometric universal covering. A singular k-simplex $\sigma : \Delta^k \longrightarrow M$ is called *straight* if its lift $\widetilde{\sigma} : \Delta^k \longrightarrow \mathbb{H}^3$ to $\mathbb{H}^3$ is *straight*, that is, $\widetilde{\sigma}$ is the affine map with respect to the Euclidean structure on Δ^3 and the quadratic model on $\mathbb{H}^3$. For any singular k-simplex $\widetilde{\sigma} : \Delta^k \longrightarrow \mathbb{H}^3$, let $\mathrm{straight}(\widetilde{\sigma}) : \Delta^k \longrightarrow \mathbb{H}^3$ be the straight map with $\mathrm{straight}(\widetilde{\sigma}(v_j)) = \widetilde{\sigma}(v_j)$ for all vertices v_j ($j = 0, 1, \dots, k$) of Δ^k. We note that the image $\mathrm{straight}(\widetilde{\sigma})(\Delta^k)$ is a (possibly degenerate) straight k-simplex in $\mathbb{H}^3$. For a singular k-simplex $\sigma : \Delta^k \longrightarrow M$, the map $\mathrm{straight}_M(\sigma) = p \circ \mathrm{straight}(\widetilde{\sigma}) : \Delta^k \longrightarrow M$ is called the k-simplex obtained by *straightening* σ, where $\widetilde{\sigma} : \Delta^k \longrightarrow \mathbb{H}^3$ is a lift of σ.

The *oriented volume* of a C^1 singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$ is defined by

$$\mathrm{Vol}(\sigma) = \int_{\Delta^3} \sigma^*(\Omega_M),$$

where Ω_M is the volume form on M. We say that σ is *non-degenerate* if $\mathrm{Vol}(\sigma) \neq 0$, and *positive* (resp. *negative*) if $\mathrm{Vol}(\sigma) > 0$ (resp. $\mathrm{Vol}(\sigma) < 0$).

Let ω_M be the 3-cocycle on M defined by

$$\omega_M(\sigma) = \mathrm{Vol}(\mathrm{straight}_M(\sigma))$$

for any singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$. Since $|\omega_M(\sigma)|$ is less the volume $\boldsymbol{v}_3$ of a regular ideal 3-simplex in $\mathbb{H}^3$ for any singular 3-simplex $\sigma : \Delta^3 \longrightarrow M$, ω_M represents an element $[\omega_M]$ of $H_b^3(M, \mathbb{R})$ with $\|[\omega_M]\| \leq \boldsymbol{v}_3$. We say that $[\omega_M]$ is the *fundamental (bounded cohomology) class* of M.

For any smooth manifold N, let $C^1(\Delta^k, N)$ be the topological space of C^1-maps $\Delta^k \longrightarrow N$ with C^1-topology. We denote by $\mathcal{C}_k(N)$ the $\mathbb{R}$-vector space consisting of Borel measures μ on $C^1(\Delta^k, N)$ with the bounded total variation $\|\mu\| < \infty$. An element of $\mathcal{C}_k(N)$ is called a *k-chain*. The boundary operator $\partial_k : \mathcal{C}_k(N) \longrightarrow \mathcal{C}_{k-1}(N)$ is defined naturally. Thus we have the chain complex $(\mathcal{C}_*(N), \partial_*)$.

Now we consider the case of $N = M$. Take the base point x_0 of $\mathbb{H}^3$ and suppose that $y_0 = p(x_0)$ is the base point of M. Let μ_{Haar} be a left-right invariant Haar measure on $\mathrm{PSL}_2(\mathbb{C})$, which is normalized so that, for any bounded Borel subset U of $\mathbb{H}^3$,

$$\mu_{\mathrm{Haar}}(\{\alpha \in \mathrm{PSL}_2(\mathbb{C}) \,|\, \alpha x_0 \in U\}) = \mathrm{Vol}(U). \tag{3.1}$$

From the invariance of μ_{Haar}, we know that the quotient map $q : \mathrm{PSL}_2(\mathbb{C}) \longrightarrow P(M) = \Gamma\backslash\mathrm{PSL}_2(\mathbb{C})$ induces the measure $\widehat{\mu}_{\mathrm{Haar}}$ on the quotient space $P(M)$. That is, $\widehat{\mu}_{\mathrm{Haar}}(q(\mathcal{A}))$ is equal to $\mu_{\mathrm{Haar}}(\mathcal{A})$ for any Borel subset $\mathcal{A}$ of $\mathrm{PSL}_2(\mathbb{C})$ with $\mathcal{A} \cap \gamma\mathcal{A} = \emptyset$ if $\gamma \in \Gamma \setminus \{1\}$. For any point $x \in \mathbb{H}^3$ and $a \in P(M)$, $a \bullet x$ denotes the point of M defined by $p(\alpha x)$ for an $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ with $q(\alpha) = a$. Note that the point does not depend on the choice of $\alpha \in q^{-1}(a)$. Thus the map

$$\bullet : P(M) \times \mathbb{H}^3 \longrightarrow M$$

is well-defined. For any singular 3-simplex $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ and $a \in P(M)$, the singular 3-simplex $a \bullet \sigma : \Delta^3 \longrightarrow M$ is defined by $p \circ (\alpha\sigma)$ for an $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ with $q(\alpha) = a$.

Let $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ be a non-degenerate straight 3-simplex. Suppose that $\mathrm{smear}_M(\sigma)$ is the Borel measure on $C^1(\Delta^3, M)$ introduced in [Th1, Section 6.1], which satisfies the following conditions.

- The support $\mathrm{supp}(\mathrm{smear}_M(\sigma))$ is $\{a \bullet \sigma \,|\, a \in P(M)\}$.
- For any closed non-empty subset $\mathcal{X}$ of $P(M)$,

$$\mathrm{smear}_M(\sigma)(\{a \bullet \sigma \,|\, a \in \mathcal{X}\}) = \widehat{\mu}_{\mathrm{Haar}}(\mathcal{X}). \tag{3.2}$$

We denote the inner center of the straight 3-simplex $\sigma(\Delta^3)$ in $\mathbb{H}^3$ by $o(\sigma)$. For any non-empty almost compact subset X of M, the restriction of $\mathrm{smear}_M(\sigma)$ to $\{a \bullet \sigma \,|\, a \in P(M)$ with $a \bullet o(\sigma) \in X\}$ is denoted by $\mathrm{smear}_X(\sigma)$. By (3.1) and (3.2), its total variation is

$$\|\mathrm{smear}_X(\sigma)\| = \mathrm{Vol}(X). \tag{3.3}$$

In particular, $\mathrm{smear}_X(\sigma)$ is an element of $\mathcal{C}_3(M)$. Set $\sigma_- = \rho \circ \sigma$ for an orientation-reversing isometry ρ on $\mathbb{H}^3$ with $\rho(o(\sigma)) = o(\sigma)$. Consider the element $z_X(\sigma)$ of $\mathcal{C}_3(M)$ defined by

$$z_X(\sigma) = \frac{1}{2}(\mathrm{smear}_X(\sigma) - \mathrm{smear}_X(\sigma_-)). \tag{3.4}$$

Then, by (3.2) and (3.3), we have $\|z_X(\sigma)\| = \mathrm{Vol}(X)$ and

$$z_X(\sigma)(\{a \bullet \sigma \,|\, a \in P(M) \text{ with } a \bullet o(\sigma) \in X\}) = \frac{1}{2}\mathrm{Vol}(X).$$

For a Borel measure ω on $C^1(\Delta^3, M)$, let $\mathrm{supp}^{(2)}(w)$ be the subset of $C^1(\Delta^2, M)$ defined by

$$\mathrm{supp}^{(2)}(w) = \left\{\tau|_D \,\middle|\, \tau \in \mathrm{supp}(w) \text{ and } D \in (\Delta^3)^{(2)}\right\}, \tag{3.5}$$

where $(\Delta^3)^{(2)}$ is the set of 2-faces of Δ^3. By the definition, $\mathrm{supp}(\partial_3 w) \subset \mathrm{supp}^{(2)}(w)$.

Let $\{\widehat{f}_n\}_{n=0}^\infty$ be the sequence of normalized maps $\widehat{f}_n : \widehat{\Sigma}_n \longrightarrow E$ given in Section 2. For any m, n with $m < n$, we denote by $E(\widehat{f}_m, \widehat{f}_n)$ the closure of the component of $E \setminus \widehat{f}_m(\widehat{\Sigma}_m) \cup \widehat{f}_n(\widehat{\Sigma}_n)$ bounded by $f_m(\widehat{\Sigma}_m) \cup \widehat{f}_n(\widehat{\Sigma}_n)$.

Lemma 3.1. *Under the notation as above, let $\widehat{X} = E(\widehat{f}_m, \widehat{f}_n)$. Then $\mathrm{supp}(\partial_3 z_{\widehat{X}}(\sigma))$ is contained in $\mathrm{supp}^{(2)}(z_{\mathcal{N}_2(\partial\widehat{X},M)}(\sigma))$ and $\|\partial_3 z_{\widehat{X}}(\sigma)\| < 8v_0(2)$ holds, where $v_0(2)$ is the constant given in Lemma 2.1 (3).*

Proof. The volume of any (real) straight 3-simplex Δ in $\mathbb{H}^3$ is less than $\boldsymbol{v}_3 = 1.014916\ldots$. On the other hand, since the volume of a 3-ball in $\mathbb{H}^3$ of radius one is $\pi(\sinh 2 - 2) = 5.11093\ldots$, the radius of the inscribed ball in Δ is less than one. Let D be any element of $(\Delta^3)^{(2)}$. For any $a \bullet \sigma$ with $a \bullet o(\sigma) \in \widehat{X}$, there exists $b \in P(M)$ with $b \bullet o(\sigma_-) \in \mathcal{N}_2(\widehat{X}, M)$ and such that $a \bullet \sigma|_D = b \bullet \sigma_-|_D$. See Figure 3.1. Similarly, we have $a \bullet \sigma_-|_D = b \bullet \sigma|_D$. In general, we can not expect that $b \bullet o(\sigma_-)$ is contained in $\widehat{X}$. However, if $a \bullet o(\sigma) \in \widehat{X} \setminus \mathcal{N}_2(\partial\widehat{X}, M)$, then $b \bullet o(\sigma_-)$ is an element of $\widehat{X}$. These facts imply that $\mathrm{supp}(\partial_3 z_{\widehat{X}}(\sigma)) \subset \mathrm{supp}^{(2)}(z_{\mathcal{N}_2(\partial\widehat{X},M)})$. Since $\partial\widehat{X} = \widehat{f}_m(\widehat{\Sigma}_m) \cup \widehat{f}_n(\widehat{\Sigma}_n)$, we have by Lemma 2.1 (3)

$$\|z_{\mathcal{N}_2(\partial\widehat{X},M)}(\sigma)\| = \mathrm{Vol}(\mathcal{N}_2(\widehat{f}_m(\widehat{\Sigma}_m) \cup \widehat{f}_n(\widehat{\Sigma}_n))) < 2v_0(2).$$

Since Δ^3 has four 2-faces, $\|\partial_3 z_{\widehat{X}}(\sigma)\| < 4 \cdot 2v_0(2) = 8v_0(2)$. □

Since the image $\tau(\Delta^3)$ of any element $\tau = a \bullet \sigma \in \mathrm{supp}\{z_{\widehat{X}}(\sigma)\}$ has 'long tails', $\tau(\Delta^3)$ is not necessarily contained in $\widehat{X}$ even if $a \bullet o(\sigma)$ is an element of $\mathrm{Int}\widehat{X}$ such that $\mathrm{dist}(a \bullet o(\sigma), \partial\widehat{X})$ is large. So we sometimes need to treat the body (inner part) and tails (outer part) of $\tau(\Delta^3)$ separately as in the next section.

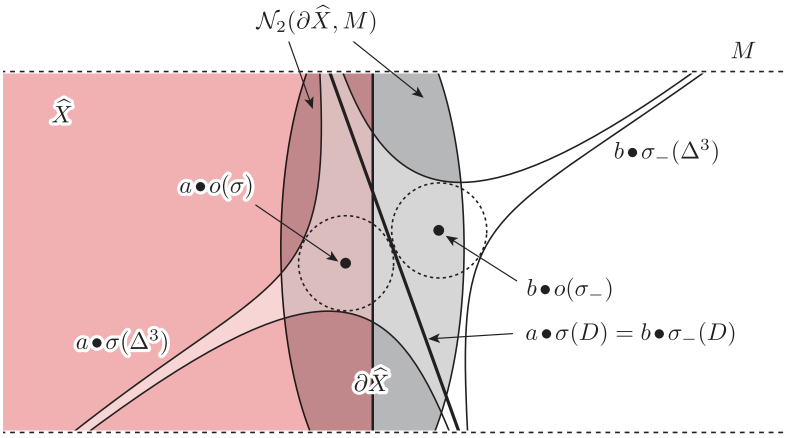

Figure 3.1

Remark 3.2. Let X be a topological space and $C_*^{\mathrm{sing}}(X)$ the singular chain group of X with real coefficients. The *Gromov norm* of an element $c = \sum_{i=1}^n r_i\sigma_i$ of $C_q^{\mathrm{sing}}(X)$ is given by $\|c\| = \sum_{i=1}^n |r_i|$. Let $C_*^{l_1}(X)$ be the norm completion of $C_*^{\mathrm{sing}}(X)$. Thus $C_*^{l_1}(X)$ is a Banach space consisting of elements $c = \sum_{i=1}^\infty r_i\sigma_i$ with $\|c\| = \sum_{i=1}^\infty |r_i| < \infty$. If an element c of $C^{l_1}(M)$ is a linear combination $\sum_{i=1}^\infty r_i\sigma_i$ of straight 3-simplices $\sigma_i : \Delta^3 \longrightarrow M$, then c is identified with the element $\sum_{i=1}^\infty r_i\delta_{\sigma_i}$ of $\mathcal{C}_3(M)$, where δ_{σ_i} is the Dirac measure on $C^1(\Delta^3, M)$ at σ_i. Then the Gromov norm $\sum_{i=1}^\infty |r_i|$ of c is equal to the total variation of $\sum_{i=1}^\infty r_i\delta_{\sigma_i}$. There exists a sequence $\{c_n\}$ of locally finite elements in $C_*^{l_1}(X)$ with $c_n = \sum_{i=1}^\infty r_i\sigma_i^n$ consisting of straight 3-simplices σ_i^n with

$$|\mathrm{Vol}(\sigma_i^n) - \mathrm{Vol}(\sigma)| < \frac{1}{n}$$

and such that $\{c_n\}$ weakly converges to $z_X(\sigma)$ and $\{\partial_3 c_n\}$ weakly converges to $\partial_3 z_X(\sigma)$. For example see the map $\mathcal{A}_*$ in [So1, Section 3]. In our arguments below, we may use the usual locally finite singular 3-chain c_n in $C^{l_1}(X)$ with sufficiently large n instead of $z_X(\sigma)$ if necessary.

4. Linear isoperimetric inequality modulo hoop families

First we define a subdivision of hyperbolic straight simplices. Let Δ be any straight 3-simplex in $\mathbb{H}^3$ with real vertices v_0, v_1, v_2, v_3 and $\mathrm{Vol}(\Delta) > 1$. Consider the inscribed sphere $S(\Delta)$ of Δ and S_i $(i = 0, 1, 2, 3)$ the round sphere in $\mathbb{H}^3$ centered at v_i and tangent to $S(\Delta)$. Each S_i intersects three edges of Δ. Let T_i be the totally geodesic triangle in Δ with the three intersection points as its vertices. We denote by $\Delta_{i,\mathrm{out}}$ the closure of the component of $\Delta \setminus T_0 \cup \cdots \cup T_3$ containing v_i and by Δ_{inn} the closure of $\Delta \setminus \Delta_{\mathrm{out}}$, where $\Delta_{\mathrm{out}} = \Delta_{0,\mathrm{out}} \cup \cdots \cup \Delta_{3,\mathrm{out}}$. We say that Δ_{inn} and Δ_{out} are the *inner* and *outer* parts of Δ, respectively. See Figure 4.1. For any small positive number ξ, say $\xi < 1/100$, there exists a simplicial triangulation $\tau(\Delta, \xi)$ of Δ satisfying the following conditions, where $\tau(\Delta, \xi)^{(i)}$ denotes the subset of $\tau(\Delta, \xi)$ consisting of i-simplices.

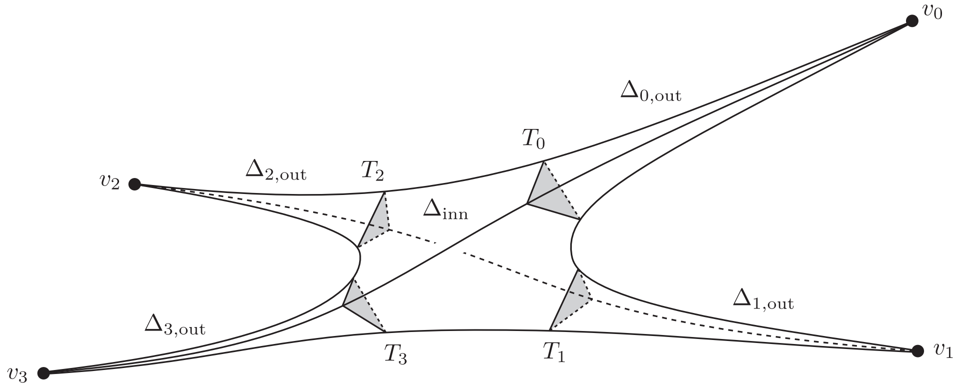

Figure 4.1

- Each element of $\tau(\Delta,\xi)^{(3)}$ is a straight simplex.
- For any $e \in \tau(\Delta,\xi)^{(1)}$ with $e \subset \Delta_{\text{inn}}$, $\delta(\xi) \le \text{length}_\Delta(e) \le \xi$, where $\delta(\xi)$ is a uniform constant with $0 < \delta(\xi) < \xi$.
- Each T_i is the underlying space of a subcomplex of $\bigcup_{i=0}^{2} \tau(\Delta,\xi)^{(i)}$. Each edge α_{ij} of T_i is evenly divided by $\tau(\Delta,\xi)^{(0)}|_{\alpha_{ij}}$.
- For each edge β_{jk} of Δ connecting v_j with v_k, $\beta_{jk} \cap \Delta_{\text{inn}}$ is evenly divided by $\tau(\Delta,\xi)^{(0)}|_{\beta_{jk}\cap\Delta_{\text{inn}}}$.
- $\Delta_{\text{out}} \setminus \big(\{v_0,\dots,v_3\} \cup T_1 \cup \dots \cup T_4\big)$ contains no elements of $\tau(\Delta,\xi)^{(0)}$.

See Figure 4.2. In fact, such triangulations can be obtained from a fixed simplicial

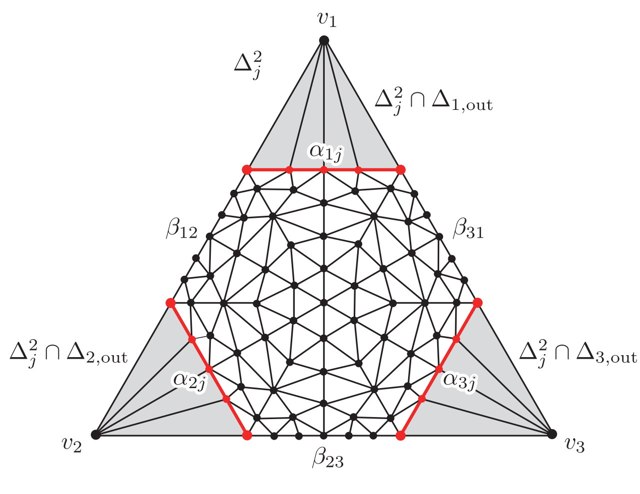

Figure 4.2. The restriction of $\tau(\Delta,\xi)$ to a 2-face Δ_j^2 of Δ in the Klein model. The inner hexagonal part is $\Delta_j^2 \cap \Delta_{\text{inn}}$.

triangulation $\tau(\Delta_\infty,\xi)$ on a regular ideal 3-simplex Δ_∞ in $\mathbb{H}^3$ satisfying the five conditions as above, where the ideal simplices of Δ_∞ are regarded as elements of $\tau(\Delta_\infty,\xi)^{(0)}$. Since $\boldsymbol{v}_3 - 0.015 < \mathrm{Vol}(\Delta) < \boldsymbol{v}_3$, there exists a κ-bi-Lipschitz map $\alpha : \Delta_{\infty,\mathrm{inn}} \longrightarrow \Delta_{\mathrm{inn}}$ for some uniform constant κ close to one such that $\alpha(\tau(\Delta_\infty,\xi)^{(0)}|_{\Delta_{\infty,\mathrm{inn}}})$ extends to a triangulation $\tau(\Delta,\xi)|_{\Delta_{\mathrm{inn}}}$ on Δ_{inn} satisfying the required conditions. We set $\tau(\Delta,\xi)_{\mathrm{inn}} = \tau(\Delta,\xi)|_{\Delta_{\mathrm{inn}}}$ and $\tau(\Delta,\xi)_{\mathrm{out}} = \tau(\Delta,\xi)|_{\Delta_{\mathrm{out}}}$.

Suppose that $\sigma' \in \mathrm{supp}(\mathrm{smear}_M(\sigma))$ and $\sigma'_- \in \mathrm{supp}(\mathrm{smear}_M(\sigma_-))$ are 3-simplices with $\sigma'|_{\Delta_j^2} = \sigma'_-|_{\Delta_j^2}$ for some 2-face Δ_j^2 of Δ. Then, for any element D of $\tau(\Delta,\xi)^{(2)}|_{\Delta_j^2}$, we have

$$\sigma'|_D - \sigma'_-|_D = 0. \tag{4.1}$$

Let $\{\widehat{f}_n\}_{n=0}^\infty$ be the sequence of normalized maps $\widehat{f}_n : \widehat{\Sigma}_n \longrightarrow E$ given in Section 2. For simplicity, we only consider here the pair $\widehat{f}_0$, $\widehat{f}_1$. Our argument works for any pair $\widehat{f}_m$, $\widehat{f}_n$ with $m < n$. By Lemma 2.1 (2), one can define an (ideal) triangulation τ_i ($i = 0,1$) on $\widehat{\Sigma}_i$ satisfying the following conditions, where $\mathcal{H}(\widehat{f}_i)$ ($i = 1,0$) is a hoop family of $\widehat{\Sigma}_i$.

(T1) Each element v of $\tau_i^{(0)}$ is either a point of $\mathcal{H}(\widehat{f}_i)$ or an ideal point of $\widehat{\Sigma}_i$. See Figure 4.3.
(T2) $\bigcup \tau_i^{(1)}$ contains $\mathcal{H}(\widehat{f}_i)$.
(T3) For any component l of $\mathcal{H}(\widehat{f}_i)$, $l \cap \bigcup \tau_i^{(0)}$ consists of just two points.
(T4) The cardinality of τ_i is uniformly bounded.
(T5) There exists a uniform constant $d_1 > 0$ such that the d_1-neighborhood of any point x of $F(\widehat{f}_i) = \widehat{f}_i^{-1}(E_{\mathrm{thick}})$ contained in $\mathrm{star}(v)$ for some $v \in \tau_i^{(0)}$, where $\mathrm{star}(v)$ is the union $\bigcup_\alpha \mathrm{Int} D_\alpha$ for all elements D_α of τ_i with v as a common vertex.

We say that τ_i is a *normalized triangulation* on $\widehat{\Sigma}_i$ with respect to $\mathcal{H}(\widehat{f}_i)$.

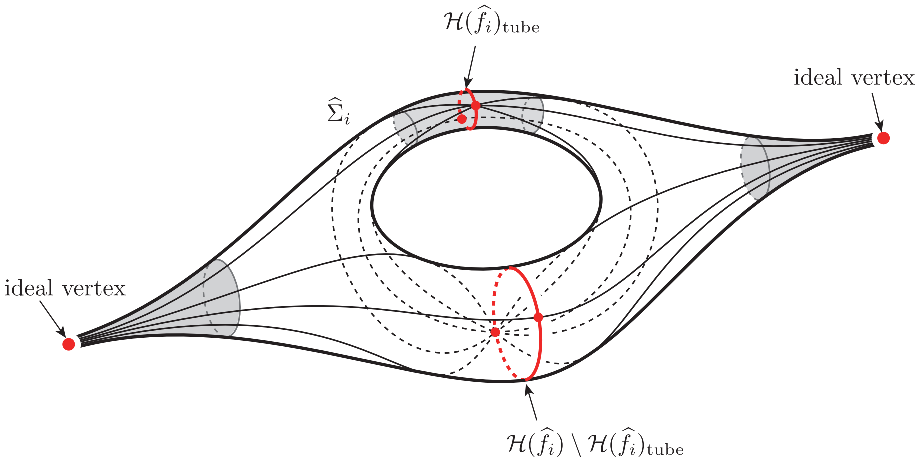

Figure 4.3. The shaded region represents $\widehat{f}_i^{-1}(E_{\mathrm{thin}})$.

Let $\mathcal{H}(\widehat{f}_i) \cap \widehat{f}_i^{-1}(E_{\mathrm{thin}}) = \mathcal{H}(\widehat{f}_i)_{\mathrm{tube}}$. We consider the unions of closed curves

$$\widehat{\mathcal{H}}_i = \widehat{f}_i(\mathcal{H}(\widehat{f}_i)) \quad \text{and} \quad \widehat{\mathcal{H}}_{i,\mathrm{tube}} = \widehat{f}_i(\mathcal{H}(\widehat{f}_i)_{\mathrm{tube}}) \tag{4.2}$$

in E.

For simplicity, throughout the remainder of this section, we set $\widehat{f}_i(\widehat{\Sigma}_i) = \widehat{f}_i(\Sigma)$ and $\widehat{f}_i(\tau_i) = \{\widehat{f}_i(\sigma); \sigma \in \tau_i\}$. A singular 2-simplex $\sigma : \Delta^2 \longrightarrow \widehat{f}_i(\Sigma)$ is called a 2-simplex *with respect to* $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ if, for any edge e of Δ^2, either $\sigma(e)$ is an element of $\widehat{f}_i(\tau_i^{(0)} \cup \tau_i^{(1)})$ (possibly an ideal vertex) or the restriction $\sigma|_e$ is an immersion into $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ connecting two points of $\widehat{f}_i(\tau_i^{(0)})$. In the latter case $\widehat{f}_i(e)$ is not necessarily contained in $\widehat{f}_i(\tau_i^{(0)} \cup \tau_i^{(1)})$. In either case, $\sigma|_e$ is called a 1-simplex *with respect to* $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$. Since $\widehat{f}_i(\Sigma)$ is not necessarily a closed surface, any simplicial 2-cycle on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ is supposed to represent a class of the locally finite homology group $H_2^{\mathrm{loc.f.}}(\widehat{f}_i(\Sigma), \mathbb{R})$.

The following lemma shows a sort of linear isoperimetric inequality for $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$.

Lemma 4.1. *There exists a uniform integer $L_0 > 0$ satisfying the following condition. Let $\widehat{c} = \widehat{e}_1 + \widehat{e}_2 + \cdots + \widehat{e}_n$ be any contractible 1-cycle on $\widehat{f}_i(\Sigma)$ such that each $\widehat{e}_j$ is a 1-simplex with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$. Then $\widehat{c}$ bounds a simplicial 2-chain $\widehat{w}$ of disk type on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ such that $\|\widehat{w}\| \le L_0\|\widehat{c}\|$.*

Proof. Let D be an (abstract) 2-disk bounded by $\widehat{c}$ and let $g : D \longrightarrow \widehat{f}_i(\Sigma)$ be a continuous map extending $\widehat{c}$. If necessary deforming g by homotopy rel. $\widehat{c}$, we may assume that D has a simplicial decomposition $\widehat{\tau}$ such that, for each element Δ of $\widehat{\tau}$, the restriction $g|_\Delta$ is a simplex on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$. Then D is divided to sub-disks $D_1, \dots, D_{k_0}$, $D_1', \dots, D_{l_0}'$ with $k_0, l_0 \le \|\widehat{c}\| = n$ and satisfying the following conditions.

- For each $k = 1, \dots, k_0$, $\mathrm{Int}D_k$ is a component of $\mathrm{Int}D \setminus g^{-1}(\widehat{\mathcal{H}}_i)$.
- For each $l = 1, \dots, l_0$, D_l' is the closure of a component of $D \setminus D_1 \cup \cdots \cup D_{k_0}$. In the degenerate case, D_l' is an arc connecting two vertices of $\widehat{c}$.

See Figure 4.4. We may assume that, if $b' = D_k \cap D_l'$ is an arc connecting two vertices of $\widehat{c}$, then $g|_{b'}$ is an immersion. It is possible unless $g(b')$ is a single point. Otherwise, one can divide $\widehat{c}$ into two contractible 1-cycles $\widehat{c}_1$ and $\widehat{c}_2$ with $\|\widehat{c}_1\| + \|\widehat{c}_2\| = \|\widehat{c}\|$ by pinching $\widehat{c}$ along $g|_{b'}$, which reduces the proof to the case of contractible 1-cycles with smaller Gromov norm.

Let $\mathcal{D} = D_1 \cup \cdots \cup D_{k_0}$ and $\mathcal{D}' = D_1' \cup \cdots \cup D_{l_0}'$. Suppose that D_k is a 'polygon' consisting of edges $b_{k,1}, \dots, b_{k,m_k}$ and arcs $b_{k,1}', \dots, b_{k,m_k}'$ such that $g(\mathrm{Int}b_{k,u}) \subset \widehat{f}_i(\Sigma) \setminus \widehat{\mathcal{H}}_i$ and $g(b_{k,u}') \subset \widehat{\mathcal{H}}_i$ for $u = 1, \dots, m_k$. Note that $b_{k,u}'$ possibly consists of a single point. Set $\mathcal{B}_k' = b_{k,1}' \cup \cdots \cup b_{k,m_k}'$. Any element of $\widehat{\tau}^{(1)}|_{D_k}$ not in $\mathcal{B}_k'$ connects distinct components of $\mathcal{B}_k'$. The number of such elements is at most $2m_k - 3$ up to proper homotopy on $(D_k, g^{-1}(\widehat{\mathcal{H}}_i) \cap D_k)$. By the property (T3) on τ_i, any property homotopy class contains at most five elements of $\widehat{\tau}^{(1)}|_{D_k}$. See Figure 4.5 for the case with maximal edges. Then we have $\#(\widehat{\tau}^{(1)}|_{D_k} \setminus \widehat{\tau}^{(1)}|_{\mathcal{B}_k'}) \le 5(2m_k - 3) \le 10m_k - 15$ and hence $\#(\widehat{\tau}^{(2)}|_{D_k}) \le 10m_k - 14$. So the inequality $\#(\widehat{\tau}^{(2)}|_{\mathcal{D}}) \le \sum_{k=1}^{k_0}(10m_k - 14) < 10n$ holds. Since each vertex of $\widehat{\tau}|_{D_k}$ not in ∂D is end points of at least two elements of $\widehat{\tau}^{(1)}|_{D_k} \setminus \widehat{\tau}^{(1)}|_{\mathcal{B}_k'}$, $\#(\widehat{\tau}^{(0)}|_{\mathcal{D} \setminus \partial D}) \le \sum_{k=1}^{k_0} 5(2m_k - 3) < 10n$. Since $\#(\widehat{\tau}^{(0)}|_{\mathcal{D}' \setminus \partial D}) = \#(\widehat{\tau}^{(0)}|_{\mathcal{D} \setminus \partial D})$, we have $\#(\widehat{\tau}^{(0)}|_{\mathcal{D}'}) < 10n + n = 11n$. It follows that $\#(\widehat{\tau}^{(2)}|_{\mathcal{D}'}) < 11n$. Thus $L_0 = 10 + 11 = 21$ is our desired uniform integer. $\square$

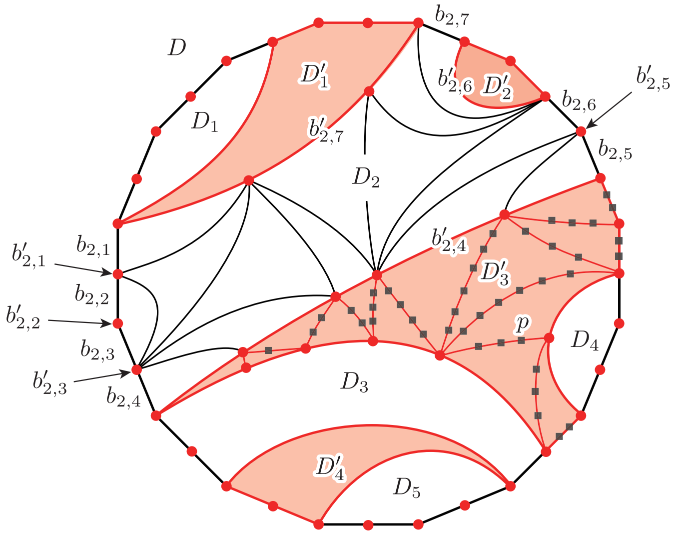

Figure 4.4. Simplicial decompositions of D_2 and D'_3 with $m_2 = 7$, $\#(\widehat{\tau}^{(0)}|_{\partial D_2}) = 16$ and $\#(\widehat{\tau}^{(0)}|_{\partial D'_3}) = 15$. Each square dot p is a point contained in an element e of $\widehat{\tau}^{(1)}|_{D'_3}$ with $p \notin \widehat{\tau}^{(0)}|_{D'_3}$ and $g(p) \in \widehat{f}_i(\tau_i^{(0)})|_{\widehat{\mathcal{H}}_{i,\mathrm{tube}}}$.

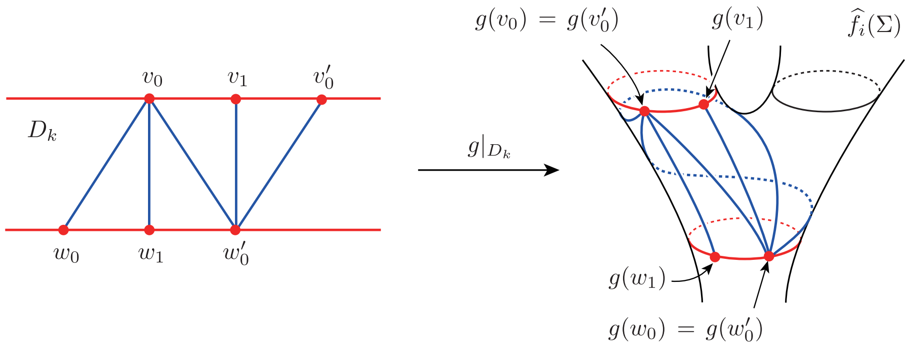

Figure 4.5. The five blue segments in D_k represent properly homotopic elements of $\widehat{\tau}^{(1)}|_{D_k}$ whose g-images are distinct edges of $\widehat{f}_i(\tau_i)$.

5. Connection of smearing 3-chains with normalized triangulations

Now we suppose that the constant R given in Section 2 is at least 4 and prove the following connecting lemma, which plays an important role in the proof of Theorem A.

Lemma 5.1 (Connecting Lemma)**.** *Let* $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ *be a straight* 3*-simplex with* $\mathrm{Vol}(\sigma) > 1$ *and* $\widehat{X} = E(\widehat{f}_0, \widehat{f}_1)$*. Then there exists a* 3*-chain* z *on* M *satisfying the following conditions.*

(1) *$z = z_{\widehat{X}}(\sigma) + \widehat{a}$, where $\widehat{a}$ is a 3-chain on M with $\|\widehat{a}\| \le b_0$ for some uniform constant $b_0 > 0$.*
(2) *For $i = 0, 1$, there exists a simplicial 2-cycle $w(\tau_i)$ on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i) \bmod \widehat{\mathcal{H}}_{i,\mathrm{tube}}$ representing the fundamental class of $\widehat{f}_i(\Sigma)$ and satisfying*

$$\partial_3 z = \mathrm{Vol}(\sigma)(w(\tau_1) - w(\tau_0)).$$

In the case when $\widehat{f}_i(\Sigma) \cap E_{\mathrm{cusp}} \neq \emptyset$, we deform $\widehat{f}_i$ temporarily by replacing the (totally geodesic) parabolic cusps of $\widehat{f}_i(\Sigma)$ by cusps of constant Gaussian curvature > -1. The modified map is still denoted by $\widehat{f}_i$. Since $r \ge 4$, one can choose such cusps so that $\mathcal{N}_4(\widehat{f}_0(\Sigma)) \cap \mathcal{N}_4(\widehat{f}_1(\Sigma)) = \emptyset$. See Figure 5.1. For $i = 0, 1$, let $\mathcal{L}_{4,i}$

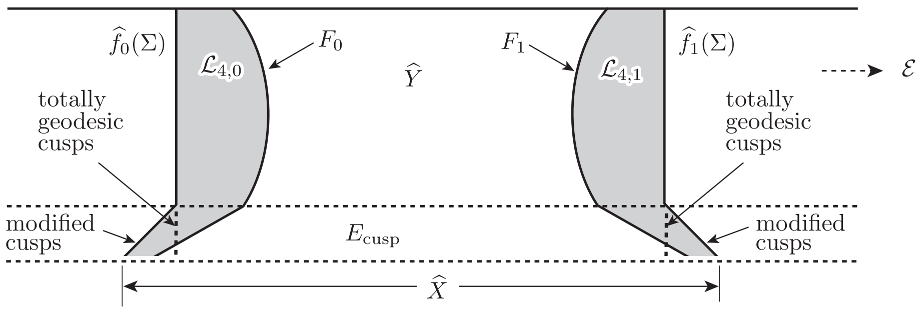

Figure 5.1

be the closure of the component of $\mathcal{N}_4(\widehat{f}_i(\Sigma)) \setminus \widehat{f}_i(\Sigma)$ contained in $\widehat{X}$ and let $\widehat{Y}$ be the closure of $\widehat{X} \setminus (\mathcal{L}_{4,0} \sqcup \mathcal{L}_{4,1})$. The intersection $F_i = \mathcal{L}_{4,i} \cap \widehat{Y}$ is the union of components of $\partial\mathcal{L}_{4,i}$ adjacent to $\widehat{Y}$.

Let w_k ($k = 2, 3$) be any k-chain with $\mathrm{supp}(w_3) \subset \mathrm{supp}(z_M(\sigma))$ and $\mathrm{supp}(w_2) \subset \mathrm{supp}^{(2)}(z_M(\sigma))$. See (3.5) for the definition of $\mathrm{supp}^{(2)}(\cdot)$. We denote by $w_3(\xi_0)$ the 3-chain obtained by replacing each 3-simplex σ' of $\mathrm{supp}(w_3)$ with the sum $\sum_{D \in \tau(\sigma,\xi_0)^{(3)}} \sigma'|_D$. The subdivision $w_2(\xi_0)$ of w_2 is defined similarly. By (4.1), $(\partial_3 w_3)(\xi_0) = \partial_3(w_3(\xi_0))$. So one can denote it as $\partial_3 w_3(\xi_0)$. For any closed subset A of M, we denote by $w_k|_A$ the sub-chain of w_k consisting of $\sigma' \in \mathrm{supp}(w_k)$ whose inner center $o(\sigma')$ is contained in A. In particular, $z_M(\sigma)|_A = z_A(\sigma)$. We denote $(\partial_3 w_3)|_A$ by $\partial_3 w_3|_A$ shortly.

Proof of Lemma 5.1. The proof is done in five steps. Figure 5.2 illustrates our process schematically, where $w_1 \xrightarrow{z} w_2$ means that $\partial_3 z = w_2 - w_1$.

Step 1. For $i = 0, 1$, we set $u_i(\xi_0) = z_{\mathcal{N}_2(\widehat{f}_i(\Sigma)) \cup \mathcal{L}_{4,i}}(\sigma)(\xi_0)$. Let σ'' be any element of $\mathrm{supp}(z_M(\sigma)(\xi_0))$ with $o(\sigma'') \in \mathcal{N}_2(\widehat{f}_i(\Sigma)) \cap \mathcal{L}_{4,i}$. See Figure 5.3, where the center region represents $\mathcal{N}_2(\widehat{f}_1(\Sigma)) \cap \mathcal{L}_{4,1}$. Note that σ'' is represented as $\sigma'|_D$ for some $\sigma' \in \mathrm{supp}(z_M(\sigma))$ and $D \in \tau(\sigma, \xi_0)^{(3)}$. From the definition of $\tau(\Delta, \xi_0)$ together with elementary hyperbolic geometry, one can prove that $\mathrm{dist}_M(o(\sigma''), o(\sigma')) < 2$. Here '2' is not essential. We just need a positive uniform constant. Then $o(\sigma')$ is contained in $\mathcal{N}_2(\widehat{f}_i(\Sigma)) \cup \mathcal{L}_{4,i}$ and hence $\sigma'' \in \mathrm{supp}(u_i(\xi_0)|_{\widehat{X}})$. Note that the point $o(\sigma')$ is not necessarily an element of $\mathcal{L}_{4,i}$, which is the reason why we use

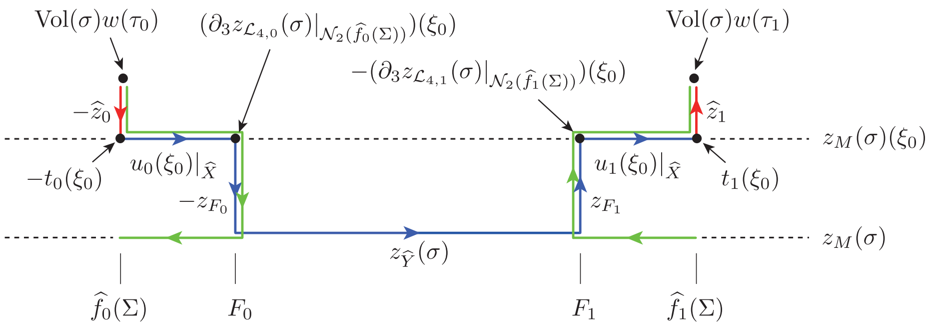

Figure 5.2. The union of blue segments represents z_{ξ_0}, that of blues and reds does $\widehat{z}_{\xi_0}$, and that of greens does $\widehat{a}$.

$z_{\mathcal{N}_2(\widehat{f}_i(\Sigma))\cup\mathcal{L}_{4,i}}(\sigma)(\xi_0)$ but not $z_{\mathcal{L}_{4,i}}(\sigma)(\xi_0)$ to define $u_i(\xi_0)$. See Figure 5.3. By Lemma 3.1,

$$\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)(\xi_0) = \big(\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)}\big)(\xi_0) + \big(\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(\widehat{f}_i(\Sigma))}\big)(\xi_0).$$

Strictly, for any 3-chain w on M and any subset A of M, $\partial_3 w|_A$ means $(\partial_3 w)|_A$.

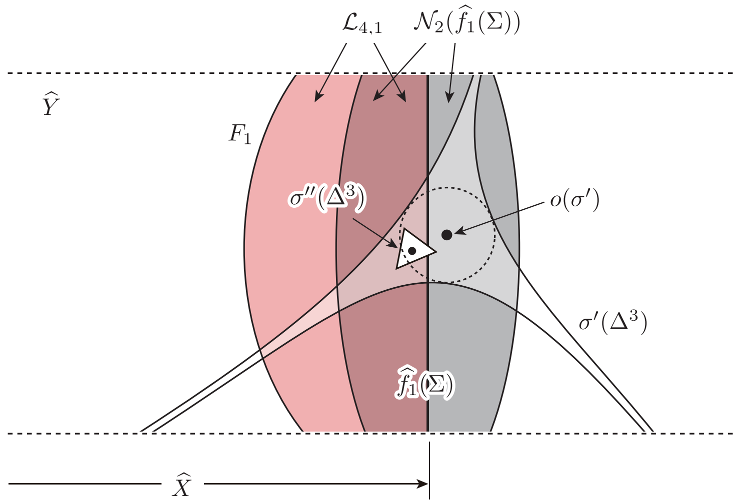

Figure 5.3. The case of $o(\sigma') \in \mathcal{N}_2(\widehat{f}_1(\Sigma)) \setminus \mathcal{L}_{4,1}$.

In general, $(\partial_3 w)|_A$ is not equal to $\partial_3(w|_A)$. Take a small $\varepsilon > 0$ arbitrarily. Since $\mathrm{Int}\mathcal{N}_2(\widehat{f}_i(\Sigma)) \cap \mathcal{N}_2(F_i) = \emptyset$, $\partial_3(u_i(\xi_0)|_{\widehat{X}})$ is represented as the sum

$$\partial_3(u_i(\xi_0)|_{\widehat{X}}) = (\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)})(\xi_0) + t_i(\xi_0)$$

such that $t_i(\xi_0)$ is the sub-chain of $\partial_3(u_i(\xi_0)|_{\widehat{X}})$ consisting of 2-simplices $\sigma'' \in \mathrm{supp}(\partial_3(u_i(\xi_0)|_{\widehat{X}}))$ with $o(\sigma'') \in \mathrm{Int}\mathcal{N}_2(\widehat{f}_i(\Sigma))$ or equivalently $o(\sigma'') \in \mathcal{N}_\varepsilon(\widehat{f}_i(\Sigma))$ if $\xi_0 = \xi_0(\varepsilon) > 0$ is taken sufficiently small. See Figure 5.2.

Step 2. Since $\mathcal{N}_2(F_i) \cap \mathrm{Int}\mathcal{N}_2(\partial \widehat{X})$ is empty,

$$\mathrm{supp}(\partial_3 z_{\mathcal{L}_{4,i} \cup \widehat{Y}}(\sigma)|_{\mathcal{N}_2(F_i)}) = \mathrm{supp}(\partial_3 z_{\widehat{X}}(\sigma)|_{\mathcal{N}_2(F_i)}) = \emptyset.$$

So, we have $\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)} = -\partial_3 z_{\widehat{Y}}(\sigma)|_{\mathcal{N}_2(F_i)}$. Here we consider a chain homotopy z_{F_i} between $\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)}$ and its subdivision $(\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)})(\xi_0)$. Since $\mathcal{N}_2(F_i) \subset \mathcal{N}_6(\widehat{f}_i(\Sigma))$, there exists a 3-chain z_{F_i} consisting of 3-simplices whose inner centers are contained in $\mathcal{N}_2(F_i)$ and satisfying

$$\begin{aligned}\partial_3 z_{F_i} &= -(\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)})(\xi_0) + \partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)} \\ &= -(\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)})(\xi_0) - \partial_3 z_{\widehat{Y}}(\sigma)|_{\mathcal{N}_2(F_i)}\end{aligned}$$

and

$$\begin{aligned}\|z_{F_i}\| &\le 3\|\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)}(\xi_0)\| \le 3n_0(\xi_0)\|\partial_3 z_{\mathcal{L}_{4,i}}(\sigma)|_{\mathcal{N}_2(F_i)}\| \\ &= 3n_0(\xi_0)\mathrm{Vol}(\mathcal{N}_2(F_i)) \le 3n_0(\xi_0)v_0(6),\end{aligned} \tag{5.1}$$

where $n_0(\xi_0)$ is the cardinality of $\tau(\Delta, \xi_0)^{(3)}$, $v_0(6)$ is the constant given in Lemma 2.1 (3) and '3' means that the triangular prism $\Delta^2 \times [0,1]$ is divided into three 3-simplices. Consider the 3-chain

$$z_{\xi_0} = u_0(\xi_0)|_{\widehat{X}} - z_{F_0} + z_{\widehat{Y}}(\sigma) + z_{F_1} + u_1(\xi_0)|_{\widehat{X}}. \tag{5.2}$$

See Figure 5.2 again. Since z_{ξ_0} consists of 3-simplices whose inner centers are contained $\mathcal{N}_\varepsilon(\widehat{X})$,

$$\mathrm{supp}(\partial_3 z_{\xi_0}) \subset \mathrm{supp}^{(2)}\big(z_{\xi_0}|_{\mathcal{N}_\varepsilon(\widehat{f}_0(\Sigma)) \sqcup \mathcal{N}_\varepsilon(\widehat{f}_1(\Sigma))}\big).$$

By (5.1),

$$\begin{aligned}\|z_{\xi_0} - z_{\widehat{Y}}(\sigma)\| &= \|u_0(\xi_0)|_{\widehat{X}}\| + \|z_{F_0}\| + \|z_{F_1}\| + \|u_1(\xi_0)|_{\widehat{X}}\| \\ &\le 2n_0(\xi_0)v_0(2) + 6n_0(\xi_0)v_0(6) \\ &= 2n_0(\xi_0)(v_0(2) + 3v_0(6)).\end{aligned} \tag{5.3}$$

In the following two steps, we will define 3-chains $\widehat{z}_0$ and $\widehat{z}_1$ with $\partial_3 \widehat{z}_i = vw(\tau_i) - t_i(\xi_0)$ $(i = 0, 1)$ for some constant $v > 0$, which is shown to equal $\mathrm{Vol}(\sigma)$ in Step 5.

Step 3. By (5.2), $\partial_3 z_{\xi_0}|_{\mathcal{N}_\varepsilon(\widehat{f}_i(\Sigma))} = \partial_3(u_i(\xi_0)|_{\widehat{X}})|_{\mathcal{N}_\varepsilon(\widehat{f}_i(\Sigma))}$. Let $w_{i,\mathrm{inn}}$, $w_{i,\mathrm{out}}$ be the 2-sub-chains of $\partial_3 z_{\xi_0}|_{\mathcal{N}_\varepsilon(\widehat{f}_i(\Sigma))}$ corresponding to elements of $\tau(\Delta, \xi_0)^{(2)}_{\mathrm{inn}}$ and $\tau(\Delta, \xi_0)^{(2)}_{\mathrm{out}}$ respectively. Since $\partial_2(\partial_3 z\xi_0|_{\mathcal{N}_\varepsilon(\widehat{f}_i(\Sigma))}) = 0$, $\partial_2 w_{i,\mathrm{inn}} = -\partial_2 w_{i,\mathrm{out}}$. By the geometrical boundedness of normalized maps in thick parts, we may assume that there exists a projection $\mathrm{pr}_i : \mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma)) \longrightarrow \widehat{f}_i(\Sigma)$ which is 2-Lipschitz on $\mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma)) \cap E_{\mathrm{thick}}$ and with $\mathrm{pr}_i(\mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma)) \cap E_{\mathrm{thin}}) \subset \widehat{f}_i(\Sigma) \cap E_{\mathrm{thin}}$ if necessary replacing ε by a smaller positive number, where '2' is taken as a constant greater than 1. Here one can retake $\xi_0 > 0$ if necessary so that $\mathrm{diam}(\sigma'(\Delta^2)) < \varepsilon$ for any 2-simplex σ' in $w_{i,\mathrm{inn}}$. Since $o(\sigma') \in \mathcal{N}_\varepsilon(\widehat{f}_i(\Sigma))$, it follows that $\sigma'(\Delta^2)$ is contained in $\mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma))$. Then there exists a 3-chain $z_{i,\mathrm{inn}}$ on $\mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma))$ with

$$\partial_3 z_{i,\mathrm{inn}} = \mathrm{pr}_{i*}(w_{i,\mathrm{inn}}) + p_i - w_{i,\mathrm{inn}}, \tag{5.4}$$

where p_i is the product simplicial complex isomorphic to $\partial_2 w_{i,\mathrm{inn}} \times [0,1]$ and with $\partial_2 p_i = \partial_2 w_{i,\mathrm{inn}} - \partial_2(\mathrm{pr}_{i*}(w_{i,\mathrm{inn}}))$. By Lemma 2.1 (3),

$$\begin{aligned}\|p_i\| &\le 2\|\partial_2 w_{i,\mathrm{inn}}\| \le 2 \cdot 3\|w_{i,\mathrm{inn}}\| \le 2 \cdot 3 \cdot 4\|z_{\xi_0}|_{\mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma))}\| \\ &\le 24n_0(\xi_0)\mathrm{Vol}(\mathcal{N}_2(\widehat{f}_i(\Sigma))) \le 24n_0(\xi_0)v_0(2),\end{aligned} \tag{5.5}$$

where '2' means that any rectangle is divided into two 2-simplices, '3' any 2-simplex has three edges, and '4' any 3-simplex has four 2-faces. We retake $\xi_0 > 0$ sufficiently small if necessary so that the diameter of $\mathrm{pr}_i(\sigma'(\Delta^2)) \cap E_{\mathrm{thick}}$ for any 2-simplex σ' in $w_{i,\mathrm{inn}}$ is less than the constant d_1 given in (T5). By (5.4) together with a standard argument of simplicial approximation of homology theory (for example see Spanier [Sp, Section 3.4]), we know that there exists a simplicial 2-chain $\widehat{w}_{i,\mathrm{inn}}$ on $\widehat{f}_i(\Sigma)$ and a simplicial 3-chain $\widehat{z}_{i,\mathrm{inn}}$ on $\mathcal{N}_{2\varepsilon}(\widehat{f}_i(\Sigma))$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ such that $\partial_3 \widehat{z}_{i,\mathrm{inn}} = \widehat{w}_{i,\mathrm{inn}} + \widehat{p}_i - w_{i,\mathrm{inn}}$, where $\widehat{p}_i$ is the product simplicial complex isomorphic to p_i with $\partial_2 \widehat{p}_i = \partial_2 w_{i,\mathrm{inn}} - \partial_2 \widehat{w}_{i,\mathrm{inn}}$. See Figure 5.4 (a). Since any triangular prism

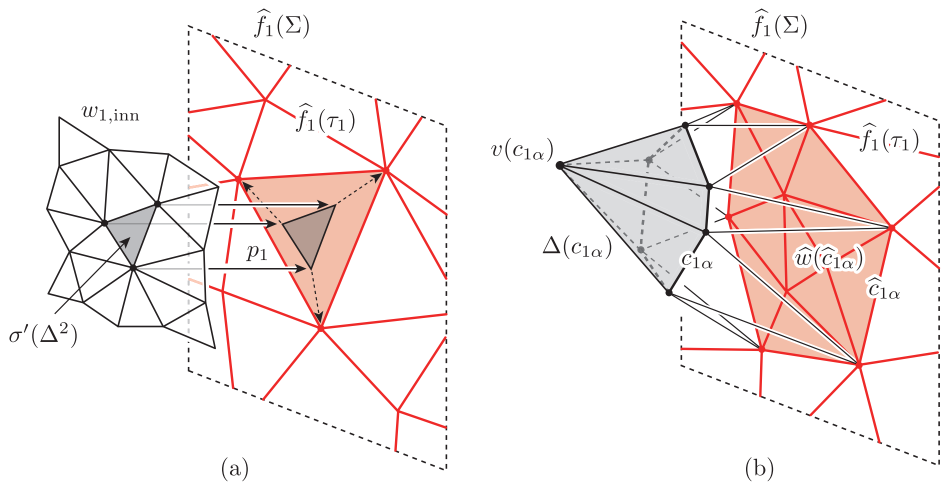

Figure 5.4

is divided into three 3-simplices, one can choose the 3-chain $\widehat{z}_{i,\mathrm{inn}}$ so that

$$\|\widehat{z}_{i,\mathrm{inn}}\| \le 3\|w_{i,\mathrm{inn}}\| \le 12 n_0(\xi_0) v_0(2) \tag{5.6}$$

holds. See Figure 5.5.

Step 4. If necessary replacing $w_{i,\mathrm{out}}$ by a usual locally finite singular 2-chain as in Remark 3.2, we may assume that $\partial_2 w_{i,\mathrm{out}}$ is a locally finite sum of 1-cycles $c_{i\alpha}$. Suppose that $c_{i\alpha}$ is represented as $e_1 + \cdots + e_n$ for $e_1, \ldots, e_n \in W^{(1)}_{i,\mathrm{out}}$, where $W_{i,\mathrm{out}}$ is the minimum simplicial complex containing the terms of $w_{i,\mathrm{out}}$. From the construction of $w_{i,\mathrm{out}}$, there exist isosceles hyperbolic 2-simplices $\Delta_1, \ldots, \Delta_n \in W^{(2)}_{i,\mathrm{out}}$ which have a common vertex v_0 and such that the sum $\Delta(c_{i\alpha}) = \Delta_1 + \cdots + \Delta_n$ is a 2-chain satisfying $\partial_2 \Delta(c_{i\alpha}) = c_{i\alpha}$. See Figure 5.4 (b). Thus $c_{i\alpha}$ is a 1-cycle in $\partial_2 w_{i,\mathrm{inn}} = -\partial_2 w_{i,\mathrm{out}}$ contractible in E. Since $\widehat{f}_i(\Sigma)$ is incompressible in E, $\partial_2 \widehat{w}_{i,\mathrm{inn}}$ is also a locally finite sum of 1-cycles $\widehat{c}_{i\alpha}$ contractible in $\widehat{f}_i(\Sigma)$. By Lemma 4.1, $\widehat{c}_{i\alpha}$ bounds a simplicial 2-chain $\widehat{w}(c_{i\alpha})$ of disk type in $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ such that $\|\widehat{w}(c_{i\alpha})\| \le L_0 \|\widehat{c}_{i\alpha}\| \le L_0 \|c_{i\alpha}\|$. Let $\widehat{w}_{i,\mathrm{out}}$ be the sum of all

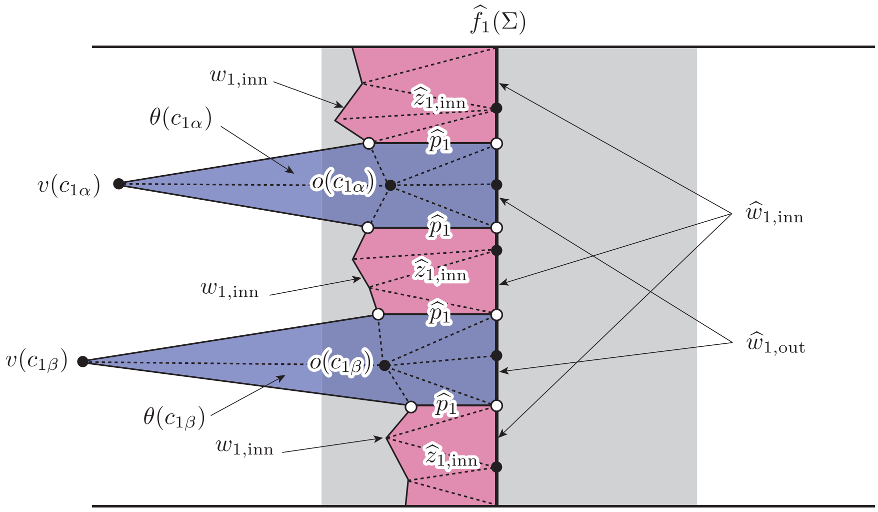

Figure 5.5. The gray region represents $\mathcal{N}_{2\varepsilon}(\widehat{f}_1(\Sigma))$. Each white dot represents a 1-cycle either in $\partial_2 w_{1,\mathrm{inn}} = -\partial_2 w_{1,\mathrm{out}}$ or in $\partial_2 \widehat{w}_{1,\mathrm{inn}} = -\partial_2 \widehat{w}_{1,\mathrm{out}}$.

$\widehat{w}(c_{i,\alpha})$'s. Then, as (5.5),

$$\begin{aligned}\|\widehat{w}_{i,\mathrm{out}}\| &\le L_0 \sum_\alpha \|c_{i\alpha}\| \le L_0 \|c_{i\alpha}\| = L_0 \|\partial_2 w_{i,\mathrm{out}}\| = L_0 \|\partial_2 w_{i,\mathrm{inn}}\| \\ &\le 4L_0 n_0(\xi_0) v_0(2).\end{aligned}$$

Thus $\widehat{w}_i = \widehat{w}_{i,\mathrm{inn}} + \widehat{w}_{i,\mathrm{out}}$ is a locally finite simplicial 2-cycle on $\widehat{f}_i(\Sigma)$ with respect to $\widehat{f}_i(\tau_i)$ mod $\widehat{\mathcal{H}}_{i,\mathrm{tube}}$ with $\|\widehat{w}_i\| \le 4(L_0+1) n_0(\xi_0) v_0(2)$.

Consider the 3-chain $\theta(c_{i\alpha})$ obtained by suspending the 2-sphere cycle $\Delta(c_{i\alpha}) - \widehat{p}(c_{i\alpha}) + \widehat{w}(c_{i\alpha})$ with a vertex $o(c_{i\alpha})$. See Figure 5.5 again. Then we have $\partial_3 \theta(c_{i\alpha}) = \Delta(c_{i\alpha}) + \widehat{p}(c_{i\alpha}) - \widehat{w}(c_{i\alpha})$ and $\|\theta(c_{i\alpha})\| \le (L_0+3)\|c_{i\alpha}\|$. Here '$3\,(=2+1)$' means that $\|\widehat{p}(c_{i\alpha})\| \le 2\|c_{i\alpha}\|$ and $\|\delta(c_{i\alpha})\| \le \|c_{i\alpha}\|$. Let $\widehat{z}_{i,\mathrm{out}}$ be the sum of all $\theta(c_{i\alpha})$'s. Then

$$\|\widehat{z}_{i,\mathrm{out}}\| \le \sum_\alpha (L_0+3)\|c_{i\alpha}\| \le 4(L_0+3) n_0(\xi_0) v_0(2). \tag{5.7}$$

We set $\widehat{z}_i = \widehat{z}_{i,\mathrm{inn}} + \widehat{z}_{i,\mathrm{out}}$. By (5.6) and (5.7),

$$\|\widehat{z}_i\| \le 12 n_0(\xi_0) v_0(2) + 4(L_0+3) n_0(\xi_0) v_0(2) \le 4(L_0+6) n_0(\xi_0) v_0(2). \tag{5.8}$$

In the last step, we will construct a 3-chain z satisfying the conditions (1) and (2) of this lemma.

Step 5. Let $\widehat{z}_{xi_0}$ be the 3-chain defined by $\widehat{z}_{\xi_0} = -\widehat{z}_0 + z_{\xi_0} + \widehat{z}_1$. See Figure 5.2. It follows from the definition that

$$\partial_3 \widehat{z}_{\xi_0} = \widehat{w}_1 - \widehat{w}_0.$$

Consider the 3-chain $\widehat{a}$ defined by $\widehat{a} = \widehat{z}_{\xi_0} - z_{\widehat{X}}(\sigma) = \widehat{z}_0 + z_{\xi_0} + \widehat{z}_1 - z_{\widehat{X}}(\sigma)$. See Figure 5.2. Since $\widehat{X} \setminus \widehat{Y} \subset \mathcal{N}_2(\widehat{f}_0(\Sigma)) \cup \mathcal{N}_2(\widehat{f}_1(\Sigma))$, by (5.3) and (5.8)

$$\begin{aligned}\|\widehat{a}\| &\leq \|z_{\widehat{X}\setminus\widehat{Y}}(\sigma)\| + \|\widehat{z}_0\| + \|z_{\xi_0} - z_{\widehat{Y}}(\sigma)\| + \|\widehat{z}_1\| \\ &\leq \mathrm{Vol}(\widehat{X} \setminus \widehat{Y}) + 8(L_0+6)n_0(\xi_0)v_0(2) + 2n_0(\xi_0)(v_0(2) + 3v_0(6)) \\ &< 2v_0(2) + (8(L_0+6)+2)n_0(\xi_0)v_0(2) + 6n_0(\xi_0)v_0(6) =: b_0.\end{aligned}$$

In the case of $\widehat{f}_i(\Sigma) \cap E_{\mathrm{cusp}} \neq \emptyset$, we deform z by a projection in E_{cusp} sending $\widehat{f}_i(\Sigma) \cap E_{\mathrm{cusp}}$ to the totally geodesic cusps in E_{cusp} without moving $\widehat{f}_i(\Sigma) \cap \partial E_{\mathrm{cusp}}$. See the E_{cusp}-part in Figure 5.1. The deformation is accomplished by a chain homotopy consisting of 3-simplices each of which has a 2-face belonging to $\widehat{f}_0(\tau_0^{(2)}) \cup \widehat{f}_1(\tau_1^{(2)})$. This shows (1).

Note that

$$z_M(\sigma) = (z_{\widehat{X}}(\sigma) + \widehat{a}) + (z_{M\setminus \mathrm{Int}\widehat{X}}(\sigma) - \widehat{a}) = \widehat{z}_{\xi_0} + (z_{M\setminus\mathrm{Int}\widehat{X}}(\sigma) - \widehat{a}).$$

See Figure 5.6. By our construction of $\widehat{z}_{\xi_0}$, the boundary $\partial_3 \widehat{z}_{\xi_0}$ has the form

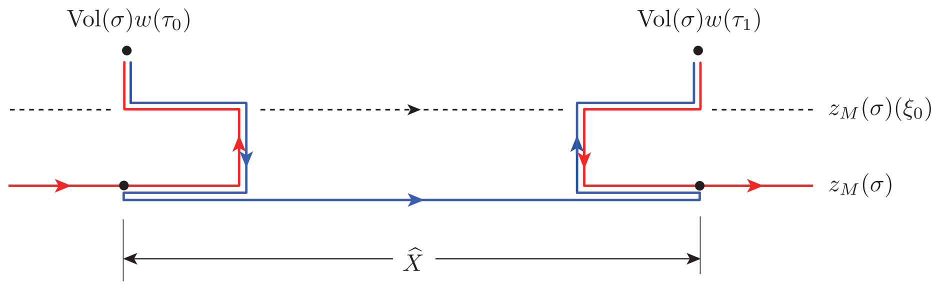

Figure 5.6. The union of blue segments represents $z_{\widehat{X}}(\sigma) + \widehat{a} = \widehat{z}_{\xi_0}$ and that of reds does $z_{M\setminus\mathrm{Int}\widehat{X}}(\sigma) - \widehat{a}$.

$v(w(\tau_1) - w(\tau_0))$ for some $v > 0$. From the definition (3.4) of $z_M(\sigma)$ and (3.3), $z_M(\sigma)$ represents the class $\mathrm{Vol}(\sigma)[M]$ of the locally finite homology group $H_3^{\mathrm{loc.f.}}(M, \mathbb{R})$. It follows that $v = \mathrm{Vol}(\sigma)$. Thus $z = \widehat{z}_{\xi_0}$ satisfies the condition (2) and hence it is our desired 3-chain. This completes the proof of Lemma 5.1. □

6. Proofs of Theorems A and C

First we recall the necessary conditions in Theorem A. Suppose that M, M' are oriented hyperbolic 3-manifolds with markings $\iota : \Sigma \longrightarrow M$ and $\iota' : \Sigma \longrightarrow M'$ which satisfy

(0.1) $$\|\iota^*([\omega_M]) - \iota'^*([\omega_{M'}])\| < \boldsymbol{v}_3$$

in $H_b^3(\Sigma, \mathbb{R})$. Since both M and M' are homeomorphic to $\Sigma \times \mathbb{R}$, there exists an orientation preserving homeomorphism $\varphi : M \longrightarrow M'$ such that $\varphi \circ \iota$ is properly homotopic to ι'. So (0.1) is rewritten as

(0.2) $$\|[\omega_M] - \varphi^*([\omega_{M'}])\| < \boldsymbol{v}_3$$

in $H_b^3(M, \mathbb{R})$. Here we consider the case that $(+)$-end $\mathcal{E}$ of M is totally degenerate. Let E be the neighborhood of $\mathcal{E}$ with respect to $\iota(\Sigma)$ and set $E' = \varphi(E)$.

For any piecewise smooth proper homotopy equivalences $f_i : \Sigma \longrightarrow M$ $(i = 0, 1)$, there exists a piecewise smooth proper continuous map $Z : \Sigma \times [0, 1] \longrightarrow E$ with $f_0 = Z|_{\Sigma\times\{0\}}$ and $f_1 = Z|_{\overline{\Sigma}\times\{1\}}$. Here $\overline{\Sigma}$ is equal to Σ as a surface but it has the orientation opposite to that on Σ. Then the *bounding volume* $\mathrm{Vol}^{\mathrm{bd}}(Z)$ of Z is defined by

$$\mathrm{Vol}^{\mathrm{bd}}(Z) = \int_{\Sigma\times[0,1]} Z^*(\Omega_E),$$

where Ω_E is the volume form on E. It is a standard fact in homology theory that $\mathrm{Vol}^{\mathrm{bd}}(Z)$ is independent of the choice of the extension Z of f_1 and f_2. Thus one can set $\mathrm{Vol}^{\mathrm{bd}}(Z) = \mathrm{Vol}^{\mathrm{bd}}(f_0, f_1)$. From the definition, $\mathrm{Vol}^{\mathrm{bd}}(f_1, f_0) = -\mathrm{Vol}^{\mathrm{bd}}(f_0, f_1)$ holds.

Now we are ready to prove Theorem A.

Proof of Theorem A. Recall that $M_{(\mathrm{cusp})}$ is the union of components of M_{cusp} meeting $\iota(\Sigma_{\mathrm{cusp}})$ non-trivially. We denote by $E_{\mathrm{cusp}*}$ the union of components V of E_{cusp} such that $\varphi(V)$ is freely homotopic into E'_{cusp} in E'. In particular, $E_{(\mathrm{cusp})}$ is a sub-union of $E_{\mathrm{cusp}*}$. One can retake the homeomorphism φ so that $E'_{\mathrm{cusp}*} = \varphi(E_{\mathrm{cusp}*})$ is a union of components of E'_{cusp}. Let C_* be a finite core of M which meets each component of $E_{\mathrm{cusp}*}$ non-trivially and is disjoint from $E_{\mathrm{cusp}} \setminus E_{\mathrm{cusp}*}$.

For any sub-end $\mathcal{E}^\flat$ of $\mathcal{E}$ with respect to C_*, let $E^\flat$ be the closure of the component of $E \setminus C_*$ adjacent to $\mathcal{E}^\flat$. Since $\mathcal{E}$ is totally degenerate, so is $\mathcal{E}^\flat$. Consider a sequence $\{\widehat{f}_n\}_{n=0}^\infty$ of normalized maps $\widehat{f}_n : \widehat{\Sigma}_n^\flat \longrightarrow E^\flat$ satisfying the conditions given in Section 2, where any $\widehat{\Sigma}_n^\flat$ admits a marking $\iota_n : \Sigma^\flat \longrightarrow \widehat{\Sigma}_n^\flat$ for a fixed complete hyperbolic surface $\Sigma^\flat$ of finite area. Note that $\Sigma^\flat$ is not homeomorphic to Σ when $E_{\mathrm{cusp}*} \neq E_{(\mathrm{cusp})}$ or equivalently the (+)-end $\mathcal{E}$ is not genuine. Let $\mathcal{H}(\widehat{f}_n)$ be a hoop family of $\widehat{\Sigma}_n^\flat$, see (4.2). Suppose that τ_n is a triangulation on $\Sigma^\flat$ such that $\widehat{\tau}_n = \iota_n(\tau_n)$ is a normalized triangulation with respect to $\mathcal{H}(\widehat{f}_n)$, which satisfies the conditions (T1)–(T5) given in Section 4. We consider the union $\widehat{\mathcal{H}}_{E^\flat} = \bigcup_{n=0}^\infty \widehat{\mathcal{H}}_n$ of $\widehat{\mathcal{H}}_n = \widehat{f}_n(\mathcal{H}(\widehat{f}_n))$. Let $\mathcal{N}(\widehat{\mathcal{H}}_{E^\flat})$ be a tubular neighborhood of $\widehat{\mathcal{H}}_{E^\flat}$ in M consisting of mutually disjoint tubular neighborhoods with $\mathrm{Vol}(\mathcal{N}(\widehat{\mathcal{H}}_{E^\flat})) = \sum_{n=0}^\infty \mathcal{N}(\widehat{\mathcal{H}}_n) < \infty$. Note that the normal radius of any components of $\mathcal{N}(\widehat{\mathcal{H}}_n)$ converges to zero as $n \to \infty$. From the definition of C_*, we know that any accidental cusp of $\mathcal{E}^\flat$ does not correspond to any cusp of M' via φ. So, if necessary removing finitely many entries from $\{\widehat{f}_n\}$, one can suppose that, for each component l of $\widehat{\mathcal{H}}_{E^\flat}$, $\varphi(l)$ is not freely homotopic into M'_{cusp}. Thus we have a continuous map $\psi : M \longrightarrow M'$ satisfying the following conditions.

(P1) $\psi|_{M\setminus\mathcal{N}(\widehat{\mathcal{H}}_{E^\flat})} = \varphi|_{M\setminus\mathcal{N}(\widehat{\mathcal{H}}_{E^\flat})}$.

(P2) For each component l of $\widehat{\mathcal{H}}_{E^\flat}$, $\psi(l)$ is a closed geodesic in M'.

Consider a piecewise totally geodesic map $f'^*_n : \Sigma'^*_n \longrightarrow M'$ properly homotopic to $\psi \circ \widehat{f}_n : \widehat{\Sigma}_n^\flat \longrightarrow M'$ and satisfying the following conditions.

- For any $v \in \tau_n^{(0)}$, $f'^*_n(v) = \psi \circ \widehat{f}_n(v)$.
- For any $e \in \tau_n^{(1)}$, $f'^*_n(e)$ is a geodesic segment in E' homotopic to $\psi \circ \widehat{f}_n(e)$ rel. ∂e.
- For any $\Delta \in \tau_n^{(2)}$, $f'^*_n(\Delta)$ is a totally geodesic triangle in E' bounded by $f'^*_n(\partial\Delta)$.

Now we need to consider the following two cases.

Case 1. $\mathcal{E}^\flat$ has no accidental cusps.

Let $\sigma : \Delta^3 \longrightarrow \mathbb{H}^3$ be any straight simplex in $\mathbb{H}^3$ with $\mathrm{Vol}(\sigma) > 1$. For any $n \in \mathbb{N}$, suppose that $\widehat{a}_{0,n}$ is the connecting 3-chain given in Lemma 5.1 (1) associated with $\widehat{X} = E^\flat(\widehat{f}_0, \widehat{f}_n)$ such that $\|\widehat{a}_{0,n}\|$ is less than a constant $b_0 > 0$ independent of n. Moreover, for the 3-chain $z_{0,n} = z(0,n) + \widehat{a}_{0,n}$ on E, $\partial_3 z_{0,n} = \mathrm{Vol}(\sigma)(w(\tau_n) - w(\tau_0))$ holds, where $w(\tau_{n_j})$ $(j = 0, 1)$ is the 2-cycle on $\widehat{f}_{n_j}(\widehat{\Sigma}^\flat_{n_j})$ as in Lemma 5.1 (2). There exists the 2-cycle $S(\tau_n)$ on $\Sigma^\flat$ with respect to τ_n mod $\mathcal{H}(\widehat{f}_n)_{\mathrm{tube}}$ satisfying $\widehat{f}_{n*}(S(\tau_n)) = w(\tau_n)$. Then $\mathrm{straight}(\psi_*(z_{0,n}))$ is a locally finite 3-chain on M' with

$$\begin{aligned}
&\partial_3\, \mathrm{straight}(\psi_*(z_{0,n})) \\
&\qquad = \mathrm{Vol}(\sigma)(\mathrm{straight}(\psi \circ \widehat{f}_n)_*(S(\tau_n)) - \mathrm{straight}(\psi \circ \widehat{f}_0)_*(S(\tau_0))) \\
&\qquad = \mathrm{Vol}(\sigma)((f_n'^*)_*(S(\tau_n)) - (f_0'^*)_*(S(\tau_0))).
\end{aligned}$$

Here the equality $\mathrm{straight}(\psi \circ \widehat{f}_n)_*(S(\tau_n)) = (f_n'^*)_*(S(\tau_n))$ is proved by the fact that $f_n'^*$ is a piecewise totally geodesic map defined as above. It follows that

$$\omega_{M'}(\psi_*(z_{0,n})) = \mathrm{Vol}(\mathrm{straight}(\psi_*(z_{0,n}))) = \mathrm{Vol}(\sigma)\mathrm{Vol}^{\mathrm{bd}}(f_0'^*, f_n'^*).$$

Hence we have

$$\begin{aligned}
\omega_{M'}(\psi_*(z(0,n))) &= \omega_{M'}(\psi_*(z_{0,n})) - \omega_{M'}(\psi_*(\widehat{a}_{0,n})) \\
&\leq \mathrm{Vol}(\sigma)\mathrm{Vol}^{\mathrm{bd}}(f_0'^*, f_n'^*) + b_0 \boldsymbol{v}_3.
\end{aligned} \tag{6.1}$$

Consider any bounded 3-cocycle η on M satisfying

$$[\omega_M] - \varphi^*([\omega_{M'}]) = [\omega_M] - \psi^*([\omega_{M'}]) = [\eta] \quad \text{in } H_b^3(M, \mathbb{R}). \tag{6.2}$$

Then there exists a bounded 2-cochain $c \in C_b^2(M)$ with $\omega_M - \psi^*\omega_{M'} + \delta^2 c = \eta$.

Let $\mathcal{E}'^\flat$ be the end of E' with respect to $\varphi(C_*)$ which corresponds to $\mathcal{E}^\flat$ via φ. Suppose that $\mathcal{E}'^\flat$ were either geometrically finite or non-genuine or a simply degenerate end with ending lamination different from the ending lamination ν of $\mathcal{E}$. Since ν is a connected full lamination of $\Sigma^\flat$, if necessary passing to a subsequence, we may assume that $\{f_n'^*\}$ either converges uniformly to a pleated map $f_\infty'^* : \Sigma^\flat \longrightarrow M'$ realizing ν or diverges to an end of M' opposite to $\mathcal{E}'^\flat$. In either case, $B = \sup\{\mathrm{Vol}^{\mathrm{bd}}(f_0'^*, f_n'^*)\} < \infty$. By (6.1) together with Lemma 3.1,

$$\begin{aligned}
\|\eta\| &\geq \frac{(\omega_M - \psi^*\omega_{M'} + \delta^2 c)(z(0,n))}{\|z(0,n)\|} \\
&\geq \frac{\mathrm{Vol}(\sigma)(\mathrm{Vol}(E^\flat(\widehat{f}_0, \widehat{f}_n)) - B) - b_0\boldsymbol{v}_3 - 8\|c\|v_0(2)}{\mathrm{Vol}(E^\flat(\widehat{f}_0, \widehat{f}_n))}.
\end{aligned}$$

Since $\lim_{n\to\infty} \mathrm{Vol}(E^\flat(\widehat{f}_0, \widehat{f}_n)) = \infty$, we have $\|\eta\| \geq \mathrm{Vol}(\sigma)$ and hence $\|\eta\| \geq \boldsymbol{v}_3$ by letting $\mathrm{Vol}(\sigma) \to \boldsymbol{v}_3$. Since η is any element of $Z_b^3(M)$ satisfying (6.2), it follows that $\|[\omega_M] - \varphi^*([\omega_{M'}])\| \geq \boldsymbol{v}_3$. This contradicts (0.2). Thus $\mathcal{E}'^\flat$ must be a simply degenerate end of M' with the ending lamination equal to ν via φ.

Case 2. $\mathcal{E}^\flat$ has an accidental cusp.

Let $\mathcal{E}_1^\flat, \ldots, \mathcal{E}_k^\flat$ $(k \geq 2)$ be the genuine sub-ends of $\mathcal{E}^\flat$ and ν_i $(i = 1, \ldots, k)$ the ending lamination of $\mathcal{E}_i^\flat$. Then $\nu_1 \cup \cdots \cup \nu_k$ is realized as a geodesic lamination in $\Sigma^\flat$. Let $\Lambda^\flat$ be a maximal union of simple closed geodesic in $\Sigma^\flat$ disjoint from $\nu_1 \cup \cdots \cup \nu_k$. Then, for any component l of $\Lambda^\flat$, $\iota_n(l)$ is homotopic to a component of $\widehat{\mathcal{H}}_{n,\mathrm{tube}}$ in $\widehat{\Sigma}_n^\flat$ (see Figure 4.3) corresponding to a accidental cusp V of $\mathcal{E}^\flat$ and hence $V \subset E_{\rm cusp} \setminus E_{\rm cusp*}$. From the definition of $E_{\rm cusp*}$, $\varphi(\iota_n(l))$ is realized as a geodesic loop in $E'^{\flat}$. It follows that $\nu_1 \cup \cdots \cup \nu_k \cup \Lambda^{\flat}$ is a maximal lamination in $\Sigma^{\flat}$ realized in $E'^{\flat}$. As in Case 1, this gives a contradiction. Thus Case 2 does not occur.

By the results of Cases 1 and 2, we have known that $E_{\rm cusp*} = E_{\rm cusp}$, $E'_{\rm cusp*} = E'_{\rm cusp}$ and any genuine sub-end of $\mathcal{E}$ and the corresponding genuine sub-end of $\mathcal{E}'$ are simply degenerate ends with the same ending lamination. Then, by Ending Lamination Theorem [Mi, BCM], φ is properly homotopic rel. $M \setminus {\rm Int}E$ to a homeomorphism $\varphi_0 : M \longrightarrow M'$ such that $\varphi_0|_E : E \longrightarrow E'$ is bi-Lipschitz. □

Proof of Theorem C. From our assumption, one can choose the homeomorphism $\varphi : M \longrightarrow M'$ so that $\varphi(M_{\rm cusp}) = M'_{\rm cusp}$. Let E be a neighborhood of any genuine end $\mathcal{E}$ of M. Then $E' = \varphi(E)$ is a neighborhood of the genuine end $\mathcal{E}'$ of M' corresponding to $\mathcal{E}$ via φ. If both $\mathcal{E}$ and $\mathcal{E}'$ are geometrically finite, then it is well known that $\varphi|_E : E \longrightarrow E'$ is properly homotopic rel. ∂E to a bi-Lipschitz map from E to E', see for example [Th1, Subsection 8.3] and Epstein-Marden [EM] for more details. So it suffices to consider the case when at least one of $\mathcal{E}$ and $\mathcal{E}'$, say $\mathcal{E}$, is simply degenerate. Let $p^{(\prime)} : \widetilde{M}^{(\prime)} \longrightarrow M^{(\prime)}$ be the covering associated with $\pi_1(E^{(\prime)}) \subset \pi_1(M^{(\prime)})$ and $\widetilde{\varphi} : \widetilde{M} \longrightarrow \widetilde{M}'$ a lift of φ. Then $\widetilde{M}^{(\prime)}$ has a submanifold $\widetilde{E}^{(\prime)}$ such that the restriction $p^{(\prime)}|_{\widetilde{E}^{(\prime)}}$ is an isometry onto $E^{(\prime)}$. Since $p^*([\omega_M] - \varphi^*([\omega_{M'}])) = [\omega_{\widetilde{M}}] - \widetilde{\varphi}^*([\omega_{\widetilde{M}'}])$, by (0.2)

$$\|[\omega_{\widetilde{M}}] - \widetilde{\varphi}^*([\omega_{\widetilde{M}'}])\| \leq \|[\omega_M] - \varphi^*([\omega_{M'}])\| < \boldsymbol{v}_3.$$

By applying Theorem A to the genuine ends of $\widetilde{M}$ and $\widetilde{M}'$ adjacent to $\widetilde{E}$ and $\widetilde{E}'$ respectively, $\widetilde{\varphi}|_{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{E}'$ is properly homotopic rel. $\partial\widetilde{E}$ to a bi-Lipschitz map and hence $\varphi|_E : E \longrightarrow E'$ is so rel. ∂E. Combining these facts, one can show that φ is properly homotopic to a bi-Lipschitz map. □

References

[BP] R. Benedetti and C. Petronio, Lectures on hyperbolic geometry, Universitext, Springer-Verlag, Berlin, 1992. MR 1219310

[Bo] F. Bonahon, Bouts des variétés hyperboliques de dimension 3, Ann. of Math. (2) **124** (1986) 71–158. MR 0847953

[BCM] J. Brock, R. Canary and Y. Minsky, The classification of Kleinian surface groups, II: The Ending Lamination Conjecture, Ann. of Math. (2) **176** (2012) 1–149. MR 2925381

[EM] D.B.A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, Fundamentals of hyperbolic geometry: selected expositions, 117–266, London Math. Soc. Lecture Note Ser. **328**, Cambridge Univ. Press, Cambridge, 2006. MR 2235711

[Far1] J. Farre, Bounded cohomology of finitely generated Kleinian groups, Geom. Funct. Anal. **28** (2018) 1597–1640. MR 3881830

[Far2] J. Farre, Borel and volume classes for dense representations of discrete groups, Int. Math. Res. Not. IMRN **15** (2022) 11891–11956. MR 4458569

[FHS] M. Freedman, J. Hass and P. Scott, Least area incompressible surfaces in 3-manifolds, Invent. Math. **71** (1983) 609–642. MR 0695910

[Gr] M. Gromov, Volume and bounded cohomology, Publ. Math. Inst. Hautes Étud. Sci. **56** (1982) 5–99. MR 0686042

[He] J. Hempel, 3-manifolds, Ann. of Math. Studies 86, Princeton Univ. Press, Princeton, N.J., Univ. of Tokyo Press, Tokyo, 1976. MR 0415619

[MT] K. Matsuzaki and M. Taniguchi, Hyperbolic manifolds and Kleinian groups, Oxford Univ. Press, New York, 1998. MR 1638795

[MC] D. McCullough, Compact submanifolds of 3-manifolds with boundary, Quart. J. Math. Oxford **37** (1986), 299–307. MR 0854628

[Mi] Y. Minsky, The classification of Kleinian surface groups I: models and bounds, Ann. of Math. **171** (2010) 1-107. MR 1859020

[OM] K. Ohshika and H. Miyachi, On topologically tame Kleinian groups with bounded geometry, Spaces of Kleinian groups, 29–48, London Math. Soc. Lecture Note Ser. **329** Cambridge University Press, Cambridge, 2006. MR 2258743

[Sc] P. Scott, Compact submanifolds of 3-manifolds, J. London Math. Soc. **7** (1973) 246–250. MR 0326737

[So1] T. Soma, A rigidity theorem for Haken manifolds, Math. Proc. Camb. Phil. Soc. **118** (1995) 141–160. MR 1329465

[So2] T. Soma, Bounded cohomology of closed surfaces, Topology **36** (1997) 1221–1246. MR 1452849

[So3] T. Soma, Bounded cohomology and topologically tame Kleinian groups, Duke Math. J. **88** (1997) 357–370. MR 1452849

[So4] T. Soma, Volume and structure of hyperbolic 3-manifolds, preprint, on line at `https://trhksoma.fpark.tmu.ac.jp/preprints.html`.

[Su] D. Sullivan, On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, Riemann surfaces and related topics, Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978) pp. 465–496, Ann. of Math. Studies 97, Princeton Univ. Press, Princeton, N.J., 1981. MR 0624833

[Sp] E. H. Spanier, Algebraic topology, Springer-Verlag, New York-Berlin, 1981. MR 0666554

[Th1] W. Thurston, The geometry and topology of three-manifolds, Lecture Notes, Princeton Univ., Princeton (1978), on line at `http://library.msri.org/books/gt3m/`.

[Th2] W. Thurston, Collected works of William P. Thurston with commentary, Vol. II, 3-manifolds, complexity and geometric group theory, American Mathematical Society, Providence, RI, 2022. MR 4556463

Department of Mathematical Sciences, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

Email address: `tsoma@tmu.ac.jp`