# A coarse Cartan-Hadamard theorem with application to the coarse Baum-Connes conjecture

FUKAYA Tomohiro
深谷友宏

Tokyo Metropolitan university
首都大学東京

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# Table of contents

Non-positively curved spaces and groups

Coarsely Convex Space

Application

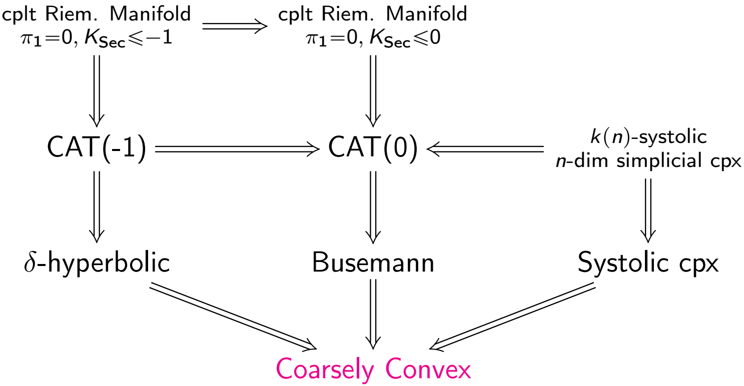
Groups acting on a coarsely convex space

# Several notions of non-positively/negatively curved spaces

| Class | by | QI-inv | Product | coarse Baum-Connes |
|---|---|---|---|---|
| Geodesic $\delta$-hyperbolic | Gromov | Yes | No | Higson-Roe, Willett |
| CAT(0) | C-A-T Gromov | No | Yes | Higson-Roe, Willett F-O |
| Busemann | Busemann | No | Yes | Higson-Roe, Willett F-O |
| Systolic complex | Chepoi J-S, H | No | No $\mathbb{R} \times \mathbb{R}^2$ | Novikov: O-P cBC: F-O |
| Coarsely Convex | F-O | Yes | Yes | F-O |

J-S: Januszkiewich-Świątkowski H: Haglund
O-P: Osajda-Przytycki

# Relations


cplt Riem. Manifold
$\pi_1=0$, $K_{Sec}\leqslant -1$
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CAT(-1)
CAT(0)
$k(n)$-systolic
$n$-dim simplicial cpx
$\delta$-hyperbolic
Busemann
Systolic cpx
Coarsely Convex

# Some notations

- Let $(X, d)$ be a metric space.
- For $p, q \in X$, we denote by $\overline{p, q} := d(p, q)$ the distance between $p$ and $q$.

**Definition**
$f : X \to Y$ is

- an $(\lambda, k)$-quasi-isometric embedding if

$$\frac{1}{\lambda}\overline{x, x'} - k \leqslant \overline{f(x), f(x')} \leqslant \lambda\overline{x, x'} + k \quad (\forall x, x' \in X).$$

- $C$-surjective if $\forall y \in Y, \exists x \in X$ s.t. $\overline{y, f(x)} \leqslant C$.
- quasi-isometry if $f$ is $(\lambda, k)$-quasi-isometric embedding and $C$-dense for some $\lambda, k, C$.

$X$ and $Y$ are quasi-isometric if $\exists f : X \to Y$ quasi-isometry .

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# Geodesic and Quasi-geodesic

### Definition

A map $\gamma\colon [a,b] \to X$ is

- a geodesic if $\gamma$ is an isometry, that is ,

$$\overline{\gamma(t),\gamma(s)} = |t-s| \quad \forall t,s \in [a,b]$$

- a $(\lambda,k)$-quasi-geodesic if $\gamma$ is $(\lambda,k)$-quasi-isometric embedding, that is,

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### Remark

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**Remark**

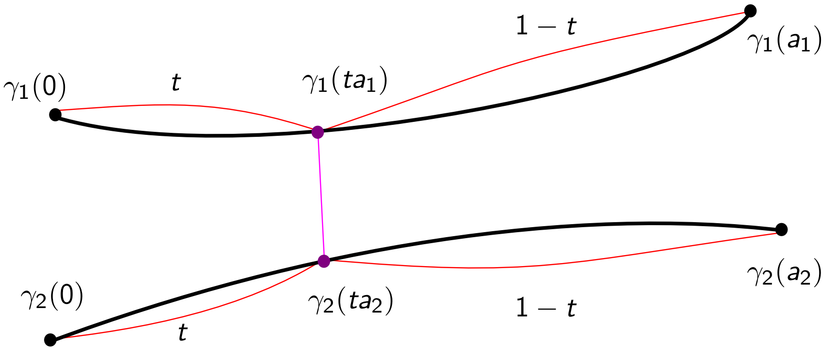
*A geodesic is a* $(1,0)$*-quasi-geodesic.*

# Convexity of Metric

The metric $d$ of $X$ is convex $\Leftrightarrow$

$\forall \gamma_i \colon [0, a_i] \to X$ geodesic segments ($i = 1, 2$), $\forall t \in [0, 1]$ we have

$$\overline{\gamma_1(ta_1), \gamma_2(ta_2)} \leqslant (1-t)\,\overline{\gamma_1(0), \gamma_2(0)} + t\,\overline{\gamma_1(a_1), \gamma_2(a_2)}\,.$$




Remark: $X$ is a Busemann space $\Leftrightarrow$ $(X, d)$ is a geodesic space and $d$ is convex.

# QI-invariance

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We want to make it QI-invariant!

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# QI-invariance: Naive Idea

Naive Idea: Replace GEODESIC by $(\lambda, k)$-QUASI-GEODESIC and introduce some constants E,C.

$\forall \gamma_i \colon [0, a_i] \to X$ $(\lambda, k)$-quasi-geodesic $(i = 1, 2)$, $\forall t \in [0, 1]$ we have

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$\cdots$ This does not work!

Consider $\mathbb{R}^2$ with $l^1$-metric (so-called, Manhattan distance.)

$d_1((x, y), (x', y')) := |x - x'| + |y - y'|.$

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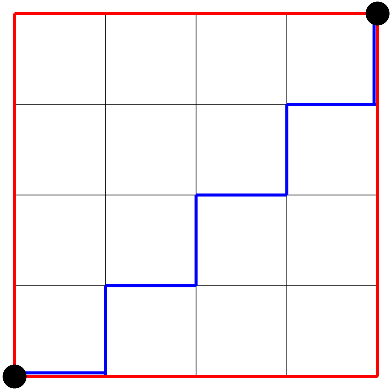
# "FAT"-bigon



"FAT"-bigon in $\mathbb{R}^2$ with $l^1$-metric

GOOD Geodesic BAD Geodesics

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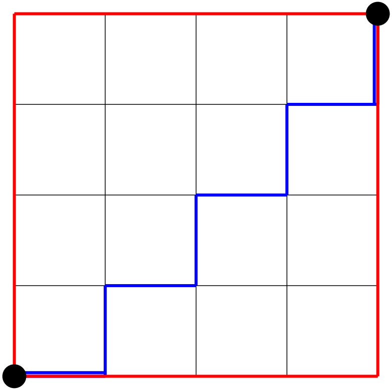

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GOOD Geodesic BAD Geodesics

# IDEA

IDEA: Consider ONLY "GOOD" quasi-geodesics.

Theorem (Osajda-Przytycki)

*Let $X$ be a systolic complex.*
*Then $X$ has a family of good geodesics.*

# Coarsely Convex space

**Definition**

- Let $X$ be a metric space.
- Let $\lambda \geqslant 1$, $k \geqslant 0$, $E \geqslant 1$, and $C \geqslant 0$ be constants.
- Let $\theta\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non-decreasing function.
- Let $\mathcal{L}$ be a family of $(\lambda, k)$-quasi-geodesic segments.

The metric space $X$ is $(\lambda, k, E, C, \theta, \mathcal{L})$-*coarsely convex*, if $\mathcal{L}$ satisfies the three $+1$ conditions in the following slides.

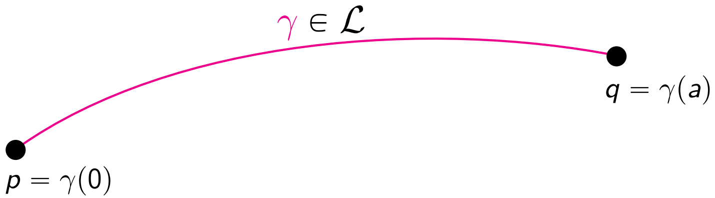
# +1: prefix-closed

$\mathcal{L}$ is prefix-closed, that is, for $\gamma\colon [0,a] \to X$ and $0 \leqslant b \leqslant a$,

$$\gamma \in \mathcal{L} \implies \gamma|[0,b] \in \mathcal{L}.$$

# First: $\mathcal{L}$-Connected

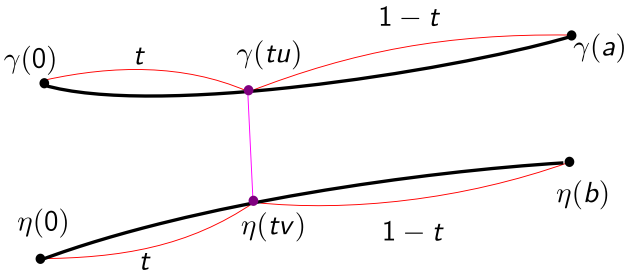
(i) $\forall p, q \in X$, $\exists \gamma \in \mathcal{L}$ with $Domain(\gamma) = [0, a]$, s.t. $\gamma(0) = p$, $\gamma(a) = q$.



# Second: Coarsely Convex Inequality

(ii) $\forall \gamma, \eta \in \mathcal{L}$ with $Domain(\gamma) = [0, a]$, $Domain(\eta) = [0, b]$.
For $0 \leqslant t \leqslant 1$, we have

$$\overline{\gamma(ta), \eta(tb)} \leqslant (1 - t)E\,\overline{\gamma(0), \eta(0)} + tE\,\overline{\gamma(a), \eta(b)} + C.$$

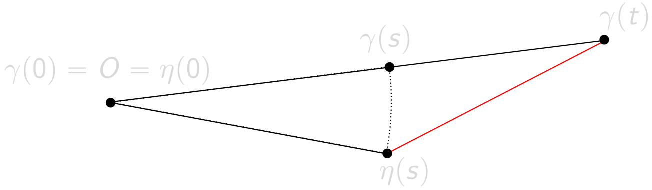


# Third: Parameters

(iii) $\forall \gamma, \eta \in \mathcal{L}$ with $Domain(\gamma) = [0, a]$, $Domain(\eta) = [0, b]$.
For $t \in [0, a]$ and $s \in [0, b]$, we have

$$|t - s| \leqslant \theta(\,\overline{\gamma(0), \eta(0)}\, + \,\overline{\gamma(t), \eta(s)}\,).$$

Consider the case $\gamma(0) = \eta(0) = O$.




If $\gamma, \eta$ are geodesic, then by triangle inequality,

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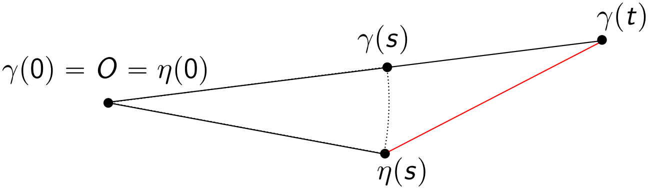


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Remark

*If $X$ is a*

- *Gromov hyperbolic space,*
- *Busemann space, or*
- *Systolic complex,*

*then we can take $\mathcal{L}$ a family of geodesic segments. Therefore the third condition is satisfied.*

# Basic properties

### Proposition (QI-invariant)

- *Let $X$ and $Y$ be metric spaces.*
- *Suppose that $X$ and $Y$ are quasi-isometric.*

*Then $X$ is coarsely convex $\Leftrightarrow$ $Y$ is coarsely convex.*

### Proposition (Stable under direct products)

- *Let $X$ and $Y$ be metric spaces.*
- *Suppose that $X$ and $Y$ are coarsely convex*

*Then the direct product $X \times Y$ is coarsely convex.*

# Examples

The following metric spaces are coarsely convex.

- Geodesic Gromov hyperbolic spaces.
- CAT(0)-spaces.
- Busemann spaces.
- Systolic complexes (Osajda-Przytycki)
- Artin groups of (almost) large type (Osajda-Huang)
- graphical $C(6)$ small cancellation groups (Osajda-Prytuła)

Moreover, the direct products of the above spaces and groups are coarsely convex!

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# Visual boundary

- Let $X$ be $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex space.
- A quasi-geodesic $\gamma\colon [0,\infty) \to X$ is $\mathcal{L}$-approximatable if $\exists \{\gamma_n\} \subset \mathcal{L}$ such that $\gamma_n$ converges to $\gamma$ uniformly on $\{0, 1, \ldots, l\}$ for all $l \in \mathbb{N}$.
- We define

$$\partial X := \{\gamma\colon [0,\infty) \to X : \gamma \text{ is } \mathcal{L}\text{-approximatable}\}/\sim$$

where $\gamma \sim \eta$ if $\sup\{\overline{\gamma(t), \eta(t)} : t \in [0,\infty)\} < \infty$.

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*If $\gamma(0) = \eta(0)$, then*

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# Topology of $\partial X$ – Gromov Product

- Let $X$ be $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex space.
- Choose a base point $O \in X$.
- Set $\mathcal{L}_O^\infty := \{\gamma\colon [0,\infty) \to X : \mathcal{L}\text{-approximatable}, \gamma(0) = O\}$.
- For $\gamma, \eta \in \mathcal{L}_O^\infty$, we define

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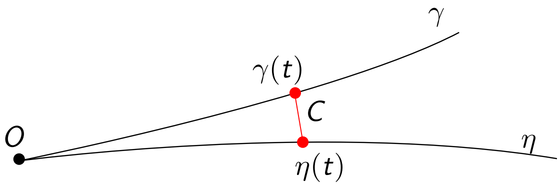

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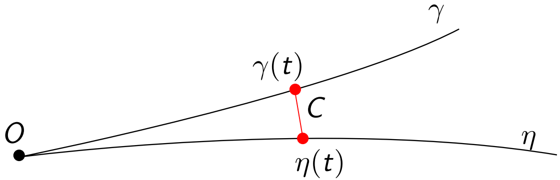


## Lemma (A)

$$\gamma \sim \eta \overset{def}{\Leftrightarrow} \sup \overline{\gamma(t), \eta(t)} < \infty \Leftrightarrow \sup \overline{\gamma(t), \eta(t)} \leqslant C \overset{def}{\Leftrightarrow} (\gamma|\eta) = \infty$$

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Lemma (B)

$\exists D > 1$ *s.t.* $\forall \gamma, \eta, \xi \in \mathcal{L}_O^\infty$, $(\gamma|\xi) \geqslant D^{-1} \min\{(\gamma|\eta), (\eta|\xi)\}$.

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## Proposition

$\exists d$ *:metric on* $\partial X$ *&* $0 < \exists \epsilon \leqslant 1$ *s.t.* $\forall [\gamma], [\eta] \in \partial X = \mathcal{L}_O^\infty / \sim$,

$$\frac{1}{2}\left(\frac{1}{D(\gamma|\eta)}\right)^\epsilon \leqslant d([\gamma],[\eta]) \leqslant \left(\frac{1}{(\gamma|\eta)}\right)^\epsilon.$$

# Coarse Cartan-Hadamard Theorem

Let $X$ be a proper coarsely convex space. The open cone over $\partial X$ is

$$\mathcal{O}\partial X := [0,\infty) \times \partial X / \{0\} \times \partial X$$

with metric: for $t, s \in [0,\infty)$; $x, y \in \partial X$

$$\overline{tx, sy} := |t-s| + \min\{t,s\} d_{\partial X}(x,y)$$

For $(t,x) \in [0,\infty) \times \partial X$, we denote $tx := [(t,x)]$.

Theorem (coarse Cartan-Hadamard)

*Let $X$ be a proper coarsely convex space. The "exponential" map*

$$\exp\colon \mathcal{O}\partial X \ni t[\gamma] \to \gamma(r(t)^{\frac{1}{\epsilon}}) \in X$$

*is coarsely homotopy equivalent map. Especially, $\mathcal{O}\partial X$ and $X$ are coarsely homotopy equivalent.*

*Here $r\colon [0,\infty) \to [0,\infty)$ is a contraction such that $r(t) \to \infty$ as $t \to \infty$.*

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## Theorem (coarse Cartan-Hadamard)

*Let $X$ be a proper coarsely convex space. The "exponential" map*

$$\exp\colon \mathcal{O}\partial X \ni t[\gamma] \to \gamma(r(t)^{\frac{1}{\epsilon}}) \in X$$

*is coarsely homotopy equivalent map. Especially, $\mathcal{O}\partial X$ and $X$ are coarsely homotopy equivalent.*

*Here $r\colon [0,\infty) \to [0,\infty)$ is a contraction such that $r(t) \to \infty$ as $t \to \infty$.*

# the Coarse Baum-Connes conjecture

- $Y$ : proper metric space
- $KX_\bullet(Y)$ : coarse $K$-homology of $Y$
  (ex. $KX_\bullet(\mathbb{Z}^n) \cong KX_\bullet(\mathbb{R}^n) \cong K_\bullet(\mathbb{R}^n)$)
- $C^*(Y)$ : a $C^*$-algebra constructed from $Y$, called Roe algebra, which is a non-equivariant analog of the reduced group $C^*$-algebra.

## Conjecture (coarse Baum-Connes)

*The following coarse assembly map is an isomorphism.*

$$\mu_Y \colon KX_\bullet(Y) \to K_\bullet(C^*(Y)).$$

# coarse homotopy invariance

## Proposition (coarse homotopy invariance)

*The coarse Baum-Connes conjecture is coarse homotopy invariant, that is, let $X$ and $Y$ be proper metric spaces, suppose*

- *$X$ and $Y$ are coarsely homotopy equivalent and,*
- *$X$ satisfies the coarse Baum-Connes conjecture,*

*then so does $Y$.*

## Proposition (Higson-Roe)

*Open cones over compact metrizable spaces satisfy coarse Baum-Connes conjecture.*

## Corollary (of Main theorem)

*Coarse Baum-Connes conjecture holds for proper coarsely convex spaces, especially, for locally finite systolic complexes.*

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## Example

The above corollary covers following spaces and groups.

- Proper Geodesic Gromov hyperbolic spaces.
- Proper CAT(0)-spaces, more generally, Busemann spaces.
- Artin groups of large types (NEW!).
- graphical $C(6)$-small cancellation groups (NEW!).
- Direct product of above spaces and groups (NEW!).

## Remark

*Osajda-Przytycki showed that Novikov conjecture for systolic groups holds.*

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# Semihyperbolic spaces and groups

## Proposition (FO)

*Let $X$ be a coarsely convex space. Then $X$ is semihyperbolic in the sense of Alonso and Bridson ('95).*

## Corollary (Aloso-Bridson, FO)

*Let $G$ be a group acting on a coarsely convex spaces $X$ properly and cocompactly by isometries. Then the following hold.*

1. *$G$ is finitely presented and of type $FP_\infty$.*
2. *$G$ satisfies a quadratic isoperimetric inequality.*

## Corollary

*The 3-dimensional discrete Heisenberg group never act geometrically on any coarsely convex space*

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# Dimension of $\partial G$ and cohomological dimension of $G$

We obtain a functional analytic characterization of the ideal boundary, and obtain the following.

## Proposition (Engel-Wulff, FO)

*Let $X$ be a coarsely convex space. Then $X$ admits an expanding and coherent combing in the sense of Engel and Wulff ('17). The ideal boundary $\partial X$ is homeomorphic to the combing corona of $X$.*

## Corollary (Engel-Wulff, FO)

*Let $G$ be a group acting geometrically on a proper coarsely convex space. If $G$ admits a finite model for the classifying space $BG$, then*

$$\mathrm{cd}(G) = \dim(\partial G) + 1.$$

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