演習問題 No. 7 の解答

1 (1) 両辺に x^4+4 を掛けると

$$\begin{aligned}4x^2 &= (Ax+B)(x^2+2x+2)+(Cx+D)(x^2-2x+2)\\ &= (A+C)x^3+(2A+B-2C+D)x^2+(2A+2B+2C-2D)x+(2B+2D)\end{aligned}$$

この式がすべての x に対して成り立たなければならないから，

$$A+C=0\,,\quad 2A+B-2C+D=4\,,\quad 2A+2B+2C-2D=0\,,\quad 2B+2D=0$$

が成り立つ．この連立方程式を解いて，$A=1,\ C=-1,\ B=D=0$ となる．

(2) (1) より，

$$\begin{aligned}\int\frac{4x^2}{x^4+4}\,dx &= \int\frac{x}{x^2-2x+2}\,dx-\int\frac{x}{x^2+2x+2}\,dx\\ &= \int\frac{(x-1)+1}{(x-1)^2+1}\,dx-\int\frac{(x+1)-1}{(x+1)^2+1}\,dx\\ &= \frac{1}{2}\log\Big(\frac{x^2-2x+2}{x^2+2x+2}\Big)+\tan^{-1}(x-1)+\tan^{-1}(x+1)\end{aligned}$$

2 (1)

$$\frac{2}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

として，A,B,C を求めると，$A=1,\ B=C=-1$ となるから，

$$\int\frac{2}{(x-1)(x^2+1)}=\int\frac{1}{x-1}\,dx-\int\frac{x+1}{x^2+1}\,dx=\log|x-1|-\frac{1}{2}\log(x^2+1)-\tan^{-1}x$$

(2)

$$\frac{3x}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

として，A,B,C を求めると，$A=-1,\ B=C=1$ となるから，

$$\begin{aligned}\int\frac{3x}{x^3+1}\,dx &= -\int\frac{1}{x+1}\,dx+\int\frac{x+1}{x^2-x+1}\,dx=-\log|x+1|+\int\frac{(x-1/2)+3/2}{(x-1/2)^2+3/4}\,dx\\ &= -\log|x+1|+\int\frac{(x-1/2)}{(x-1/2)^2+3/4}\,dx+\frac{3}{2}\int\frac{1}{(x-1/2)^2+3/4}\,dx\\ &= -\log|x+1|+\frac{1}{2}\log|(x-1/2)^2+3/4|+\frac{3}{2}\frac{1}{\sqrt{3}/2}\tan^{-1}\Big(\frac{x-1/2}{\sqrt{3}/2}\Big)\\ &= -\log|x+1|+\frac{1}{2}\log(x^2-x+1)+\sqrt{3}\tan^{-1}\Big(\frac{2x-1}{\sqrt{3}}\Big)\end{aligned}$$

(3) $x^4+x^2+1=(x^2+1)^2-x^2=(x^2-x+1)(x^2+x+1)$ であるから，

$$\frac{2}{x^4+x^2+1}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2+x+1}$$

として，A, B, C を求めると，$A=-1,\ B=C=D=1$ となるから，

$$\begin{aligned}\int\frac{2}{x^4+x^2+1}\,dx&=\int\frac{-x+1}{x^2-x+1}\,dx+\int\frac{x+1}{x^2+x+1}\,dx\\&=\int\frac{-(x-1/2)+1/2}{(x-1/2)^2+3/4}\,dx+\int\frac{(x+1/2)+1/2}{(x+1/2)^2+3/4}\,dx\\&=-\frac{1}{2}\log(x^2-x+1)+\frac{1}{2}\frac{1}{\sqrt{3}/2}\tan^{-1}\Bigl(\frac{x-1/2}{\sqrt{3}/2}\Bigr)\\&\quad+\frac{1}{2}\log(x^2+x+1)+\frac{1}{2}\frac{1}{\sqrt{3}/2}\tan^{-1}\Bigl(\frac{x+1/2}{\sqrt{3}/2}\Bigr)\\&=\frac{1}{2}\log\Bigl(\frac{x^2+x+1}{x^2-x+1}\Bigr)+\frac{1}{\sqrt{3}}\tan^{-1}\Bigl(\frac{2x-1}{\sqrt{3}}\Bigr)+\frac{1}{\sqrt{3}}\tan^{-1}\Bigl(\frac{2x+1}{\sqrt{3}}\Bigr)\end{aligned}$$

3 (1) $t=\tan\dfrac{x}{2}$ とおくと，

$$\sin x=\frac{2t}{1+t^2},\qquad \cos x=\frac{1-t^2}{1+t^2},\qquad \frac{dx}{dt}=\frac{2}{1+t^2}$$

となるから，

$$\begin{aligned}\int\frac{1}{1+\sin x+\cos x}\,dx&=\int\frac{1}{1+\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt=\int\frac{1}{t+1}\,dt\\&=\log(t+1)=\log\Bigl(\tan\frac{x}{2}+1\Bigr)\end{aligned}$$

(2) $t=\tan x$ とおくと，

$$\sin^2 x=\frac{t^2}{1+t^2},\qquad \cos^2 x=\frac{1}{1+t^2},\qquad \frac{dx}{dt}=\frac{1}{1+t^2}$$

となるから，

$$\int\frac{\cos^2 x}{\sin^4 x}\,dx=\int\frac{\dfrac{1}{1+t^2}}{\Bigl(\dfrac{t^2}{1+t^2}\Bigr)^2}\frac{1}{1+t^2}\,dt=\int\frac{1}{t^4}\,dt=-\frac{1}{3}\frac{1}{t^3}=-\frac{1}{3}\frac{1}{\tan^3 x}$$

(3) $t=\sqrt{x+1}$ とおくと，$x=t^2-1$ より，$\dfrac{dx}{dt}=2t$ となるから，

$$\begin{aligned}\int\frac{\sqrt{x+1}}{x}\,dx&=\int\frac{t}{t^2-1}2t\,dt=\int\frac{2t^2}{t^2-1}\,dt=\int\Bigl(2+\frac{1}{t-1}-\frac{1}{t+1}\Bigr)\,dt\\&=2t+\log\Bigl|\frac{t-1}{t+1}\Bigr|=2\sqrt{x+1}+\log\frac{|\sqrt{x+1}-1|}{\sqrt{x+1}+1}\end{aligned}$$

(4) $t=\sqrt{e^x-1}$ とおくと，$e^x=t^2+1$ より，

$$e^x\frac{dx}{dt}=2t,\qquad \frac{dx}{dt}=\frac{2t}{t^2+1}$$

となるから，

$$\int\frac{1}{\sqrt{e^x-1}}\,dx=\int\frac{1}{t}\frac{2t}{t^2+1}\,dt=2\int\frac{1}{t^2+1}\,dt=2\tan^{-1}t=2\tan^{-1}(\sqrt{e^x-1})$$

4 (1) $t = \sqrt[3]{x+1}$ とおくと，$x = t^3 - 1$ より，$\dfrac{dx}{dt} = 3t^2$ となるから，

$$\begin{aligned}\int_{-1}^{7} \frac{1}{1+\sqrt[3]{x+1}}\,dx &= \int_0^2 \frac{3t^2}{1+t}\,dt \\ &= 3\int_0^2 \left(t - 1 + \frac{1}{1+t}\right) dt \\ &= 3\left[\frac{t^2}{2} - t + \log|1+t|\right]_0^2 = 3\log 3\end{aligned}$$

(2) $t = \tan\dfrac{x}{2}$ とおくと，

$$\sin x = \frac{2t}{1+t^2}\,, \qquad \cos x = \frac{1-t^2}{1+t^2}\,, \qquad \frac{dx}{dt} = \frac{2}{1+t^2}$$

となるから，

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx &= \int_0^1 \frac{1}{2+\dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int_0^1 \frac{2}{3+t^2}\,dt \\ &= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1 = \frac{\pi}{3\sqrt{3}}\end{aligned}$$