フルスクリーンモード対応

既約3次元多様体の幾何とトポロジー

相馬 輝彦（首都大学東京）

日本数学会秋季総合分科会

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• By Jaco-Shalen-Johannson decomposition + Thurston's Uniformization Theorem, any closed Haken manifolds M admits a *torus decomposition*, which is uniquely determined up to ambient isotopy.

Torus decomposition: $M = \mathcal{H}(M) \cup \mathcal{S}(N)$,

$$\mathcal{H}(M) = H_1 \sqcup \cdots \sqcup H_m, \quad \mathcal{S}(M) = S_1 \cup \cdots \cup S_n$$

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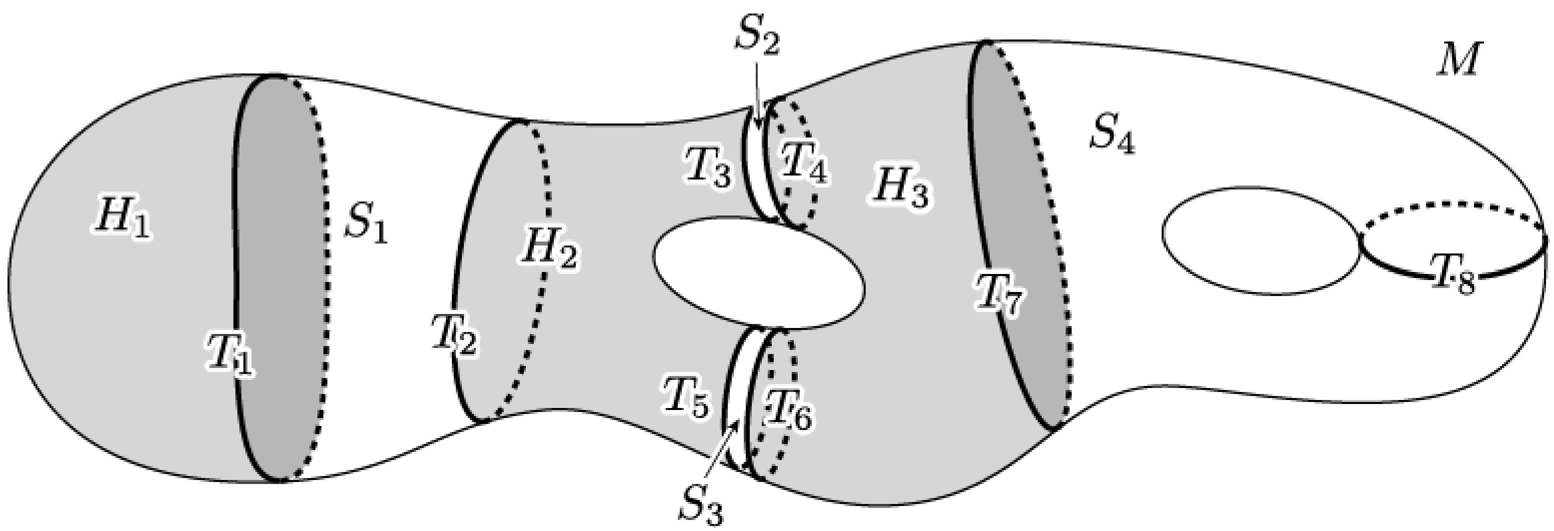

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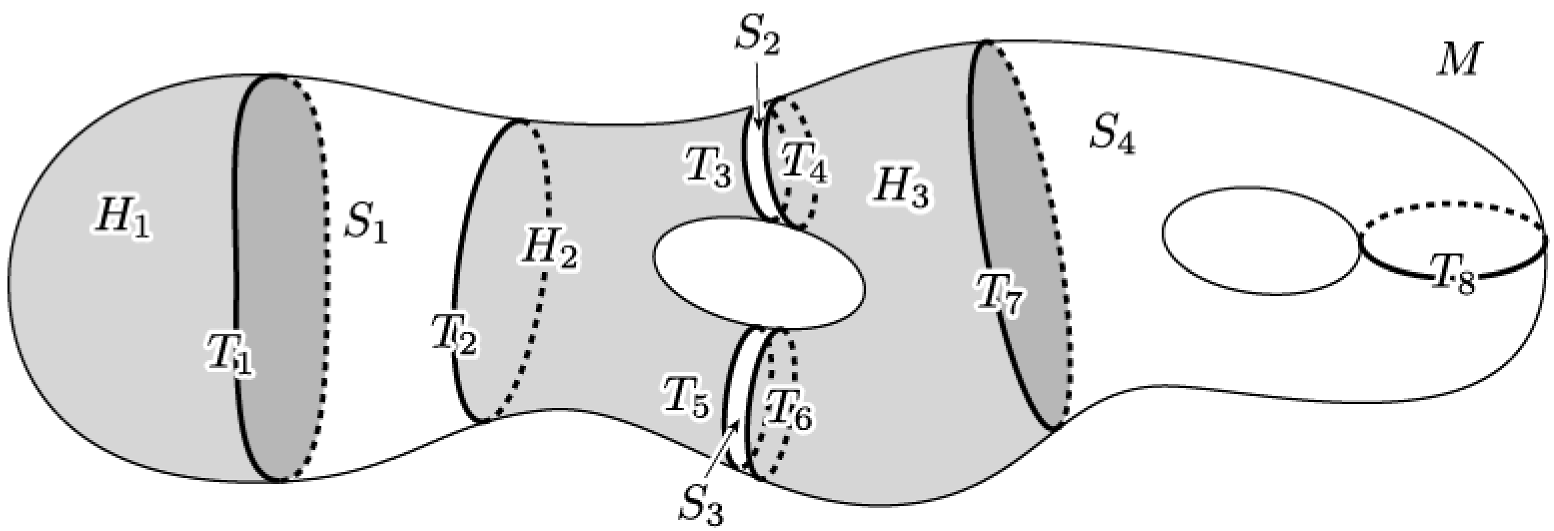

In particular, $M = \mathcal{H}(M)$ if M is hyperbolic, and $M = \mathcal{S}(M)$ if M is Seifert fibered.

3次元既約閉多様体

Closed irreducible 3-manifolds

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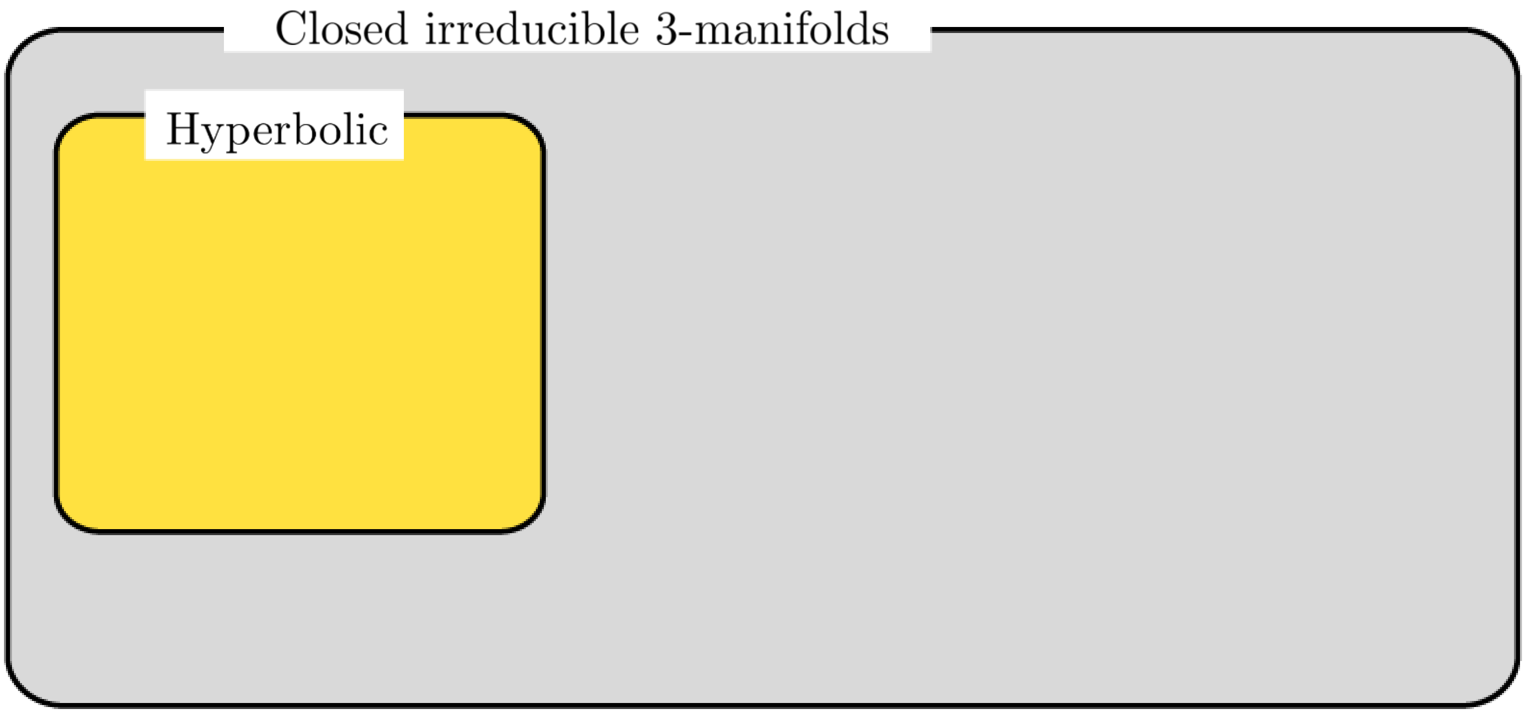

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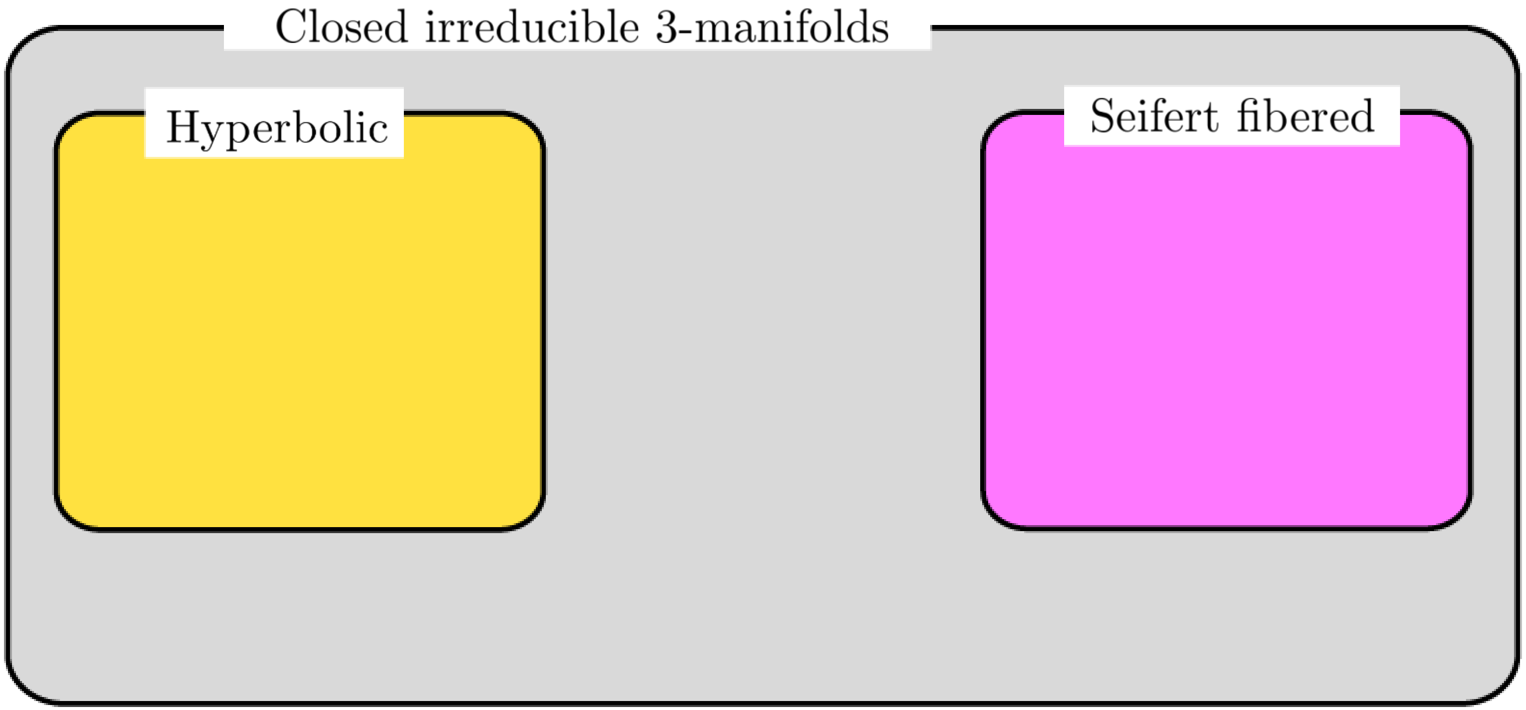
Closed irreducible 3-manifolds
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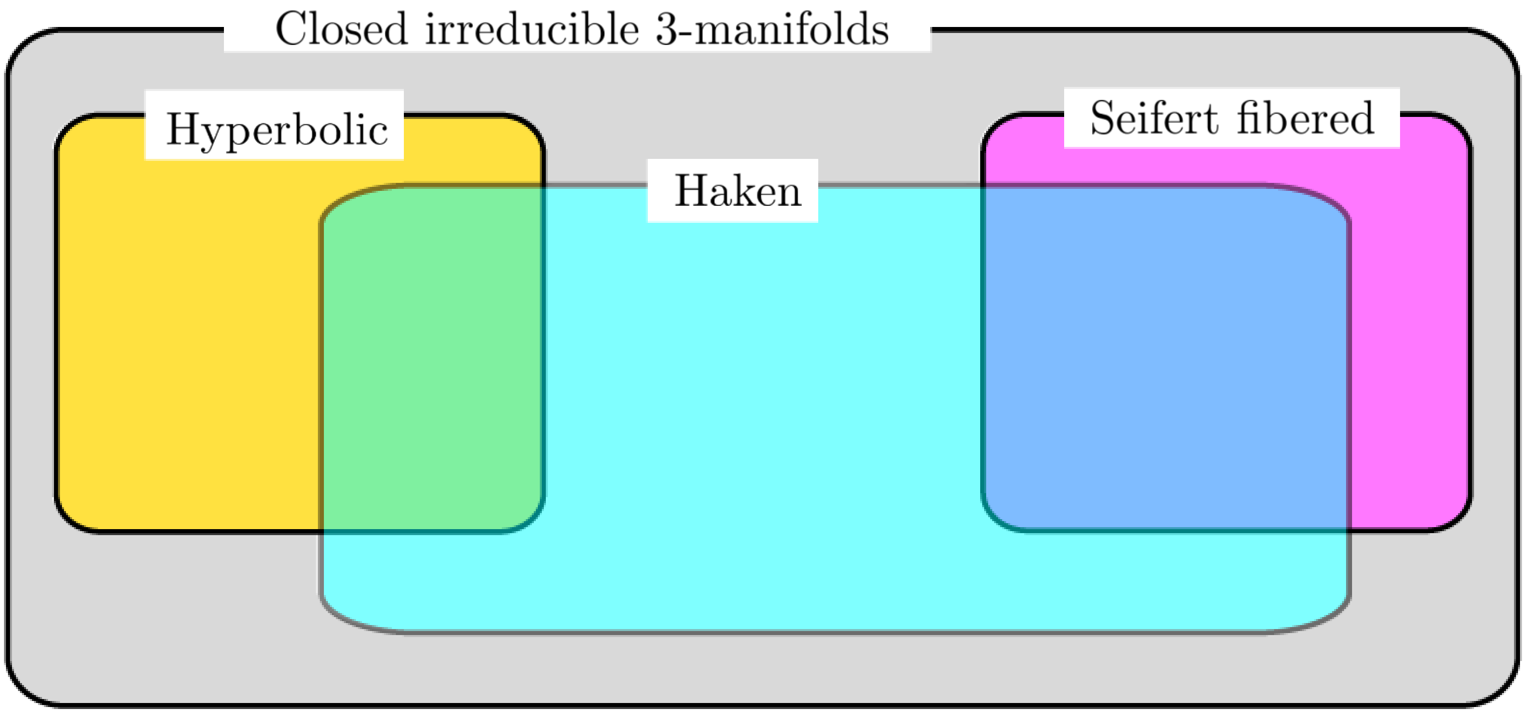
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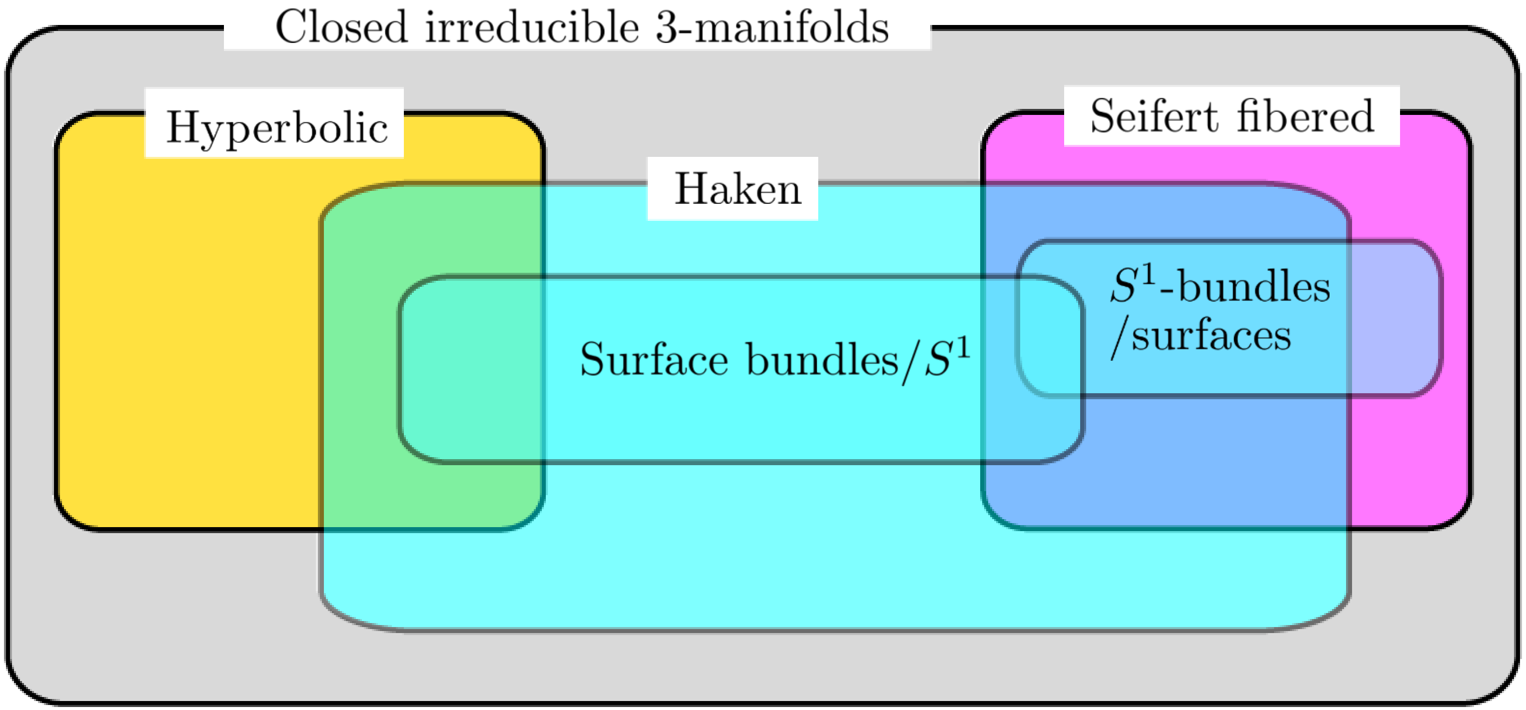
Closed irreducible 3-manifolds
Hyperbolic
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S^1-bundles
/surfaces
Surface bundles/S^1

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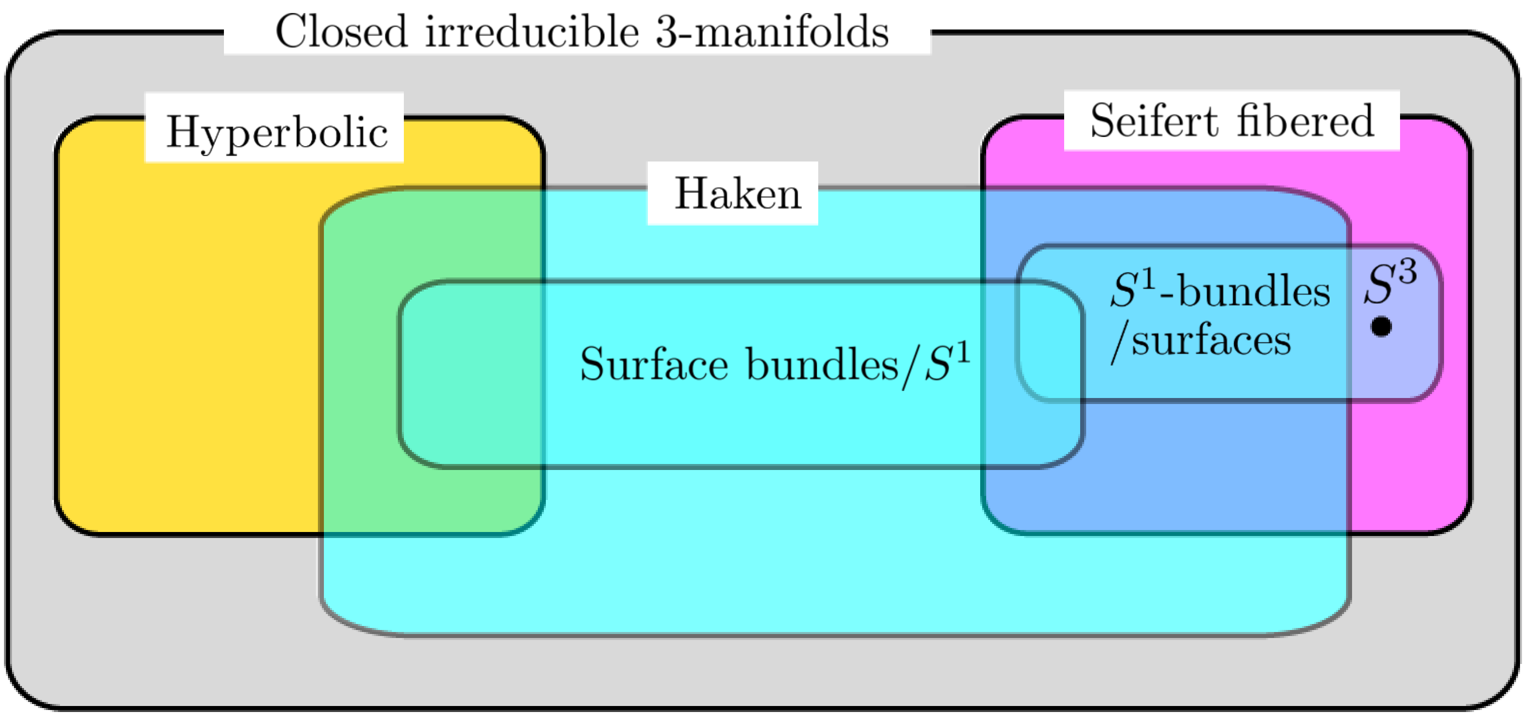

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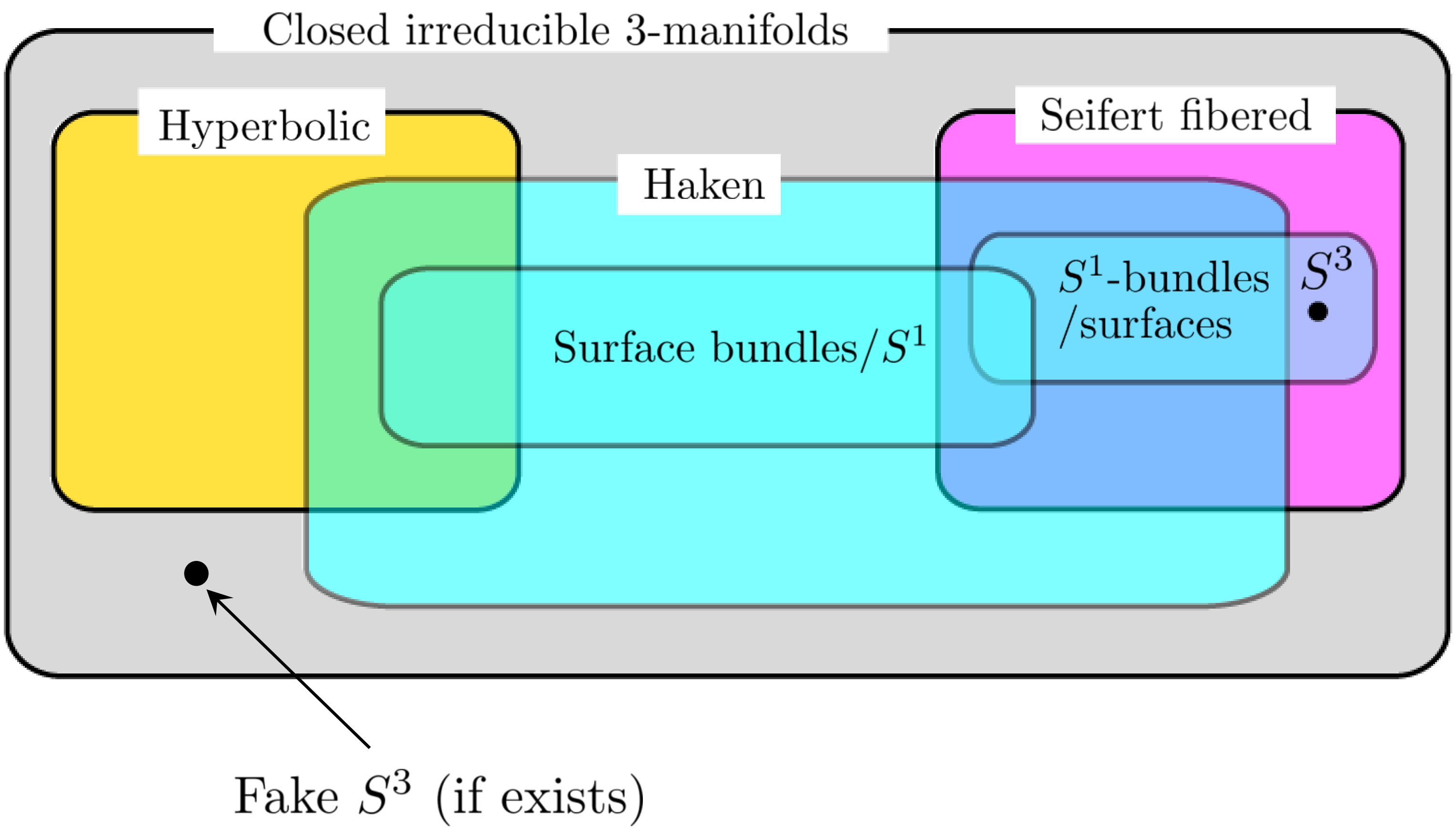

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Fake S^3 (if exists)

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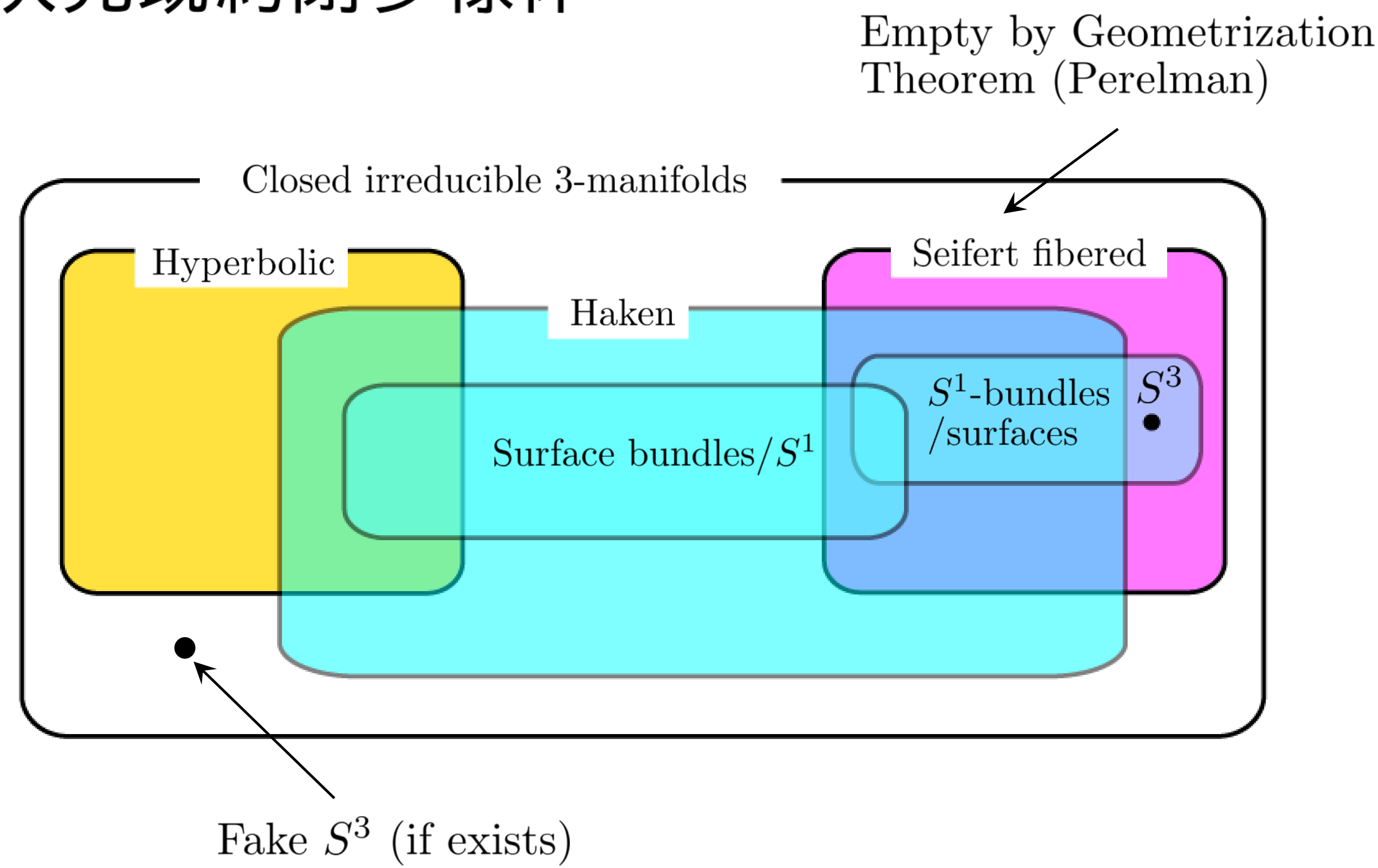

Empty by Geometrization Theorem (Perelman)
Closed irreducible 3-manifolds
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For any closed orientable manifold M, $\|[M]\|$ is called the *simplicial volume* of M and denoted by $\|M\|$.

When $\dim M = 3$ and ∂M consists of tori, the simplicial volume $\|M\|$ of M is defined similarly.

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If $\mathrm{Int} M$ is a hyperbolic 3-manifold of finite volume, then

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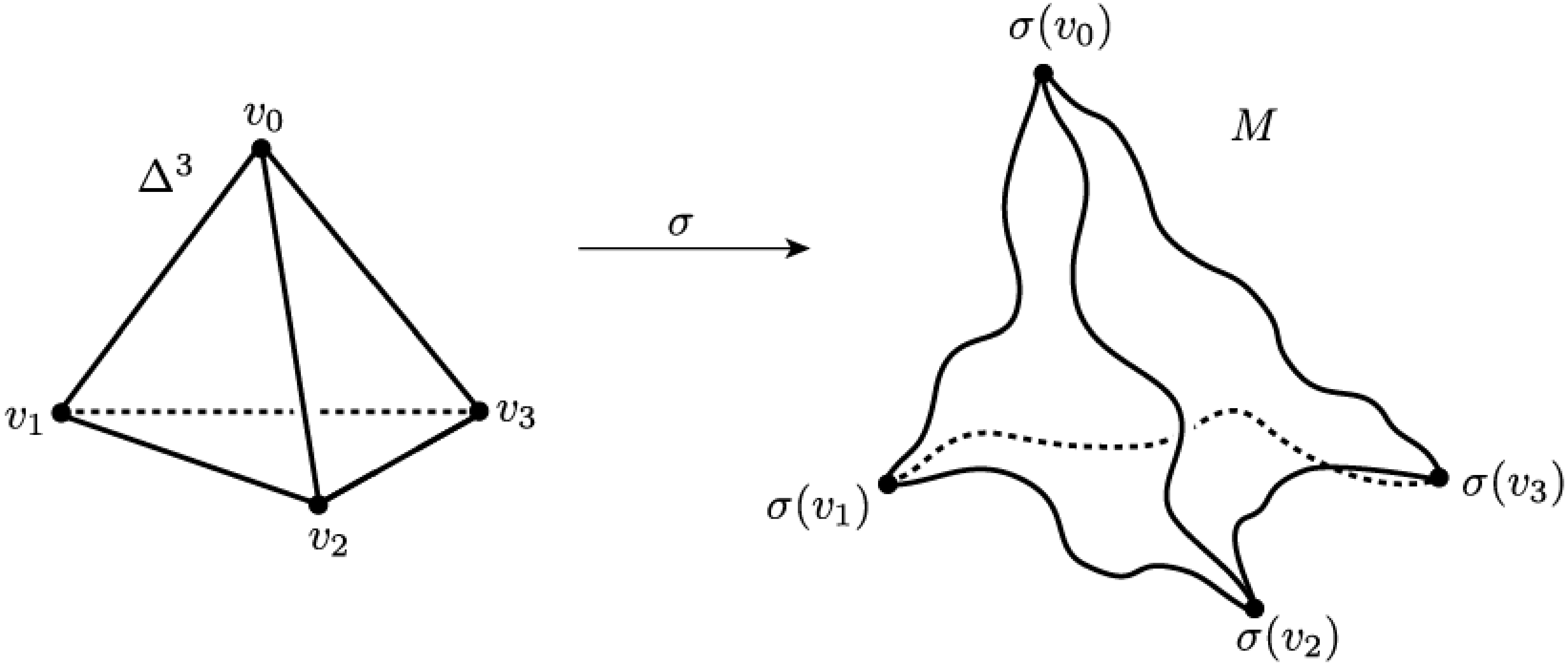
In the proof of Theorem 1.1, the *straightening* of a singular 3-simplex $\sigma : \Delta^3 \to M$ (denoted by straight(σ)) is crucial.


v_0
Δ^3
v_1
v_3
v_2
σ
$\sigma(v_0)$
M
$\sigma(v_1)$
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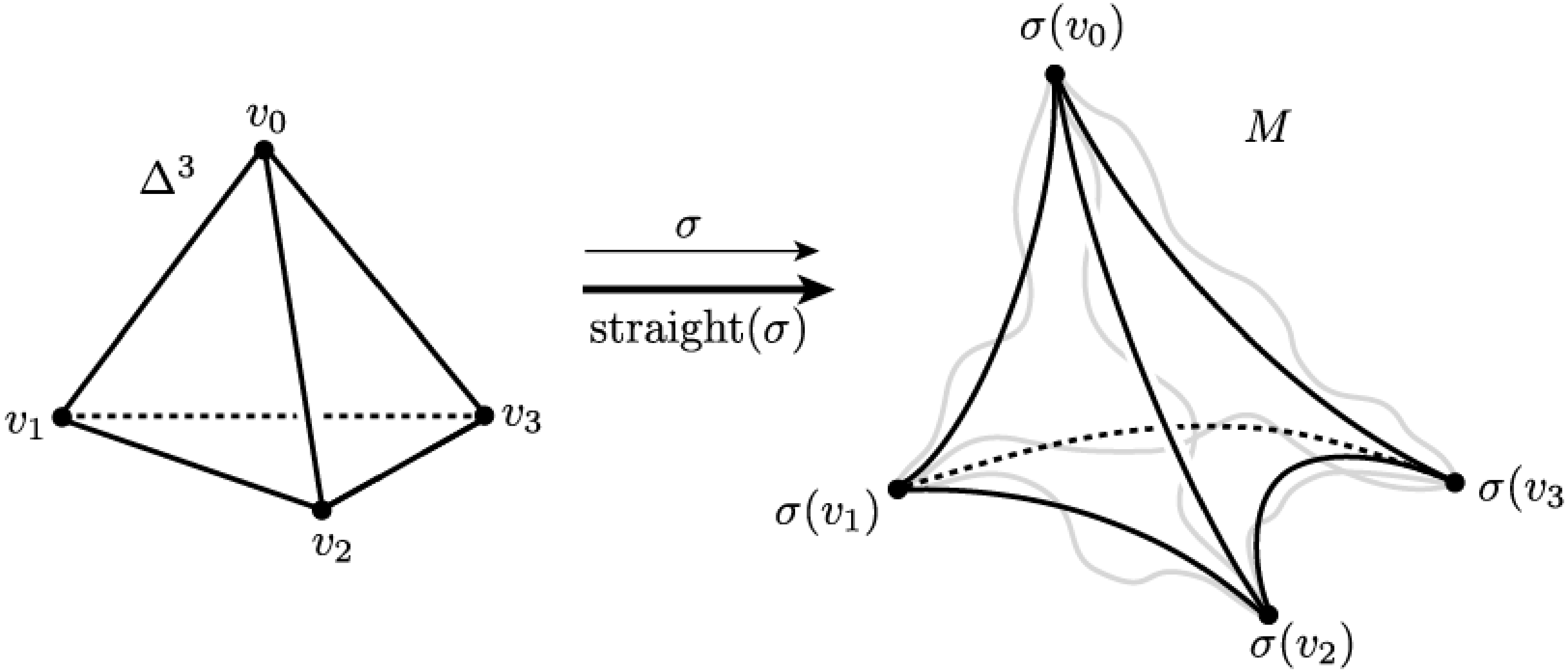

Δ^3
v_0
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straight(σ)
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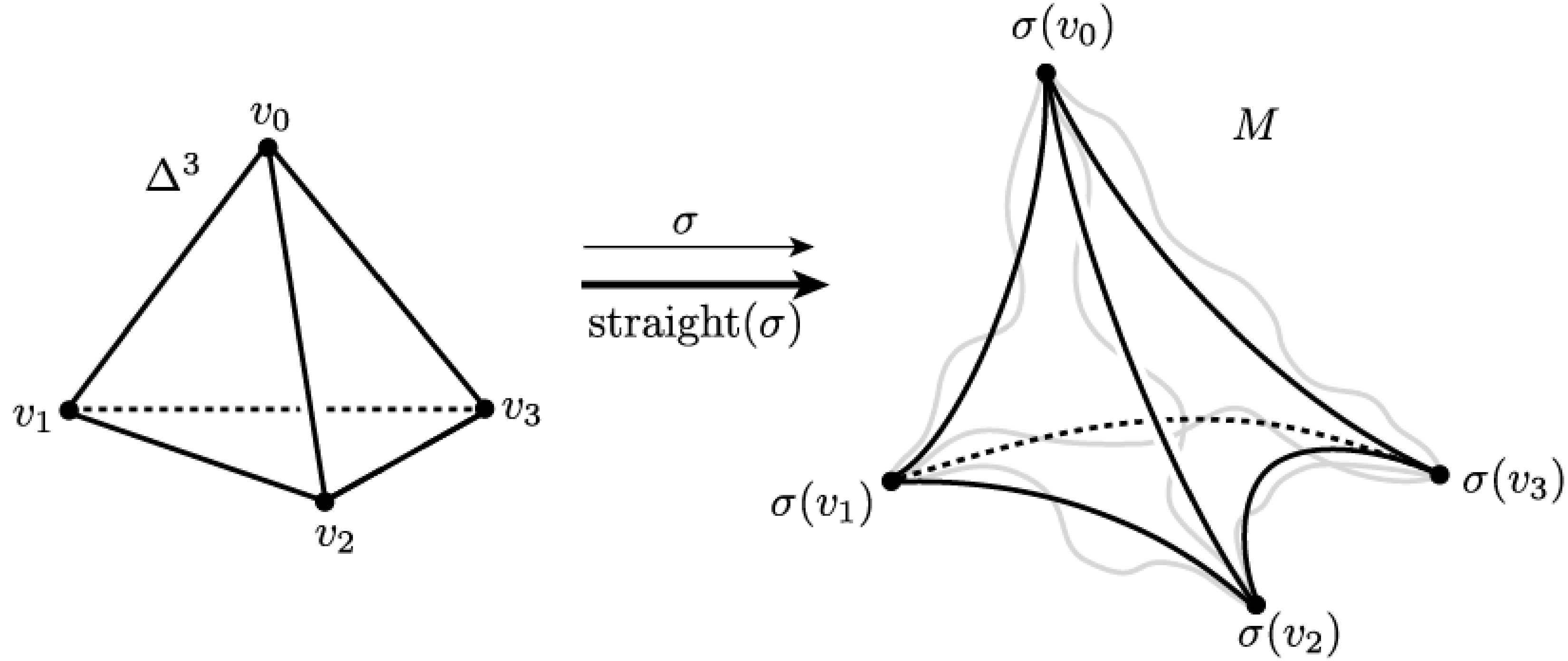

In the case of M being irreducible, by Gromov's Cutting-off Theorem [Gromov (1982)], [S (1981)],

$$\|M\| = \|H_1\| + \cdots + \|H_m\| = \frac{\mathrm{Vol}(\mathrm{Int} H_1)}{\boldsymbol{v}_3} + \cdots + \frac{\mathrm{Vol}(\mathrm{Int} H_m)}{\boldsymbol{v}_3}.$$

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Theorem 1.3 (Rigidity Theorem for irreducible 3-manifolds) [S (1995)]
Let $f : M \to N$ be degree-one maps between closed irreducible 3-manifolds. Then f is homotopic to a map g such that $g(\mathcal{H}(M)) = \mathcal{H}(N)$ and the restriction $g|_{\mathcal{H}(M)} : \mathcal{H}(M) \to \mathcal{H}(N)$ is a homeomorphism if and only if $\|M\| = \|N\|$.

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The cochain complex $(C_b^*(X), \delta_b^*)$ defines the *bounded cohomology* $H_b^*(X; \mathbb{R})$.

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In other words, is $N^k(X) = \{\alpha \in H_b^k(X\,;\mathbb{R})\,;\, \|\alpha\| = 0\}$ the zero-space?

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- By T. Yoshida (1986), $\dim H_b^3(\Sigma_g\,;\mathbb{R}) \geq \#\mathbb{N}$.

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Γ is a *Kleinian surface group* of type $\pi_1(\Sigma_g)$ if Γ is isomorphic to $\pi_1(\Sigma_g)$.

Ends of Kleinian surface groups (the case without parabolic elements)

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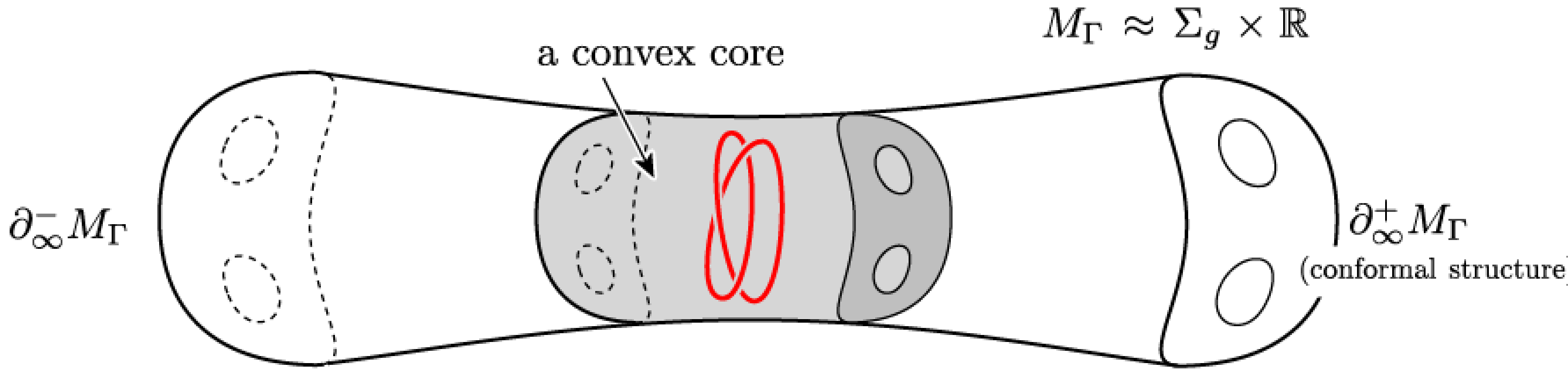
Geometrically finite (a quasi-Fuchsian group)



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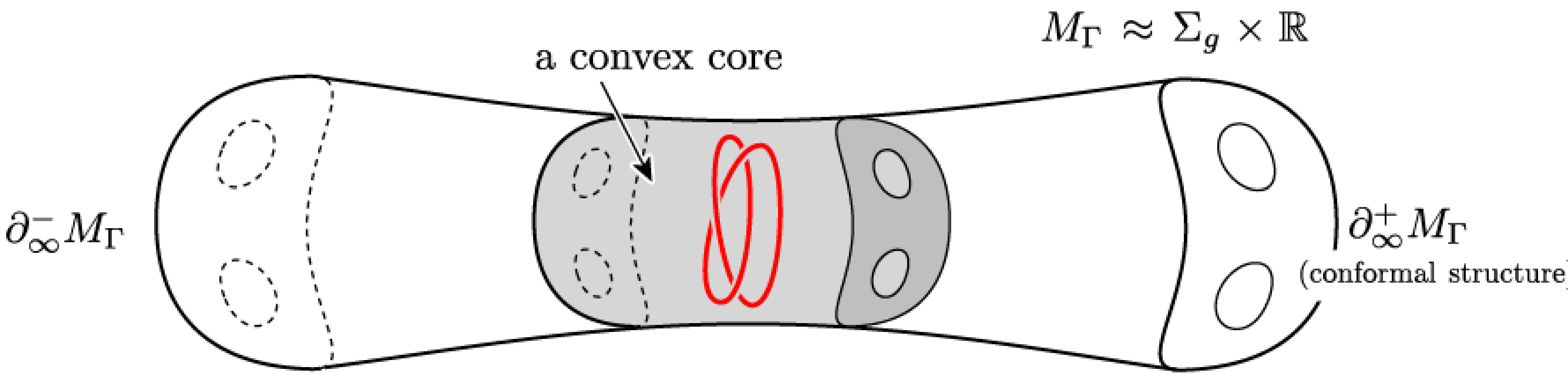
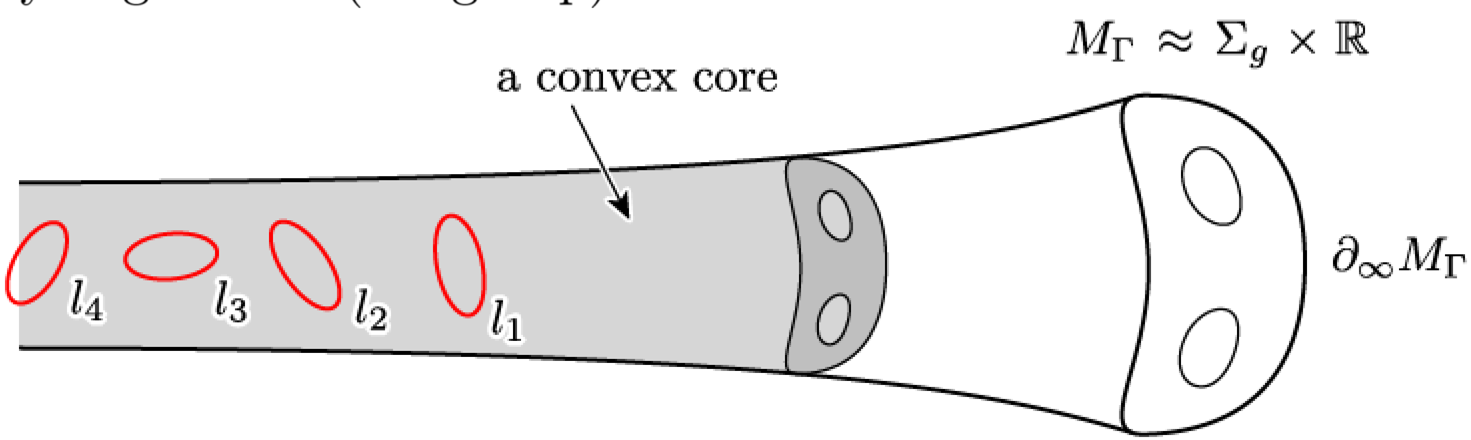
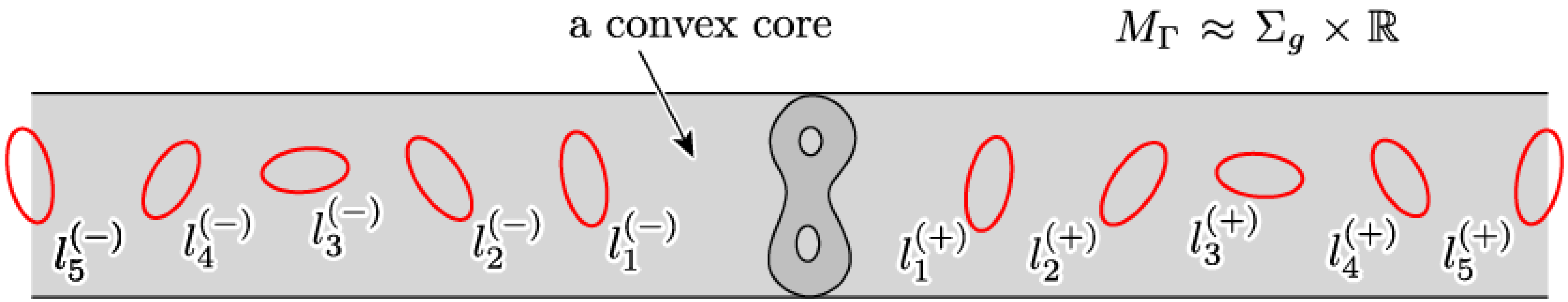


Simply degenerate (a b-group)



Doubly degenerate



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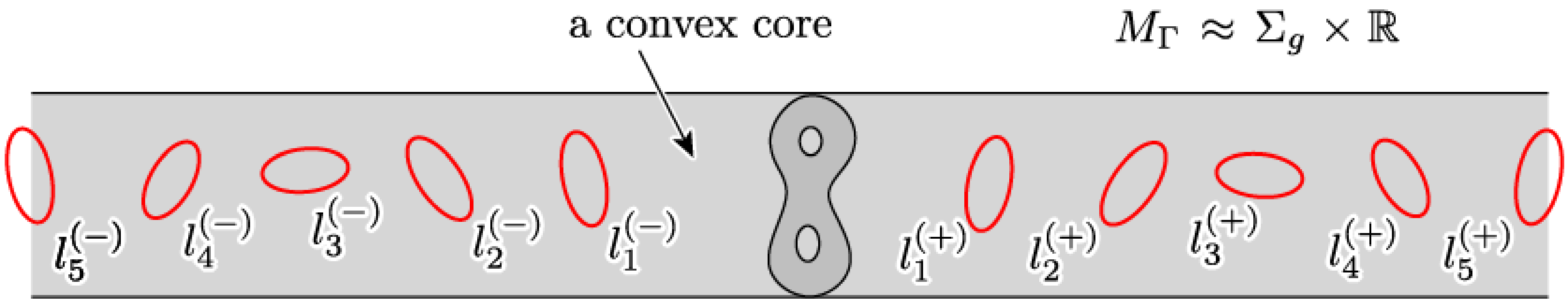
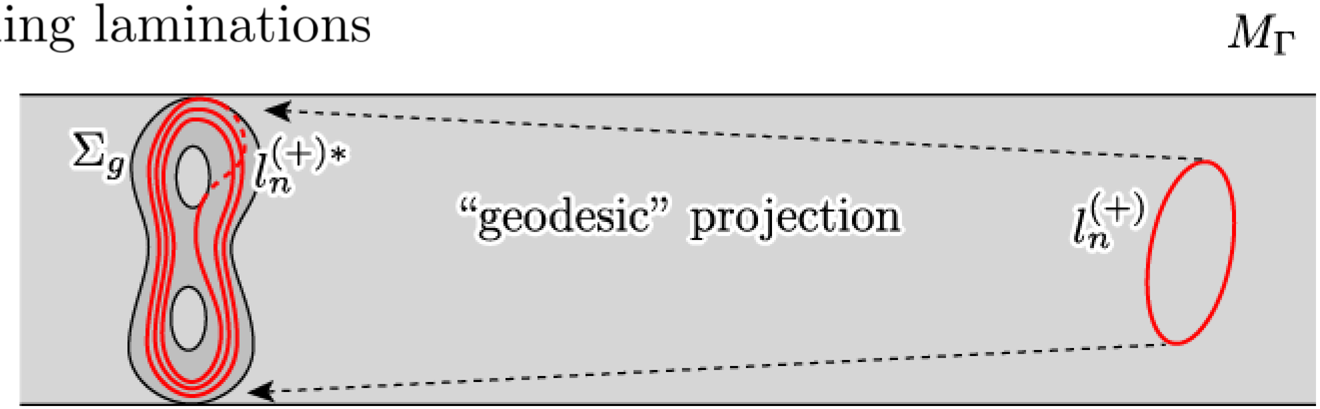


Ending laminations



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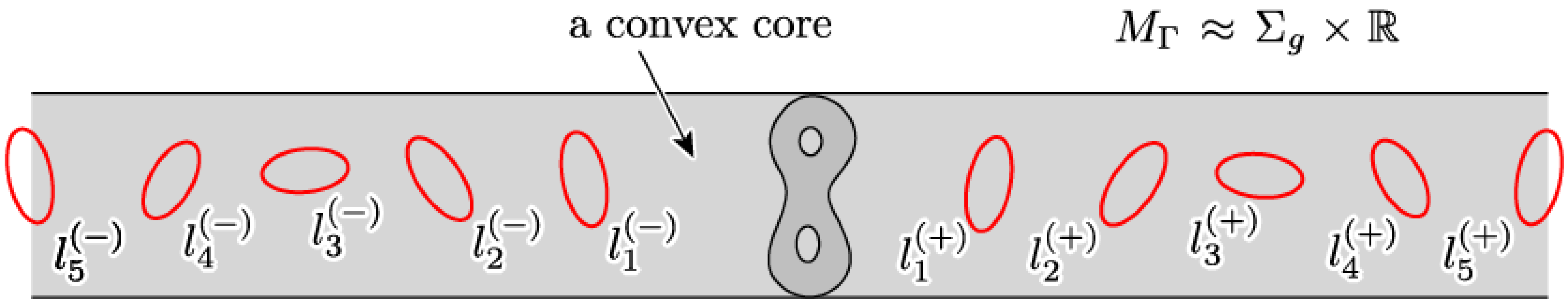

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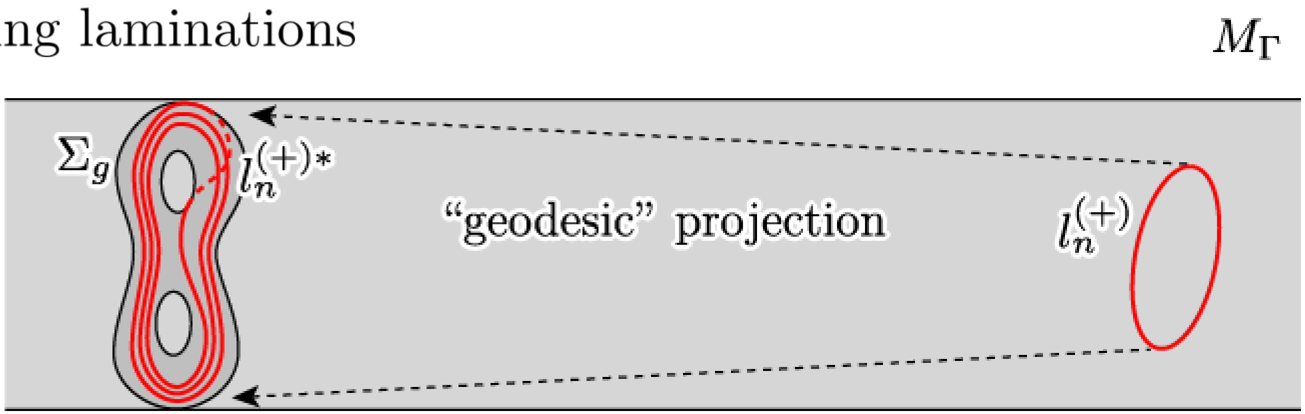

$l_n^{(\pm)*} \to \lambda^\pm$ in Σ_g : the $(\pm)$-ending laminations

(a canonical invariant of simply degenerate ends)

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for any singular 3-simplex $\sigma : \Delta^3 \to M_\Gamma$.

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Theorem 1.4 (A characterization of Kleinian groups by the fundamental classes) [S (1997)]
For any finitely generated Kleinian group Γ, the following three conditions are equivalent to each other.
(1) $[\omega_\Gamma] \neq 0$ in $H_b^3(M_\Gamma; \mathbb{R})$.
(2) $\|[\omega_\Gamma]\| = \boldsymbol{v}_3$.
(3) Either M_Γ is geometrically infinite or $\mathrm{Vol}(M_\Gamma) < \infty$.

The 3-dimensional “Teichmüller space” $\mathcal{T}^{(3)}(\Sigma_g)$ of Kleinian surface groups of type $\pi_1(\Sigma_g)$ is the set of equivalence classes (M_Γ, f) such that $f : \Sigma_g \to M_\Gamma$ is a homotopy equivalence.

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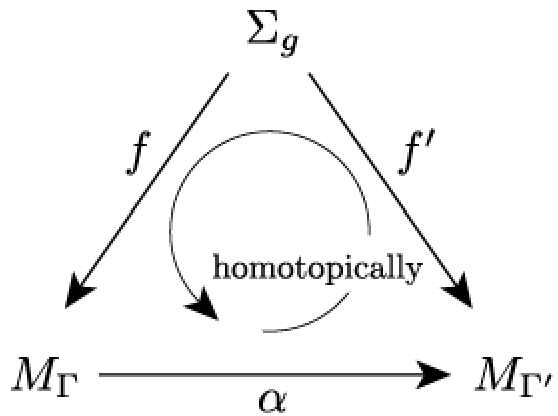
Here (M_Γ, f) and $(M_{\Gamma'}, f')$ are *equivalent* to each other in $\mathcal{T}^{(3)}(\Sigma_g)$ if there exists an orientation-preserving isometry $\alpha : M_\Gamma \to M_{\Gamma'}$ with $\alpha \circ f \simeq f'$.



Theorem 1.7 (Rigidity of Kleinian surface groups) [S (1997), (—)]
Let (M_Γ, f), $(M_{\Gamma'}, f')$ be totally degenerate elements of $\mathcal{T}^{(3)}(\Sigma_g)$. Then $(M_\Gamma, f) = (M_{\Gamma'}, f')$ in $\mathcal{T}^{(3)}(\Sigma_g)$ if and only if $f^*([\omega_\Gamma]) = f'^*([\omega_{\Gamma'}])$ in $H_b^3(\Sigma_g, \mathbb{R})$.

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- Extra assumption:

(IRC) $\quad \inf \operatorname{inj\,rad}(M_\Gamma) > 0, \ \inf \operatorname{inj\,rad}(M_{\Gamma'}) > 0.$

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Remark 2.3 Y. Liu (2012) proved the problem by using results of M. Boileau, S. Wang and others.

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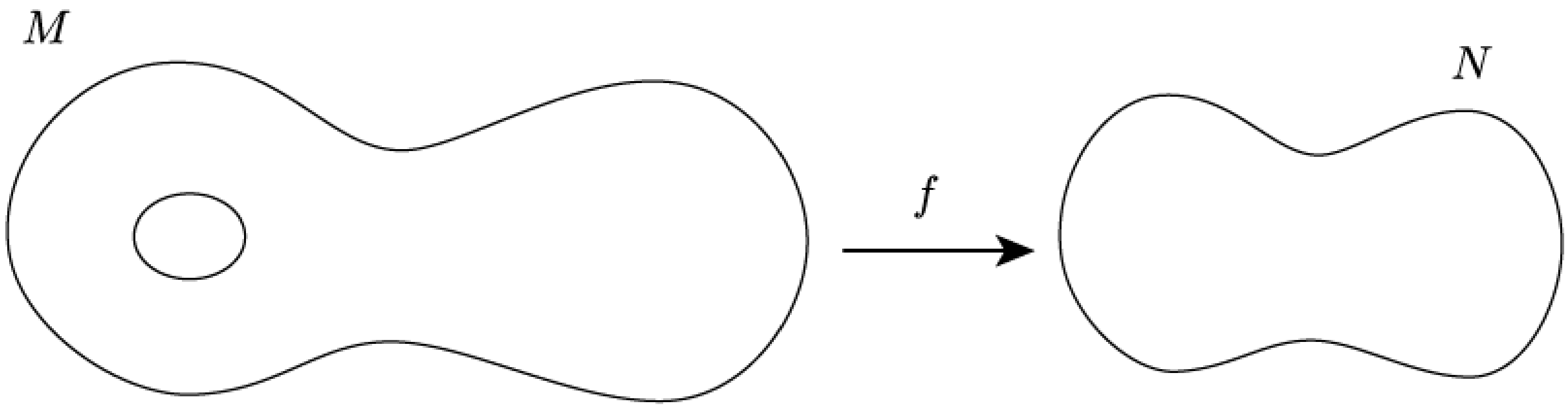

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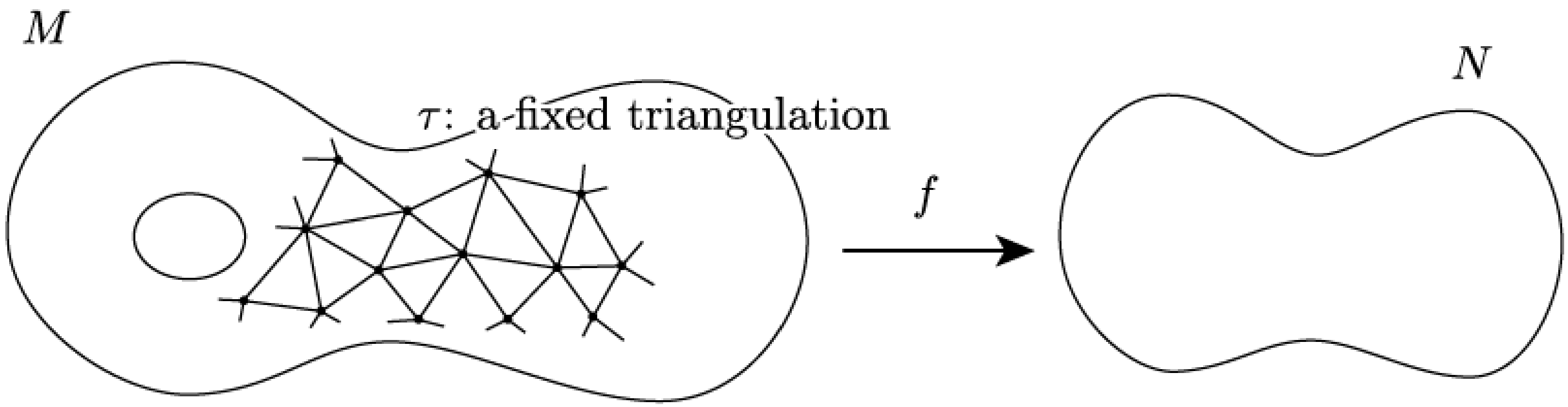

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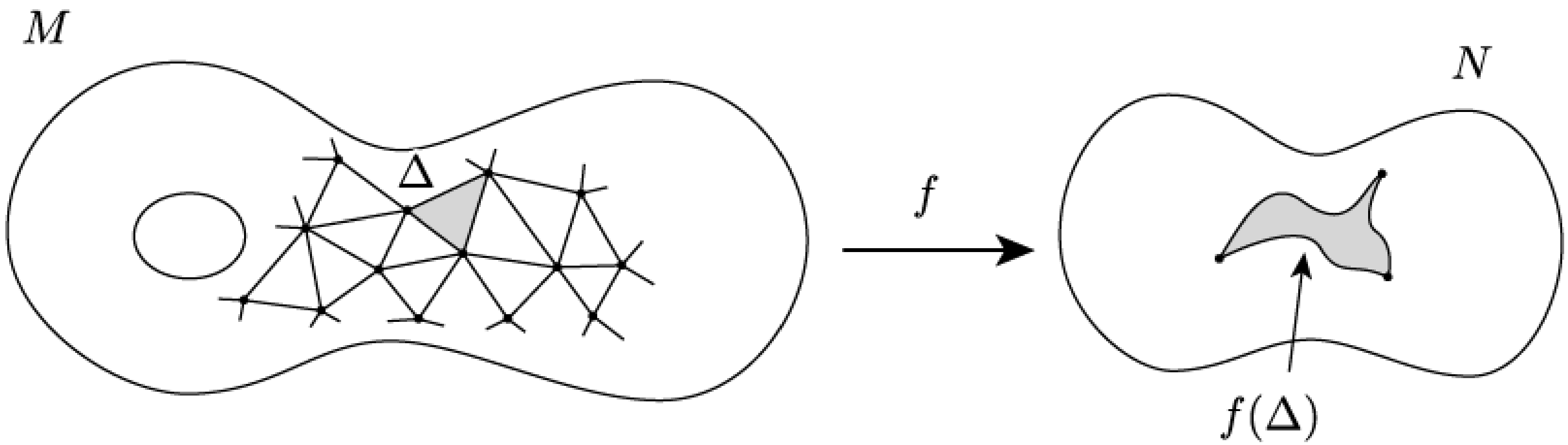

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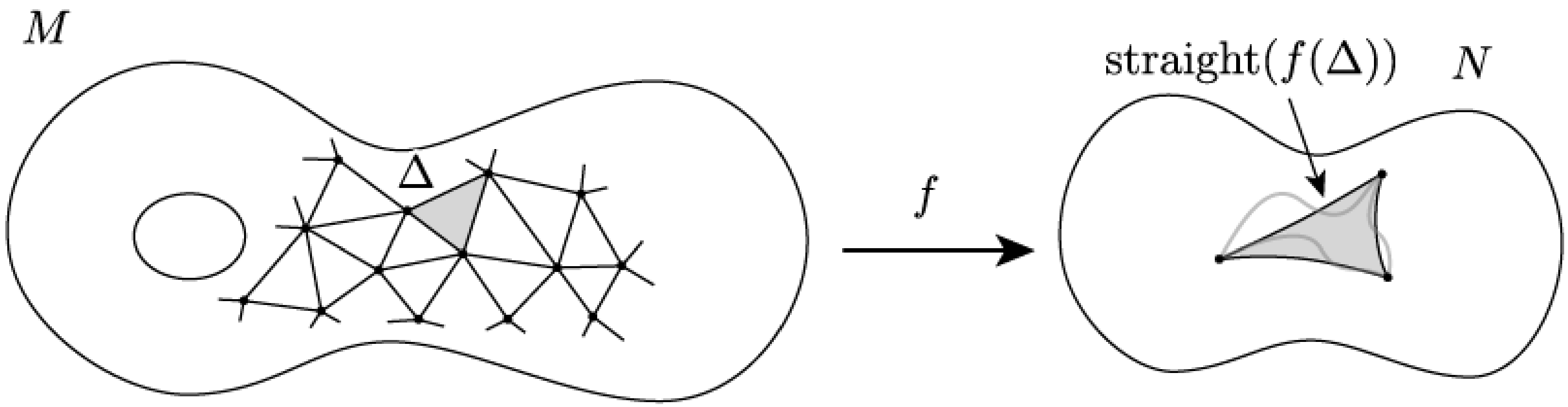

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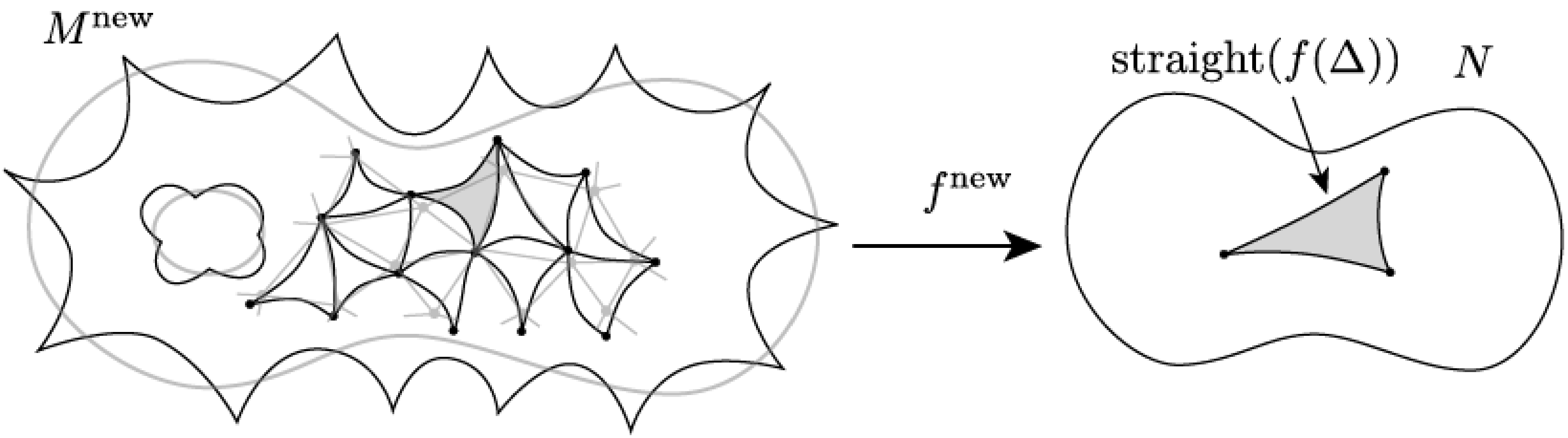

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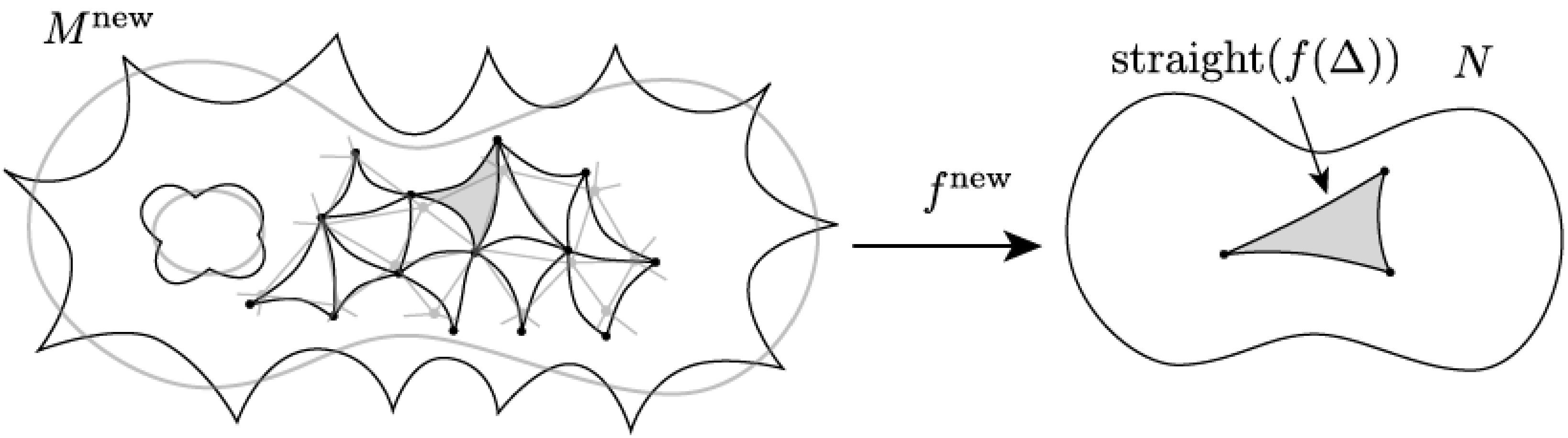

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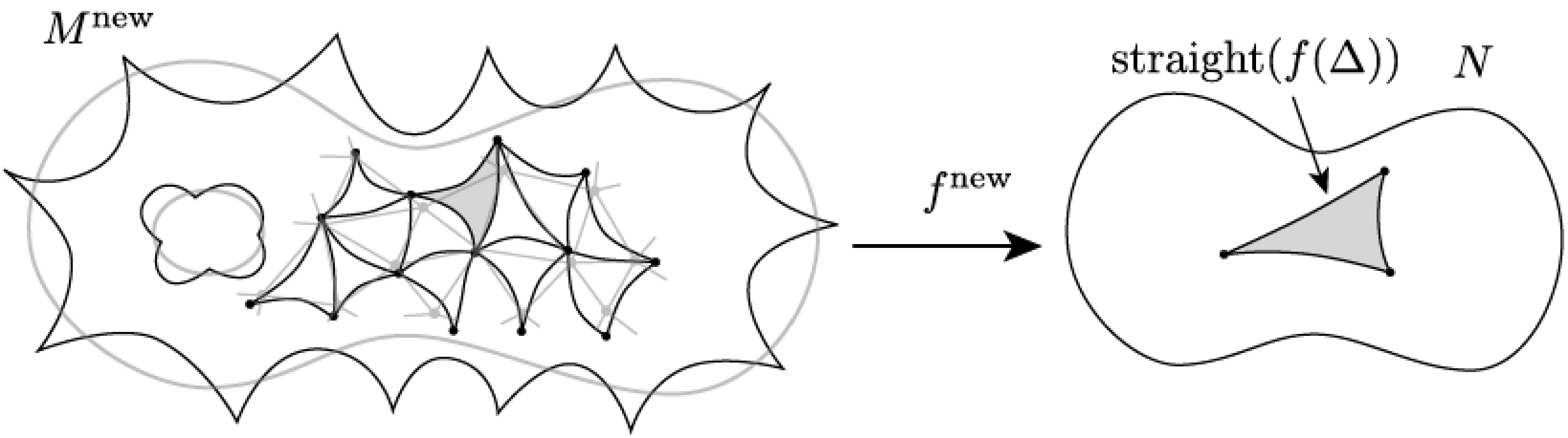

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By applying the idea of [Thurston (1986)] (the compactness of acylindrical hyperbolic structures), one can show that these virtual hyperbolic structures on (M, τ) are bounded.

It follows the finiteness of dominated hyperbolic 3-manifolds. **Q.E.D**

§3. 3次元幾何多様体の微分同相群

3.1 3次元トポロジー的観点からの最小面積曲面

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In fact, he showed that, for any any co-compact metric μ on $\mathbb{H}^3$ and smooth simple curve λ in S^2_∞, there exists a D^2-limit lamination in $\mathbb{H}^3$ consisting of μ-least area planes and bounding λ

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The following implies Gabai’s conjecture (1997).

Theorem 2.1 [S (2004), (2005)] Let X be a Gromov hyperbolic 3-space with a co-compact Riemannian metric μ. Then any Jordan curve in ∂X spans a properly embedded μ-least area plane in X.

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Generalized Smale conjecture. For any closed geometric 3-manifolds M, the inclusion $i_0 : \mathrm{Isom}_0(M) \to \mathrm{Diff}_0(M)$ is a homotopy equivalence.

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Theorem 3.3 [McCullough-S (2013)] Yes if M is an $\widetilde{SL}_2$-manifold.

Corollary. (Kirby's Problem 3.47(A3)) If M is a Seifert manifold with hyperbolic base orbifold, then the space $SF(M)$ of Seifert fibrations on M is contractible.

Flow to the proof of Theorem 3.3.

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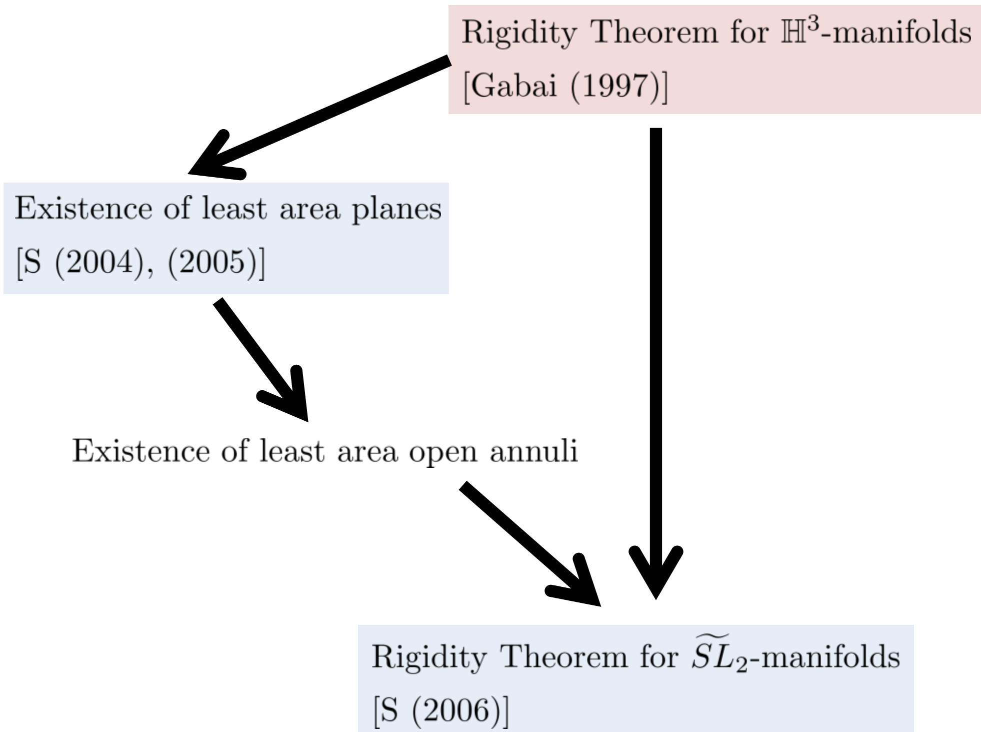

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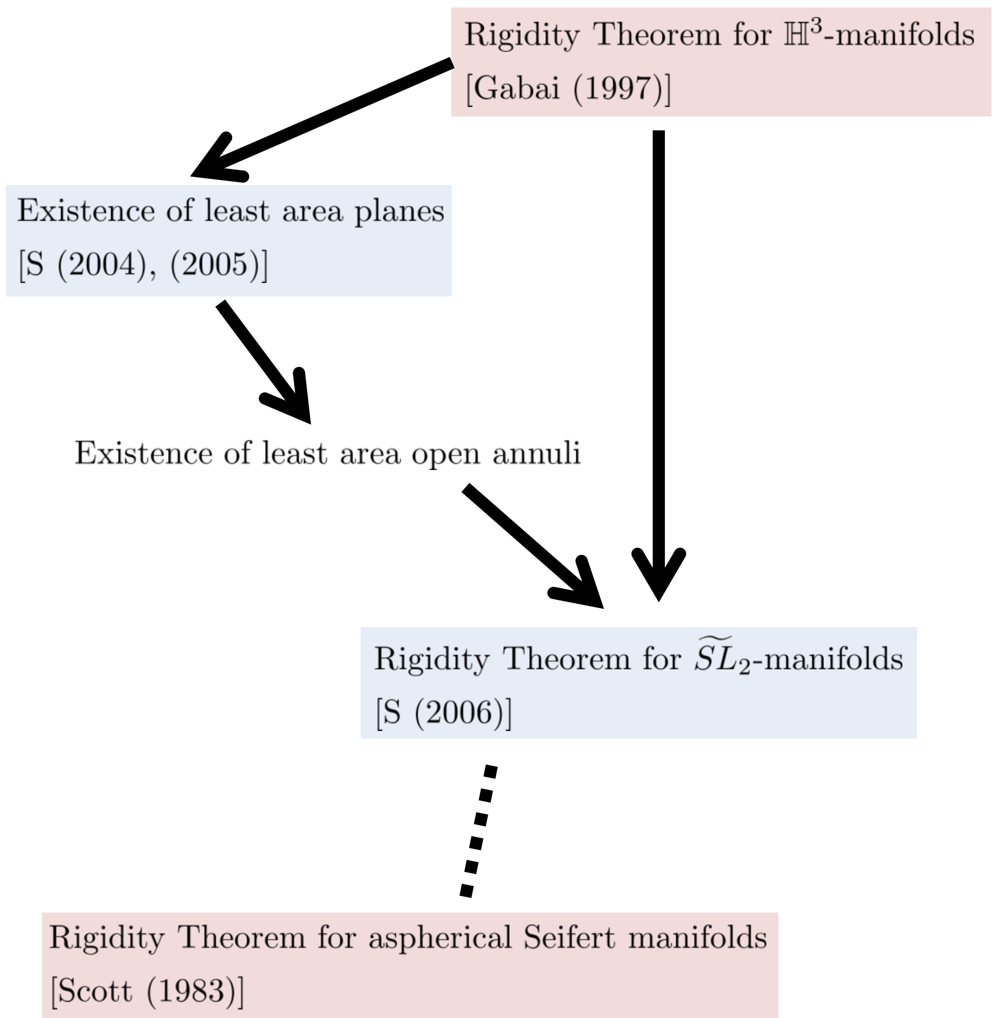

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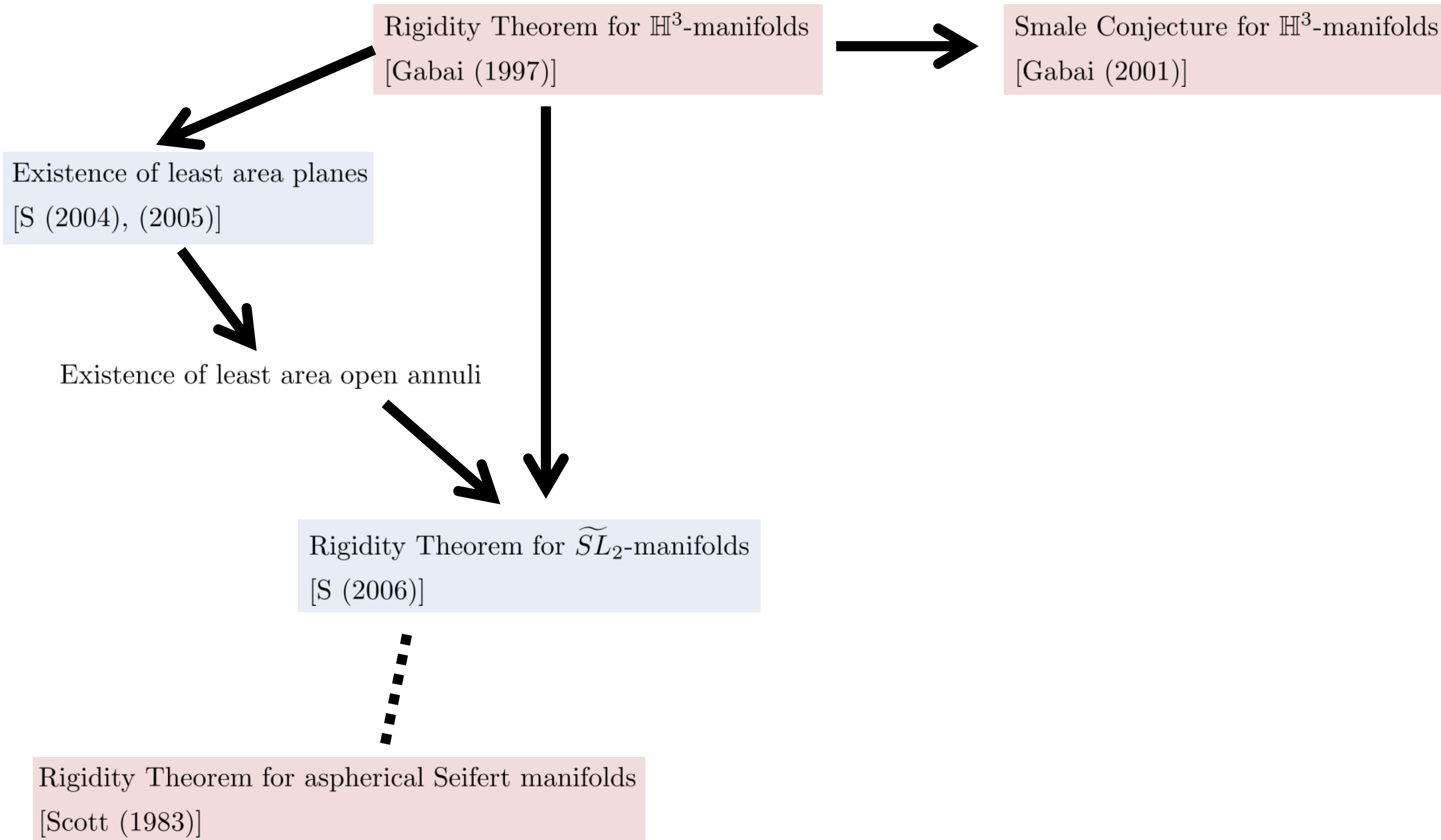

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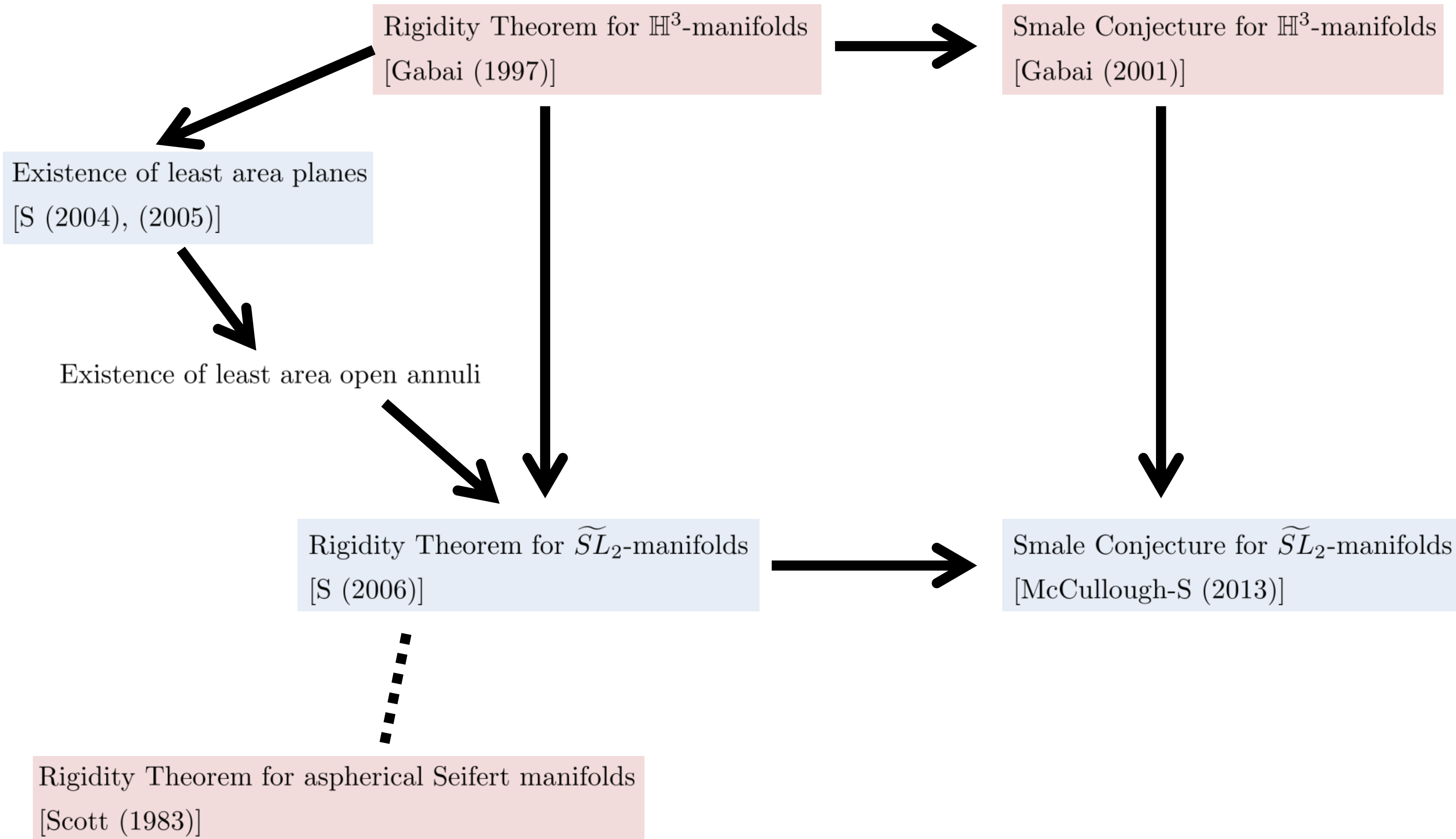

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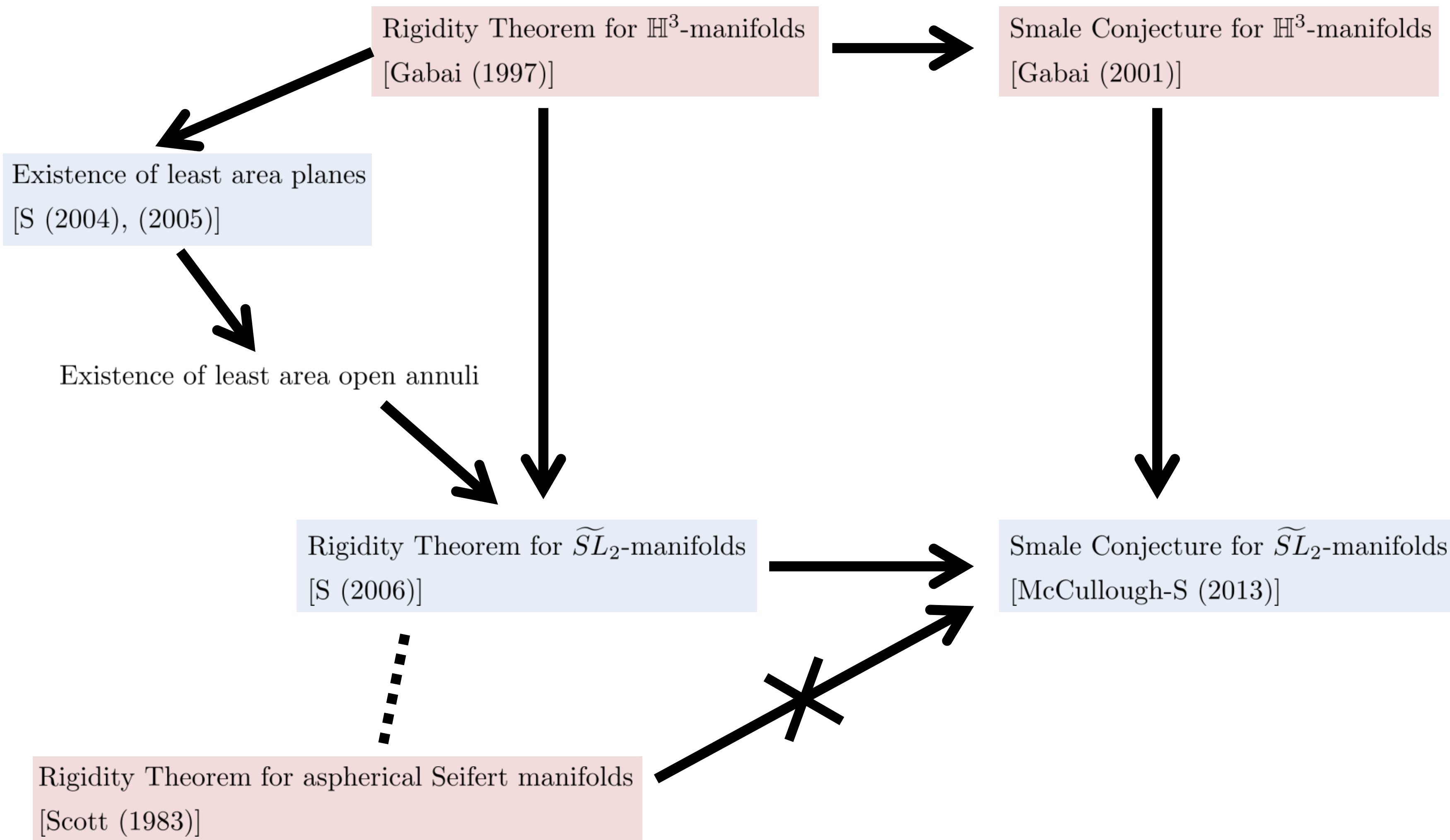

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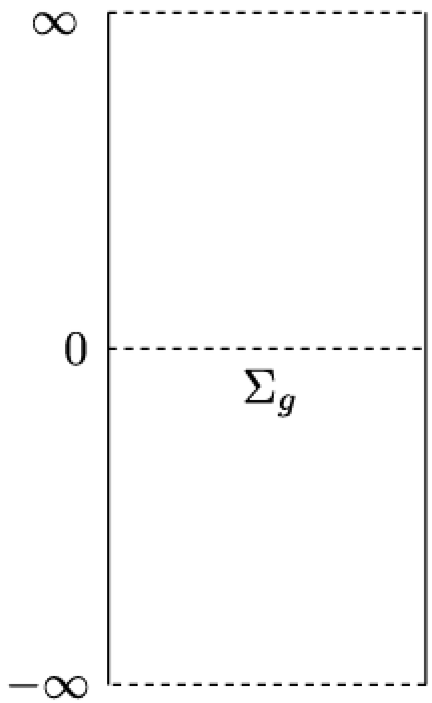

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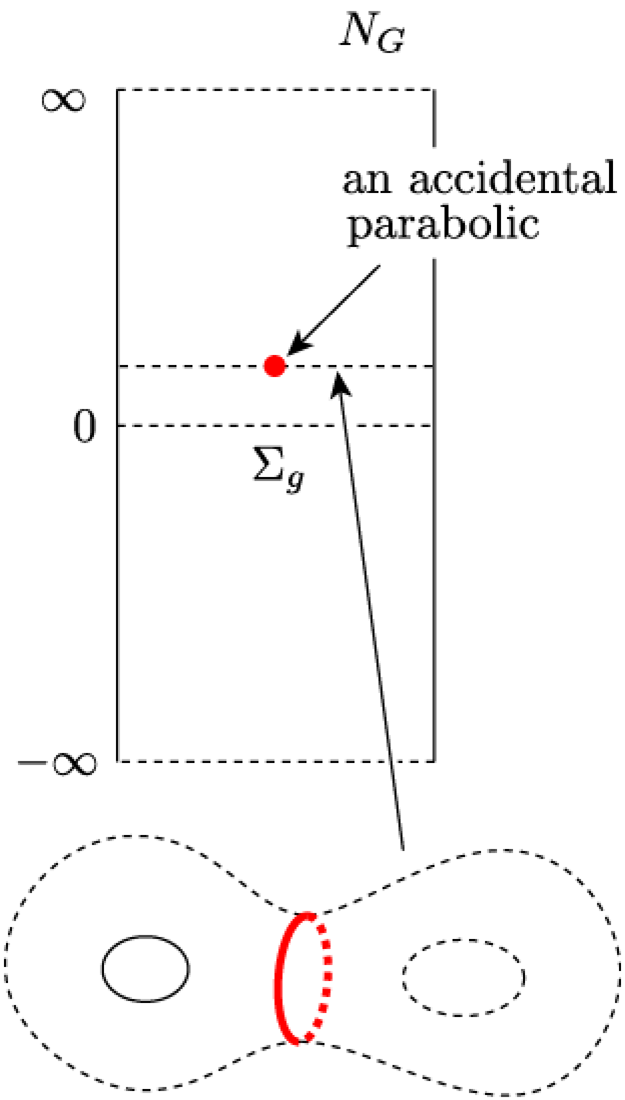

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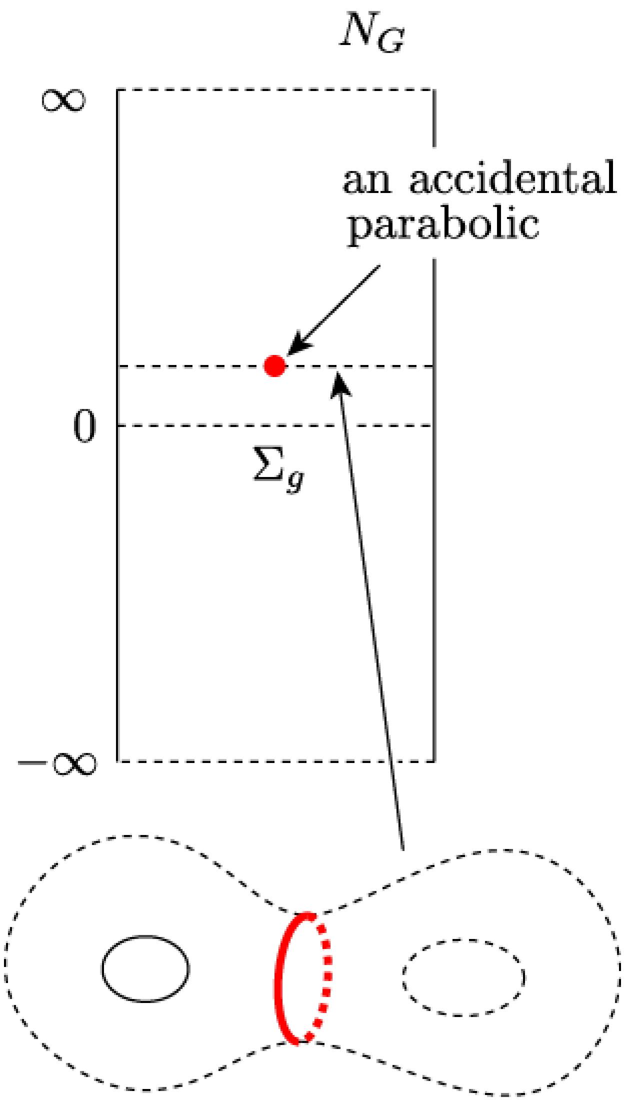

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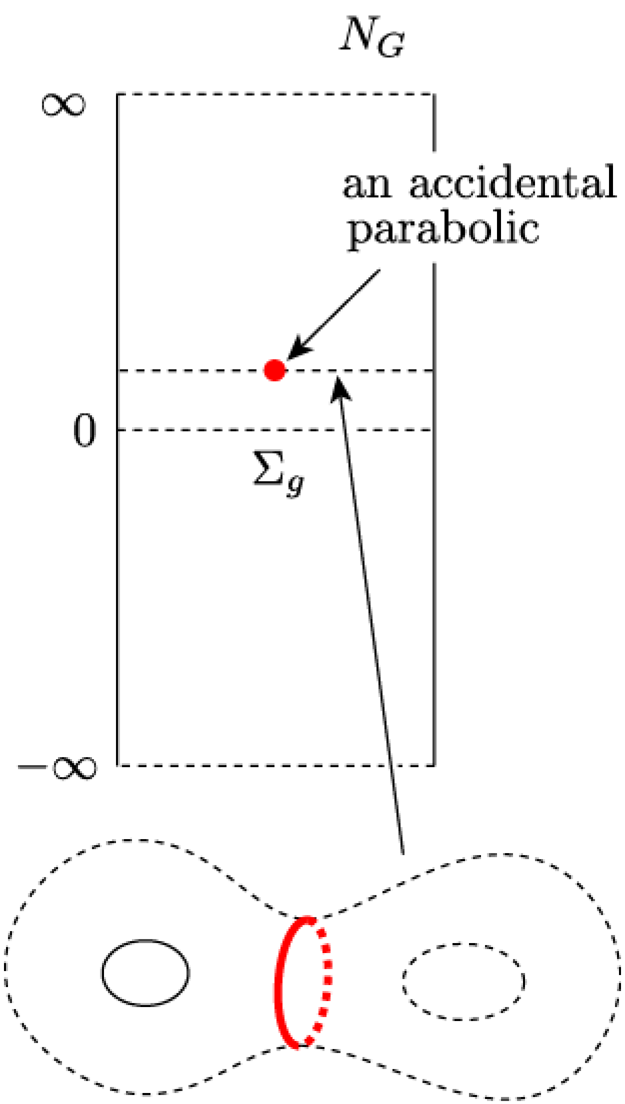

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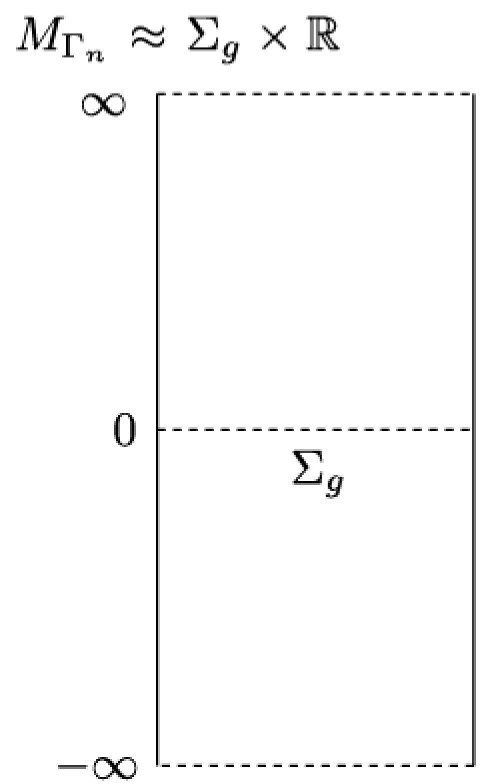

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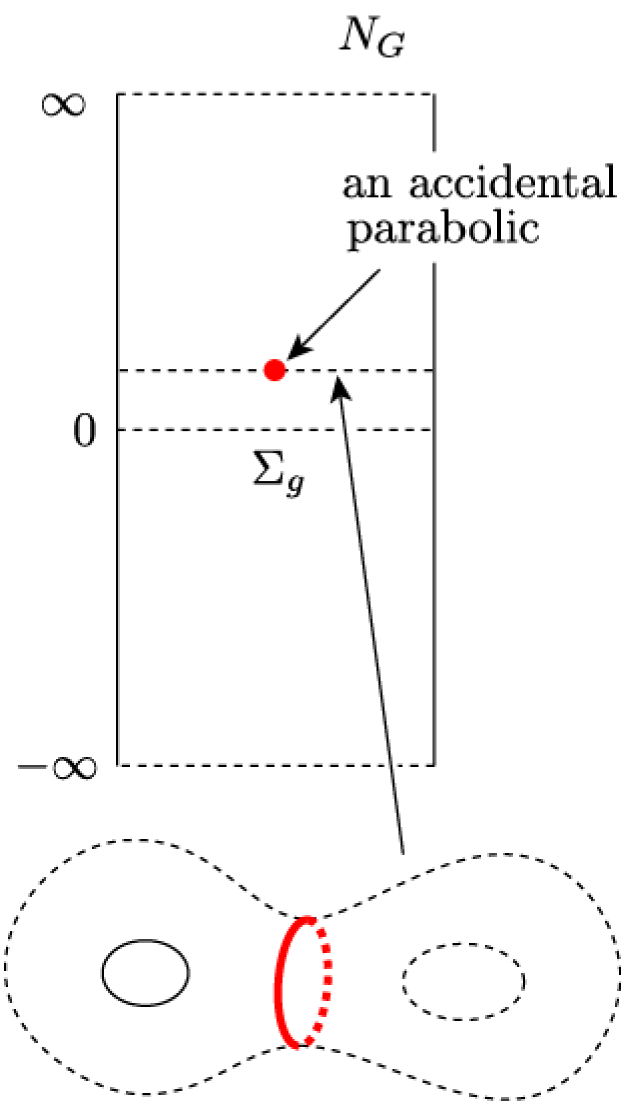

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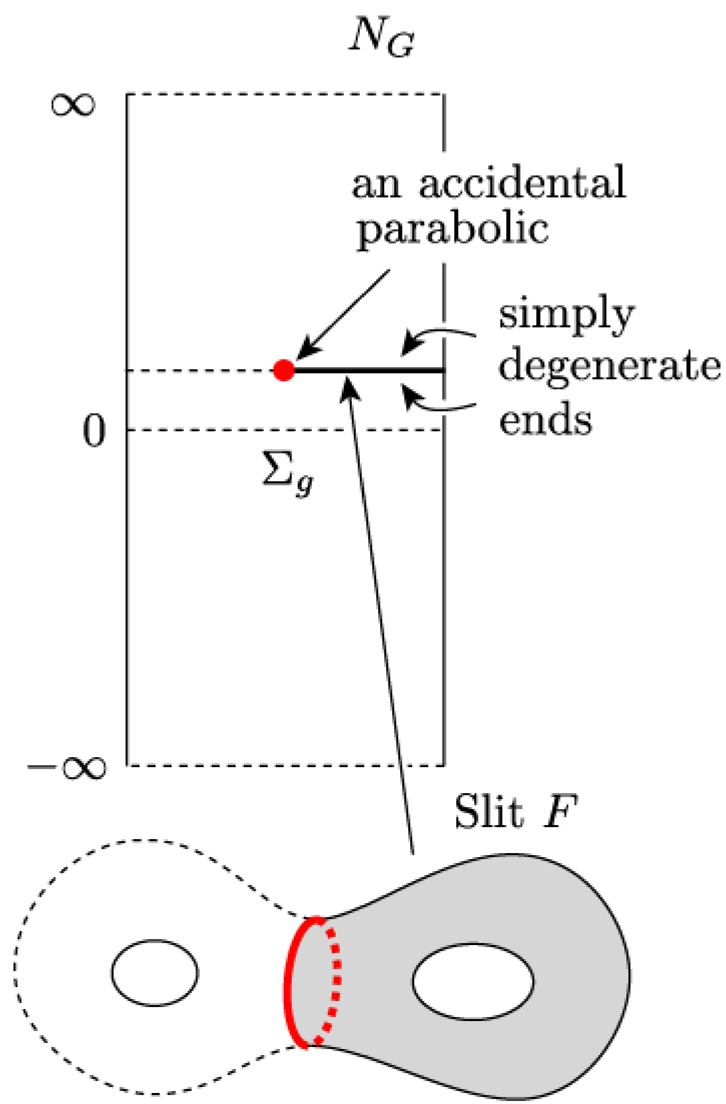

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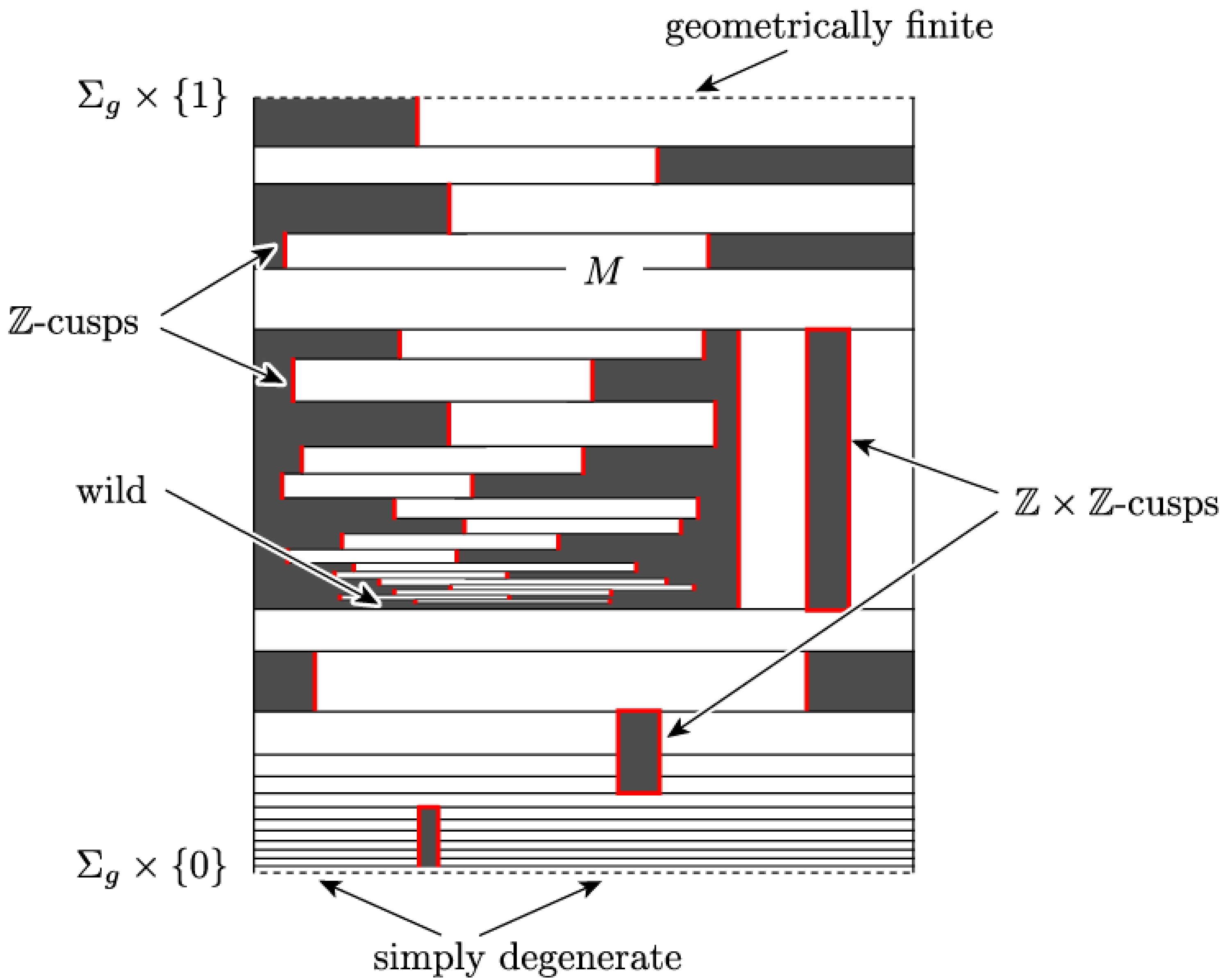
(ii) The boundary components of any incompressible annulus in M are contained in the same component of ∂M.

(iii) There is no half-open essential annulus $S^1 \times [0, \infty)$ in M converging to a wild end of M.

(iv) There exists an embedding $\psi : M \to \Sigma_g \times (0, 1)$ such that the restriction on any brick preserves the direct product structure and the ψ-image of any geometrically finite end of M is contained in $\Sigma_g \times \{0, 1\}$.

Extra condition (v) In the case when M is a doubly degenerate brick manifold of type $F \times \mathbb{R} = \bigcup_{n=-\infty}^{\infty} F \times [n.n+1]$, the $(\pm)$-ending laminations of M are not parallel in M.

Brick manifold embedded in $\Sigma_g \times (0,1)$:



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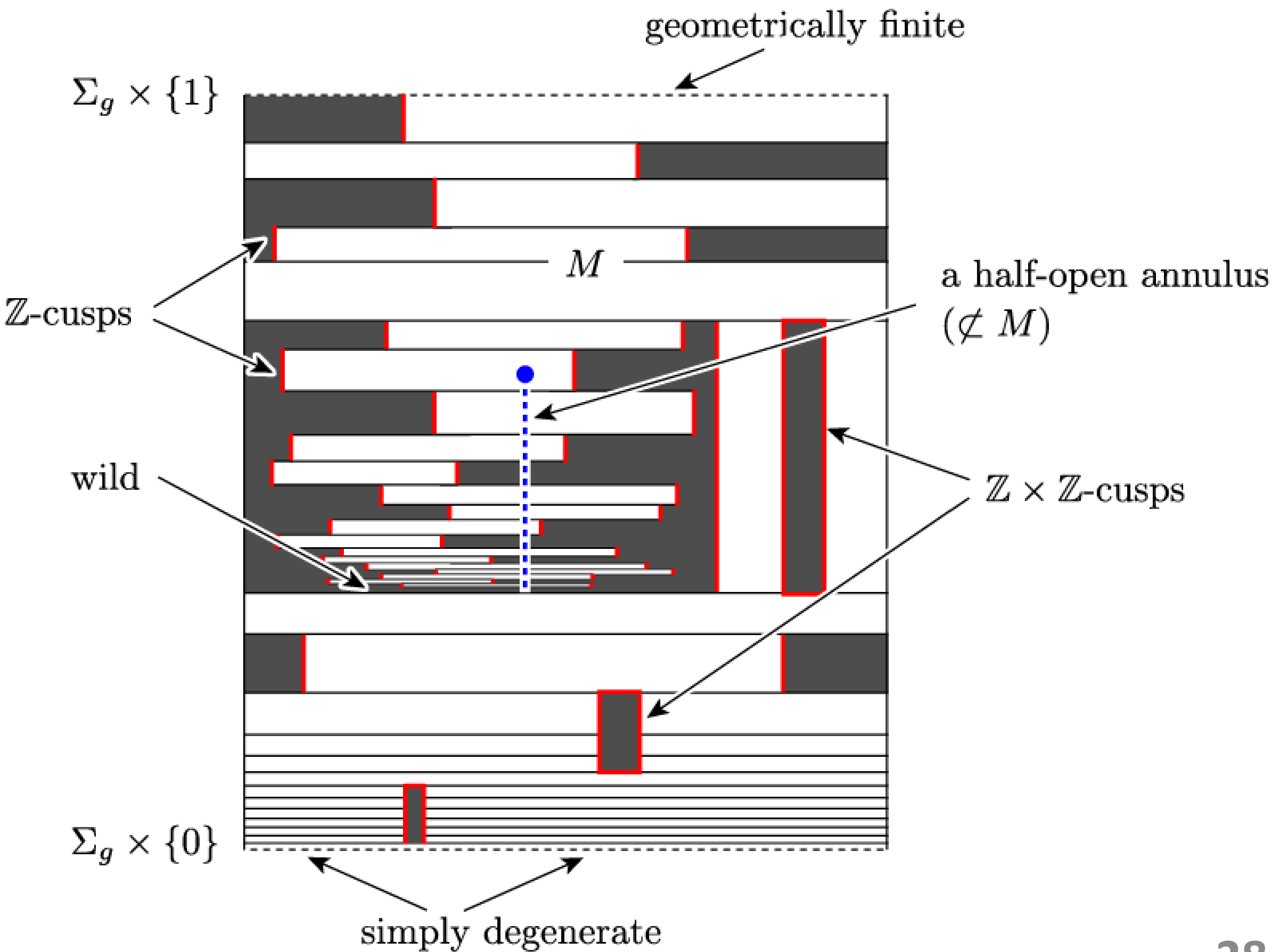

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$$\psi_n : \mathcal{N}_{R_n}(y_0, N_G) \xrightarrow{\varphi_n^{-1}} \mathcal{N}_{R_n}(x_n, M_{\Gamma_n}) \subset M_{\Gamma_n} \approx \Sigma_g \times (0,1)$$

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Theorem C. (Sufficient conditions to be geometric limits) Let M be any brick manifold satisfying the conditions (i)–(iv) (and (v) in the doubly degenerate case). Then there exists a geometric limit G of quasi-Fuchsian groups of type $\pi_1(\Sigma_g)$ which admits a bi-Lipschitz map $f : M \to N_G^{\text{n.c.}}$ extended to a conformal map $f_\infty : \partial_\infty M \to \partial_\infty N_G^{\text{n.c.}}$.

By Theorem C, there exists a geometric limit manifold N_G of quasi-Fuchsian groups which has infinitely many simply degenerate ends and infinitely many wild ends simultaneously.

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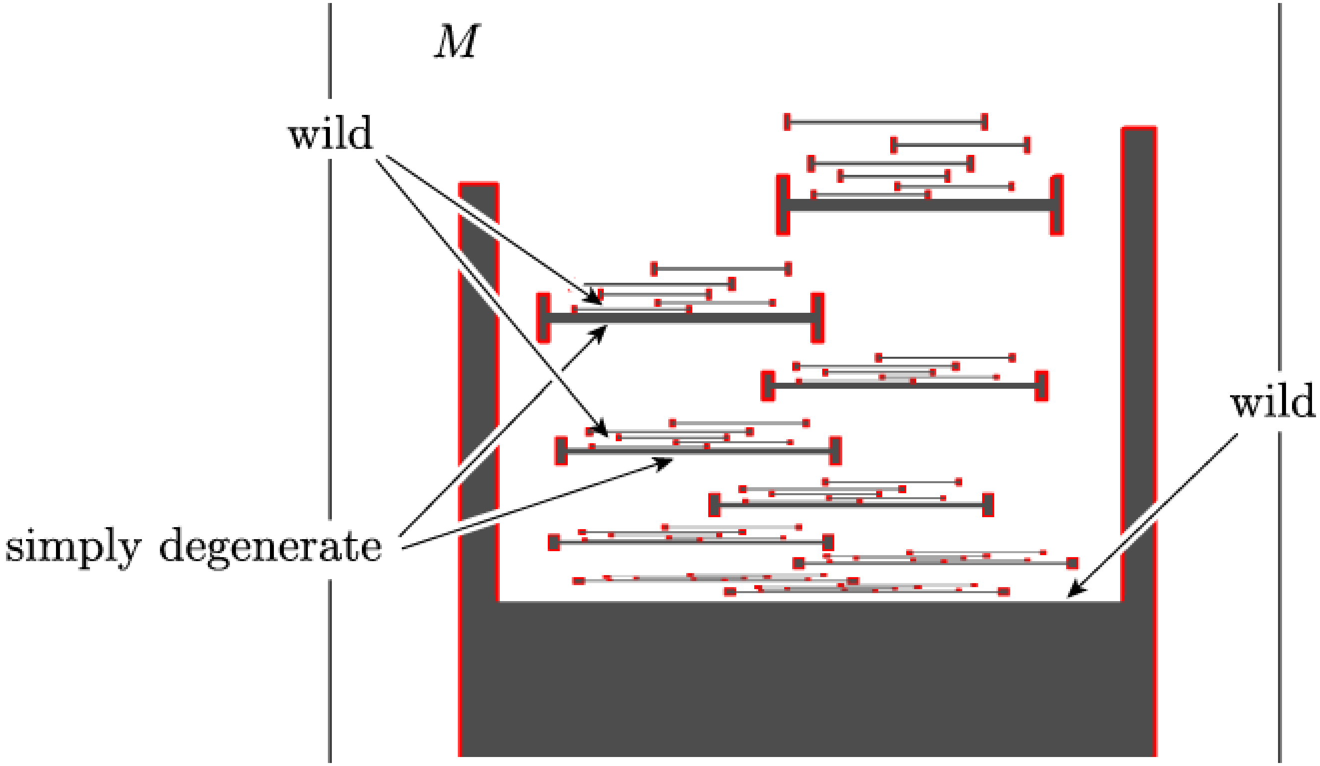

Theorem D. (Rigidity theorem for geometric limits) Let G_1, G_2 be non-elementary geometric limits of Kleinian surface groups of type $\pi_1(\Sigma_g)$. If there exists a homeomorphism $f : N_{G_1} \to N_{G_2}$ preserving the end invariants, then f is properly homotopic to an isometry.

One can prove Theorem D by using Sullivan's Rigidity Theorem.

Thank you