演習問題 No. 13 の解答

1　(1) 合成関数の微分法を用いて，

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=a\frac{\partial z}{\partial x}+c\frac{\partial z}{\partial y}\,,\quad \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=b\frac{\partial z}{\partial x}+d\frac{\partial z}{\partial y}$$

(2) 合成関数の微分法をもう一度用いると，$f(x,y)$ は C^2 級関数より $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$ が成り立つことから，

$$\begin{aligned}
\frac{\partial^2 z}{\partial u^2}&=a\frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial u}\Big)+c\frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial u}\Big)=a\frac{\partial}{\partial x}\Big(a\frac{\partial z}{\partial x}+c\frac{\partial z}{\partial y}\Big)+c\frac{\partial}{\partial y}\Big(a\frac{\partial z}{\partial x}+c\frac{\partial z}{\partial y}\Big)\\
&=a^2\frac{\partial^2 z}{\partial x^2}+2ac\frac{\partial^2 z}{\partial x\partial y}+c^2\frac{\partial^2 z}{\partial y^2}\\
\frac{\partial^2 z}{\partial u\partial v}&=a\frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial v}\Big)+c\frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial v}\Big)=a\frac{\partial}{\partial x}\Big(b\frac{\partial z}{\partial x}+d\frac{\partial z}{\partial y}\Big)+c\frac{\partial}{\partial y}\Big(b\frac{\partial z}{\partial x}+d\frac{\partial z}{\partial y}\Big)\\
&=ab\frac{\partial^2 z}{\partial x^2}+(ad+bc)\frac{\partial^2 z}{\partial x\partial y}+cd\frac{\partial^2 z}{\partial y^2}\\
\frac{\partial^2 z}{\partial v^2}&=b\frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial v}\Big)+d\frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial v}\Big)=b\frac{\partial}{\partial x}\Big(b\frac{\partial z}{\partial x}+d\frac{\partial z}{\partial y}\Big)+d\frac{\partial}{\partial y}\Big(b\frac{\partial z}{\partial x}+d\frac{\partial z}{\partial y}\Big)\\
&=b^2\frac{\partial^2 z}{\partial x^2}+2bd\frac{\partial^2 z}{\partial x\partial y}+d^2\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

(3) A は直交行列であるから，

$$A\,{}^tA=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}a&c\\b&d\end{bmatrix}=\begin{bmatrix}a^2+b^2&ac+bd\\ac+bd&c^2+d^2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

よって，$a^2+b^2=c^2+d^2=1,\ ac+bd=0$ が成り立つから，

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=(a^2+b^2)\frac{\partial^2 z}{\partial x^2}+2(ac+bd)\frac{\partial^2 z}{\partial x\partial y}+(c^2+d^2)\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$

2　(1) $f_x=5x^4y^4,\ f_y=4x^5y^3$ より

$$f_{xx}=20x^3y^4\,,\qquad f_{xy}=f_{yx}=20x^4y^3\,,\qquad f_{yy}=12x^5y^2$$

(2) $f_x=2e^{2x}\cos y,\ f_y=-e^{2x}\sin y$ より

$$f_{xx}=4e^{2x}\cos y\,,\qquad f_{xy}=f_{yx}=-2e^{2x}\sin y\,,\qquad f_{yy}=-e^{2x}\cos y$$

(3) $f_x=yx^{y-1},\ f_y=\dfrac{\partial}{\partial y}(e^{y\log x})=x^y\log x$ より

$$\begin{aligned}
&f_{xx}=y(y-1)x^{y-2}\\
&f_{xy}=f_{yx}=yx^{y-1}\log x+x^y\,\frac{1}{x}=x^{y-1}(y\log x+1)\\
&f_{yy}=x^y(\log x)^2
\end{aligned}$$

(4)

$$f_x = \frac{1}{1+(y/x)^2}\Big(-\frac{y}{x^2}\Big) = -\frac{y}{x^2+y^2}\,, \qquad f_y = \frac{1}{1+(y/x)^2}\Big(\frac{1}{x}\Big) = \frac{x}{x^2+y^2}$$

より

$$\begin{aligned} f_{xx} &= -y\frac{\partial}{\partial x}\Big(\frac{1}{x^2+y^2}\Big) = \frac{2xy}{(x^2+y^2)^2} \\ f_{xy} &= f_{yx} = -\frac{1}{x^2+y^2} - y\frac{\partial}{\partial y}\Big(\frac{1}{x^2+y^2}\Big) = \frac{y^2-x^2}{(x^2+y^2)^2} \\ f_{yy} &= x\frac{\partial}{\partial y}\Big(\frac{1}{x^2+y^2}\Big) = -\frac{2xy}{(x^2+y^2)^2} \end{aligned}$$

3 (1) 合成関数の微分法を用いて

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x\cdot 2 + (-2y)\cdot 3 = 4(2t+1) - 6(3t-5) = -10t+34$$

(2) 合成関数の微分法を用いて

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{x^2+y^2}\sinh t + \frac{y}{x^2+y^2}\cosh t = \frac{2\sinh t\cosh t}{\cosh^2 t + \sinh^2 t} = \tanh 2t$$

4 (1) $z = x^2 - y^2$ とおくと，合成関数の微分法より

$$w_x = g'(z)\frac{\partial z}{\partial x} = 2xg'(z)\,, \qquad w_y = g'(z)\frac{\partial z}{\partial y} = -2yg'(z)$$

となるから，

$$yw_x + xw_y = 2xyg'(z) - 2xyg'(z) = 0$$

(2) $z = \dfrac{y}{x}$ とおくと，合成関数の微分法より

$$w_x = nx^{n-1}g(z) + x^n g'(z)\frac{\partial z}{\partial x} = nx^{n-1}g(z) - x^{n-2}yg'(z)\,, \qquad w_y = x^n g'(z)\frac{\partial z}{\partial y} = x^{n-1}g'(z)$$

となるから，

$$xw_x + yw_y = nx^n g(z) - x^{n-1}yg'(z) + x^{n-1}yg'(z) = nx^n g(z) = nw$$

5 合成関数の微分法より

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = r\Big(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\Big) \end{aligned}$$

が成り立つ（例題 5.5）．合成関数の微分法をもう一度用いると，$f(x,y)$ が C^2 級であることより，

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\Big(\frac{\partial z}{\partial x}\Big)\cos\theta + \frac{\partial}{\partial r}\Big(\frac{\partial z}{\partial y}\Big)\sin\theta \\ &= \Big[\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\partial x}\sin\theta\Big]\cos\theta + \Big[\frac{\partial^2 z}{\partial x\partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\Big]\sin\theta \\ &= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta \end{aligned}$$

$$
\begin{aligned}
\frac{1}{r}\frac{\partial^2 z}{\partial \theta^2} &= -\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)\sin\theta + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)\cos\theta - \frac{\partial z}{\partial x}\frac{\partial}{\partial \theta}\left(\sin\theta\right) + \frac{\partial z}{\partial y}\frac{\partial}{\partial \theta}\left(\cos\theta\right) \\
&= -r\left[-\frac{\partial^2 z}{\partial x^2}\sin\theta + \frac{\partial^2 z}{\partial y \partial x}\cos\theta\right]\sin\theta + r\left[-\frac{\partial^2 z}{\partial x \partial y}\sin\theta + \frac{\partial^2 z}{\partial y^2}\cos\theta\right]\cos\theta \\
&\quad - \frac{\partial z}{\partial x}\cos\theta - \frac{\partial z}{\partial y}\sin\theta \\
\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial^2 z}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 z}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\cos^2\theta - \frac{1}{r}\left[\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right] \\
&= \frac{\partial^2 z}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 z}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\cos^2\theta - \frac{1}{r}\frac{\partial z}{\partial r}
\end{aligned}
$$

が成り立つ．よって，

$$
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}
$$

が得られる．