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Sound Attenuation in a One-Dimensional Periodic Inhomogeneous Medium

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The thermal transport process plays an important role in the sound attenuation in inhomogeneous media. This problem is best exemplified by the sound attenuation in a one-dimensional space with periodically alternating media. Using the transfer matrix technique, we give here a general theory of the sound propagation and attenuation in such a periodic system.

List of Symbols

ρ	density
P	pressure
T	temperature
s	entropy per unit mass
k	wave number of compressional wave
q	wave number of thermal wave
K	Bloch's wave number of compressional wave
Q	Bloch's wave number of thermal wave
λ_c	wavelength of compressional wave
λ_t	wavelength of thermal wave
v	velocity
α	coefficient of thermal expansion
β	isothermal compressibility
$\gamma=c_P/c_V$	ratio of specific heats at constant pressure and constant volume
$\tau=\beta/\gamma$	adiabatic compressibility
C	velocity of sound
δ	sound attenuation coefficient
d	dimension of the inhomogeneity
ε	volume fraction of the first component
κ	thermal conductivity
$\chi=\kappa/\rho c_P$	thermometric conductivity
$n=\sqrt{\omega/2\chi}$	
ω	angular frequency

§1. Introduction

The problem of the sound wave propagation in a suspension, where solid material is suspended in a liquid, and in an emulsion, where liquid particles are suspended in a liquid of different sort, has received great attention in connection with acoustics in ocean, biological systems and chemical systems, such as blood and latex. This problem has been also frequently an object of theoretical investigations.[1-5]

Allinson and Richardson[6] studied the ultrasonic attenuation in emulsions of benzene in water and water in benzene. They found that the attenuation is greater than the values estimated on the basis of the mechanism of the scattering and the viscous drag, which are crucial to the sound attenuation in suspensions. By that time, Isakovich[7] had already shown that there must be a special mechanism for the attenuation of sound in an emulsion. He reasoned that sound propagation in an emulsion can produce temperature difference at the interface between the particle and the suspending fluid, causing a heat flow from one component to another. Such a thermal diffusion process leads to entropy production, and this in turn gives rise to a considerable sound attenuation. This is an important mechanism which is absent in the homogeneous fluid.

Physically speaking, at low frequencies, the temperature difference between the particle and the suspending fluid will equilibrate in a unit cycle of sound wave oscillation. On the other hand, at high frequencies, only a small portion of the particle volume near the surface takes part in the thermal conduction process. Hence the temperature difference will not disappear sufficiently during the oscillation in this case.

Isakovich's consideration is indeed very ingenious but his formulation was rather intuitive. An extended theoretical description of the problem was given later by Epstein and Carhart[8] in 1952. They considered a single fluid sphere suspended in a fluid and included both viscous and thermal dissipations. A similar problem of a solid suspended particle was solved by Allerga and Hawley[9] on the basis of Epstein and Carhart's formalism. These authors pointed out that Isakovich's prediction is justified in the long-wavelength limit.

The analyses given by the above authors, however, have been limited to the case where the frequency is so small that the wavelength of the compressional wave is much larger than those of the thermal and viscous waves as well as the particle size. It is worthwhile to give a theory which enables us to calculate the propagation and the attenuation of the sound wave with shorter wavelengths. In fact, such a general theory is possible if we restrict ourselves to one-dimensional periodic systems. The aim of this paper is to solve rigorously the one-dimensional problem to work out dispersion relation of the sound wave and the coefficient of sound attenuation in a higher frequency regime as well. Our result includes Isakovich and Epstein-Carhart's ones in a lower frequency region. The one-dimensional periodic system can be realized, for example, by a multi-layer of a lipid-water system.

The sound wave propagation through such a system is described exactly by means of the transfer matrix formalism as shown in §2. The one-dimensional problem and its prediction can be used as a criterion for the existing approximation methods. In §3, Isakovich's theory is examined on the basis of our rigorous predictions. We show also a discrepancy between Isakovich's results and ours in a higher frequency range. A brief explanation of Isakovich's theory is included in Appendices A and B.

§2. Method for Calculating Sound Velocity and Attenuation

We present here a general method for calculating the sound velocity and attenuation in a one-dimensional periodic inhomogeneous medium. This system provides us a rigorously solvable example, which can be used as a criterion for examining the approximation methods developed such as by Isakovich and Epstein-Carhart.

We start with the fundamental equations of fluid mechanics.[10] For simplicity we neglect the bulk viscosity. Actually, our formalism will not be affected by an inclusion of the viscous effect. The fundamental equations describe the conservation of mass (continuity equation) and momentum (Euler's equation), and also the entropy balance:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial P}{\partial x}, \tag{2}$$

$$\rho T\left(\frac{\partial s}{\partial t}+v\frac{\partial s}{\partial x}\right)=\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right), \tag{3}$$

where ρ denotes the density of fluid, v the velocity, P the pressure, T the temperature, s the entropy per unit mass and κ the thermal conductivity. Since the oscillations caused by the sound wave are small, we can linearize these equations. We put

$$P=P_0+\delta P,\quad T=T_0+\delta T,\quad v=\delta v,$$
$$\rho=\rho_0+\delta\rho,\quad s=s_0+\delta s. \tag{4}$$

In the above, P_0, T_0, $\cdots$ stand for the values in equilibrium (in the absence of the acoustic field), and δP, δT, $\cdots$ are their variations when the sound wave is applied ($\delta P \ll P_0$, $\delta T \ll T_0$, $\cdots$). Retaining in (1), (2) and (3) only to the first-order terms, we obtain

$$\frac{\partial \delta\rho}{\partial t}+\rho\frac{\partial v}{\partial x}=0, \tag{5}$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial \delta P}{\partial x}, \tag{6}$$

$$\frac{\partial \delta s}{\partial t}=\frac{1}{\rho T}\frac{\partial}{\partial x}\left(\kappa\frac{\partial \delta T}{\partial x}\right), \tag{7}$$

(hereafter the suffix 0 attached to the constant quantities P_0 etc. is omitted after the linearization). Elimination of v from (5) and (6), then, yields

$$\frac{\partial^2 \delta\rho}{\partial t^2}=\rho\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \delta P}{\partial x}\right). \tag{8}$$

Note that in our medium κ, ρ etc. still depend on x. Our next objective is to eliminate the variables ρ and s. To do so, we make use of two additional equations:

$$\rho=\rho(T, P), \quad s=s(T, P), \tag{9}$$

namely, the equations of state. Then it follows that

$$\delta s=\left(\frac{\partial s}{\partial T}\right)_{\mathrm{P}}\delta T+\left(\frac{\partial s}{\partial P}\right)_{\mathrm{T}}\delta P=\frac{c_{\mathrm{P}}}{T}\delta T-\frac{\alpha}{\rho}\delta P, \tag{10}$$

$$\delta\rho=\left(\frac{\partial \rho}{\partial T}\right)_{\mathrm{P}}\delta T+\left(\frac{\partial \rho}{\partial P}\right)_{\mathrm{T}}\delta P=-\rho\alpha\delta T+\rho\beta\delta P, \tag{11}$$

where c_{P} is the specific heat at constant pressure; $\alpha=-(1/\rho)(\partial\rho/\partial T)_{\mathrm{P}}$ the coefficient of thermal expansion, and $\beta=(1/\rho)(\partial\rho/\partial P)_{\mathrm{T}}$ the isothermal compressibility. This isothermal compressibility is related to the adiabatic compressibility τ through the well-known thermodynamic formula

$$\tau=\frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_{\mathrm{s}}=\frac{\beta}{\gamma}=\beta-\frac{T\alpha^2}{\rho c_{\mathrm{P}}}. \tag{12}$$

Substituting (10) into (7) and also (11) into (8), we arrive at

$$\frac{\partial \delta T}{\partial t}-\frac{1}{\rho c_{\mathrm{P}}}\frac{\partial}{\partial x}\left(\kappa\frac{\partial \delta T}{\partial x}\right)-\frac{T\alpha}{\rho c_{\mathrm{P}}}\frac{\partial \delta P}{\partial t}=0, \tag{13}$$

and

$$\alpha\frac{\partial^2 \delta T}{\partial t^2}-\beta\frac{\partial^2 \delta P}{\partial t^2}+\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \delta P}{\partial x}\right)=0. \tag{14}$$

These two equations are the fundamental equations of our problem. Let us restrict our attention to the periodic system of two alternating media (1 and 2) as illustrated in Fig. 1. Hereafter we shall denote by the indices 1 and 2 the quantities referring, respectively, to the first and second media. The period D is the sum of the thickness of each medium d_1, $d_2 (D=d_1+d_2)$. Our main purpose is to seek for the solution of the two coupled equations and to get δP in the form $\delta P=f(x)e^{iKx}$ where $f(x+D)=f(x)$. Here K corresponds to Bloch's wave number in solid state physics or, to be more specific, in the Kronig-Penny model. The coefficient of the sound attenuation is provided by its imaginary part.

It is reasonable to postulate the following forms of δT and δP for a homogeneous medium:

$$\delta T,\ \delta P \propto \exp\{i(px-\omega t)\}. \tag{15}$$

Then (13) and (14) reduce, respectively to

$$\left(p^2-\frac{i\omega\rho c_{\mathrm{P}}}{\kappa}\right)\delta T+\frac{i\omega\alpha T}{\kappa}\delta P=0, \tag{16}$$

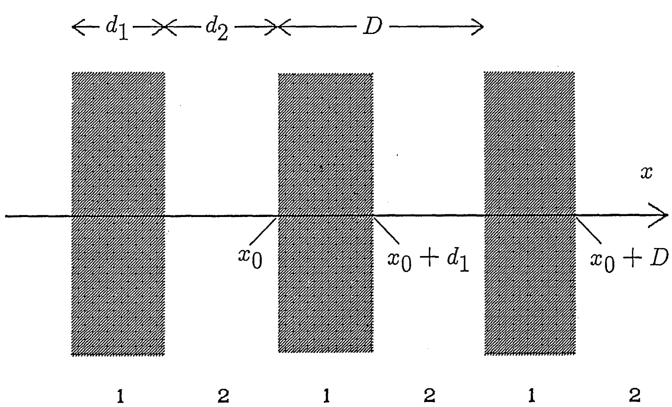

Fig. 1. Geometry of the one-dimensional periodic inhomogeneous system.

$$\rho\omega^2\alpha\delta T+\left(p^2-\frac{\omega^2}{C_T^2}\right)\delta P=0, \tag{17}$$

where we have employed the notations

$$C_s^2=\left(\frac{\partial P}{\partial\rho}\right)_s=\frac{1}{\rho\tau},\quad C_T^2=\left(\frac{\partial P}{\partial\rho}\right)_T=\frac{C_s^2}{\gamma}=\frac{1}{\rho\beta}, \tag{18}$$

for the adiabatic and isothermal sound velocity. If we put $\alpha=0$, equations (16) and (17) become mutually independent. They are, respectively, the heat-conduction equation for the thermal wave and the wave equation for the compressional wave. In this case, the wave number of the compressional wave k takes on the simple form

$$k=\frac{\omega}{C_T}=\frac{\omega}{C_s}, \tag{19}$$

which indicates the absence of the sound attenuation. In contrast to this, the wave number q of the thermal wave is

$$q=(1+i)\sqrt{\frac{\omega\rho c_P}{2\kappa}}=(1+i)\sqrt{\frac{\omega}{2\chi}}. \tag{20}$$

This shows that the thermal wave is rapidly damped with increasing x. When α is not equal to zero, the compressional wave (pressure mode) and the heat wave (temperature mode) are coupled. The compatibility condition for the two equations for δT and δP ((16) and (17)) reads

$$p^4-p^2\left(\frac{\omega^2}{C_T^2}+\frac{i\omega}{\chi}\right)+\frac{i\omega^3}{\chi C_s^2}=0, \tag{21}$$

providing the relation between p and ω. For a given ω, eq. (21) is a quadratic equation for p^2. The solution of eq. (21) which becomes identical to ω^2/C_T^2 in the limit $\alpha\to 0$ is written simply as k^2 (compressional mode), while the other solution, which approaches $i\omega/\chi$ as $\alpha\to 0$ is denoted by q^2 (thermal mode).

It is instructive to examine the asymptotic expressions of k and q in high and low frequency limits (see ref. 10 for details).

(1) In the case of $\omega\ll C^2/\chi$,

$$k=\frac{\omega}{C_s}+i\frac{\omega^2\chi}{2C_s}\left(\frac{1}{C_T^2}-\frac{1}{C_s^2}\right), \tag{22}$$

and

$$q=(1+i)\sqrt{\frac{\omega}{2\chi}}. \tag{23}$$

Equation (22) indicates the propagation of the sound with the adiabatic velocity C_s. The imaginary part of (22) is nothing but Kirchhoff's formula for the sound attenuation.

(2) In the case of $\omega\gg C^2/\chi$,

$$k=\frac{\omega}{C_T}+i\frac{C_T}{2\chi C_s^2}(C_s^2-C_T^2), \tag{24}$$

and

$$q=(1+i)\sqrt{\frac{\omega c_V}{2\chi c_P}}. \tag{25}$$

In this case the sound propagates with the isothermal velocity C_T, which is always less than C_s (see (18)). The attenuation coefficient is independent of the frequency and is inversely proportional to the thermal conductivity.

Both the compressional mode and the thermal mode consists of two components, δT and δP. The ratio between these components is determined by eq. (16) (or, equivalently by eq. (17)). For the compressional mode and the thermal mode, we define

$$\delta T=a(\omega)\delta P,\quad b(\omega)\delta T=\delta P, \tag{26}$$

respectively. In the above,

$$a(\omega)=-\frac{i\omega\alpha T/\kappa}{k^2-(i\omega\rho c_P/\kappa)}, \tag{27}$$

$$b(\omega)=-\frac{\rho\omega^2\alpha}{q^2-(\omega^2/C_T^2)}. \tag{28}$$

Again, notice that in the case of $\alpha=0$ both $a(\omega)$ and $b(\omega)$ vanish. Using thus defined $a(\omega)$ and $b(\omega)$, we introduce the following four base vectors:[1]

$$u(k)=\begin{pmatrix}1\\a(\omega)\end{pmatrix}\exp(ikx), \tag{29}$$

$$u(-k)=\begin{pmatrix}1\\a(\omega)\end{pmatrix}\exp(-ikx), \tag{30}$$

$$w(q)=\begin{pmatrix}b(\omega)\\1\end{pmatrix}\exp(iqx), \tag{31}$$

$$w(-q)=\begin{pmatrix}b(\omega)\\1\end{pmatrix}\exp(-iqx). \tag{32}$$

[1] If the medium is semi-finite, $0\leq x<\infty$, the solutions (30) and (32) should be excluded. Our system considered in what follows is, however, inhomogeneous and hence the solutions (30) and (32) representing the backward propagation should also be considered.

With these notations, the coupled state of the compressional and heat waves is represented as

$$\psi=\begin{pmatrix}\delta P\\ \delta T\end{pmatrix}=c_1u(k)+c_2u(-k)+c_3w(q)+c_4w(-q). \tag{33}$$

The set of coefficients (c_1, c_2, c_3, c_4) uniquely specifies the state.

Now we are in a position to deal with the specific problem of the one-dimensional periodic inhomogeneous medium as already illustrated in Fig. 1. The boundary conditions at the interfaces are

$$\delta P_1=\delta P_2,\quad \frac{1}{\rho_1}\frac{\partial\delta P_1}{\partial x}=\frac{1}{\rho_2}\frac{\partial\delta P_2}{\partial x}, \tag{34}$$

$$\delta T_1=\delta T_2,\quad \kappa_1\frac{\partial\delta T_1}{\partial x}=\kappa_2\frac{\partial\delta T_2}{\partial x}. \tag{35}$$

In (34) and (35), the latter conditions can be easily obtained by integrating eqs. (13) and (14) for inhomogeneous media. Especially, the latter condition of (34) enforces the continuity of the velocity.

Our problem is now reduced to find the transformation of the wave function (33) after the waves are propagated over a single period D. In view of (33), it is enough to know how the coefficients (c_1, c_2, c_3, c_4) at $x=x_0$ are transformed into the new set of coefficients (c_1', c_2', c_3', c_4') at $x=x_0+D$. To this end, we define the next two vectors:

$$\Phi_i(x)=\begin{pmatrix}\delta P_i\\ \dfrac{1}{\rho_i}\dfrac{\partial}{\partial x}\delta P_i\\ \delta T_i\\ \kappa_i\dfrac{\partial}{\partial x}\delta T_i\end{pmatrix}, \tag{36}$$

and

$$\phi_i(x)=\begin{pmatrix}c_1\exp(ik_ix)\\ c_2\exp(-ik_ix)\\ c_3\exp(iq_ix)\\ c_4\exp(-iq_ix)\end{pmatrix}, \tag{37}$$

where $i=1,2$ corresponding to the first and second component respectively. We then find from (33) that $\Phi_i(x)$ and $\phi_i(x)$ are related to each other by

$$\Phi_i(x)=M_i\phi_i(x),\quad \phi_i(x)=M_i^{-1}\Phi_i(x), \tag{38}$$

with

$$M_i=\begin{pmatrix}1 & 1 & b_i & b_i\\ \dfrac{ik_i}{\rho_i} & -\dfrac{ik_i}{\rho_i} & \dfrac{ib_iq_i}{\rho_i} & -\dfrac{ib_iq_i}{\rho_i}\\ a_i & a_i & 1 & 1\\ i\kappa_ia_ik_i & -i\kappa_ia_ik_i & i\kappa_iq_i & -i\kappa_iq_i\end{pmatrix}, \tag{39}$$

and its inverse matrix

$$M_i^{-1}=\frac{1}{2(1-a_ib_i)}\begin{pmatrix}1 & \dfrac{\rho_i}{ik_i} & -b_i & -\dfrac{b_i}{i\kappa_ik_i}\\ 1 & -\dfrac{\rho_i}{ik_i} & -b_i & \dfrac{b_i}{i\kappa_ik_i}\\ -a_i & -\dfrac{a_i\rho_i}{iq_i} & 1 & \dfrac{1}{i\kappa_iq_i}\\ -a_i & \dfrac{a_i\rho_i}{iq_i} & 1 & -\dfrac{1}{i\kappa_iq_i}\end{pmatrix}. \tag{40}$$

It is advantageous to introduce the transfer matrix M defined by

$$\phi_1(x_0+D+0^+)=M\phi_1(x_0+0^+), \tag{41}$$

or equally

$$\begin{pmatrix}c_1'\exp\{ik_1(x_0+D+0^+)\}\\ c_2'\exp\{-ik_1(x_0+D+0^+)\}\\ c_3'\exp\{iq_1(x_0+D+0^+)\}\\ c_4'\exp\{-iq_1(x_0+D+0^+)\}\end{pmatrix}=M\begin{pmatrix}c_1\exp\{ik_1(x_0+0^+)\}\\ c_2\exp\{-ik_1(x_0+0^+)\}\\ c_3\exp\{iq_1(x_0+0^+)\}\\ c_4\exp\{-iq_1(x_0+0^+)\}\end{pmatrix}, \tag{42}$$

where 0^+ is an infinitesimal positive quantity (x_0 is shown in Fig. 1). Imposition of the boundary conditions (34) and (35) produces

$$M=M_1^{-1}M_2E_2M_2^{-1}M_1E_1, \tag{43}$$

where

$$E_i=\begin{pmatrix} \exp(ik_id_i) & 0 & 0 & 0 \\ 0 & \exp(-ik_id_i) & 0 & 0 \\ 0 & 0 & \exp(iq_id_i) & 0 \\ 0 & 0 & 0 & \exp(-iq_id_i) \end{pmatrix}, \tag{44}$$

(see Appendix C).

The 4×4 matrix M is diagonalized. The diagonalized matrix can be written in the following form;

$$X^{-1}MX=\begin{pmatrix} \exp(iKD) & 0 & 0 & 0 \\ 0 & \exp(-iKD) & 0 & 0 \\ 0 & 0 & \exp(iQD) & 0 \\ 0 & 0 & 0 & \exp(-iQD) \end{pmatrix}, \tag{45}$$

where K and Q are complex wave numbers corresponding to the compressional and heat wave, respectively, determining the dispersion relation of each mode. Both K and Q are analogous to Bloch's wave number. We choose Im $K>0$ and Im $Q>0$. From now on, we consider the semi-infinite medium, $0\leq x<\infty$. Then the solutions corresponding to e^{-iKx} and e^{-iQx} should be excluded from the boundary condition at $x\to\infty$. The imaginary part of K gives the sound attenuation. Within our formalism, $\cos(KD)$ and $\cos(QD)$ can be obtained by solving the next two equations

$$\cos(KD)+\cos(QD)=\frac{1}{2}\,\mathrm{Tr}(M), \tag{46}$$

$$\cos^2(KD)+\cos^2(QD)=1+\frac{1}{4}\,\mathrm{Tr}(M^2). \tag{47}$$

The case of $\alpha_1=\alpha_2=0$ is much simpler. In this case $a_i(\omega)=b_i(\omega)=0$ (see (27) and (28)), so that each of M_i and M_i^{-1} splits into two 2×2 matrices. Then we can readily write down the dispersion relations of the compressional and thermal waves:

$$\cos(KD)=\cos(k_1d_1)\cos(k_2d_2)-\frac{1}{2}\frac{(k_1/\rho_1)^2+(k_2/\rho_2)^2}{(k_1/\rho_1)(k_2/\rho_2)}\sin(k_1d_1)\sin(k_2d_2), \tag{48}$$

and

$$\cos(QD)=\cos(q_1d_1)\cos(q_2d_2)-\frac{1}{2}\frac{(\kappa_1q_1)^2+(\kappa_2q_2)^2}{\kappa_1q_1\kappa_2q_2}\sin(q_1d_1)\sin(q_2d_2). \tag{49}$$

In the above use has been made of the following definitions:

$$k_i=\frac{\omega}{C_{Ti}}=\frac{\omega}{C_{si}},\quad q_i=\sqrt{\frac{i\omega}{\chi_i}}, \tag{50}$$

($i=1,2$; see (19) and (20)).

The expression of the attenuation coefficient for the case of non-zero α_1 and α_2 is also available. However, it is too lengthy to be written out explicitly here. Instead we are contended with the presentation of some graphical plots, which is relegated to §3.

In connection with Isakovich's theory, we have to be careful about the choice of parameter values. As shown in Appendix A, Isakovich predicted the behavior of the attenuation coefficient δ as $\delta\propto\sqrt{\omega}$ in a high-frequency region. In that region, the thickness of each layer $d(\sim d_1, d_2)$ is much larger than the wavelength of the thermal wave, but it must be much shorter than the wavelength of the compressional wave. Therefore, the frequency

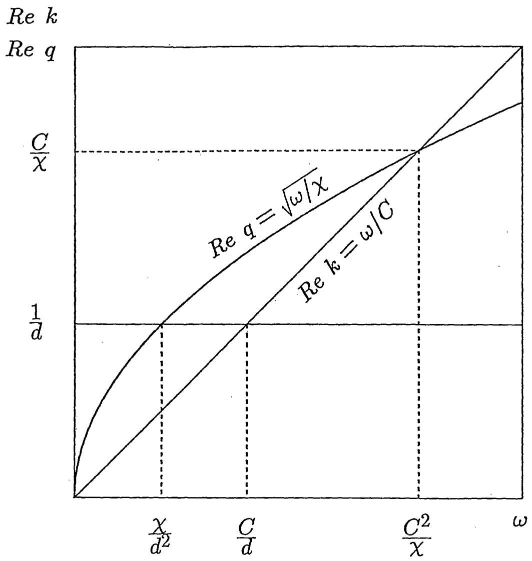

Fig. 2. The dispersion relation of the compressional wave k and the thermal wave q. The parameter d gives the dimension of the inhomogeneity. There are four different frequency regions divided by three critical frequencies.

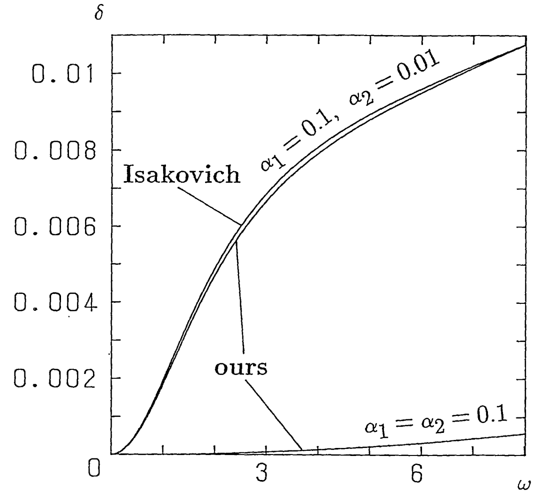

Fig. 3. The attenuation coefficient δ as a function of frequency ω in an inhomogeneous system. The parameters are set as $T=1.0$, $d_1=d_2=0.5$, $\rho_1=\rho_2=1.0$, $c_{P1}=c_{P2}=2.0$, $C_{T1}=C_{T2}=10.0$, $\kappa_1=\kappa_2=0.1$. The frequency covers regions 1 and 2 in this graph. This graph is written in arbitrary scales.

region, where $\delta \propto \sqrt{\omega}$ holds, should be confined to the range

$$\mathrm{Re}\, k \ll \frac{1}{d} \ll \mathrm{Re}\, q. \tag{51}$$

In other words,

$$\frac{\chi}{d^2} \ll \omega \ll \frac{C}{d}. \tag{52}$$

This implies that Isakovich's result $\delta \propto \sqrt{\omega}$ can be obtained in a system such that

$$\frac{1}{d} \ll \frac{C}{\chi}. \tag{53}$$

In the above, C is the velocity of the sound ($\sim C_T, C_s$). In addition to the two frequencies appearing in (52), there exists another critical frequency C^2/χ at which the wavelength of the thermal wave is equal to that of the compressional wave. For the system satisfying equation (53) we get

$$\frac{\chi}{d^2} \ll \frac{C}{d} \ll \frac{C^2}{\chi}. \tag{54}$$

All of these facts are well illustrated in Fig. 2.

Details are discussed in the following section.

§3. Discussions

In this section, we examine the validity of Isakovich's predictions by comparing with our rigorous formalism.

Figure 3 is a graph of the attenuation coefficient δ plotted against the frequency ω for typical values of α_1 and α_2. For the simplicity we consider the case where $d_1=d_2=d$, $\rho_1=\rho_2=\rho$, $c_{P1}=c_{P2}=c_P$ and $\kappa_1=\kappa_2=\kappa$. The results of Isakovich's theory is also shown. Within the framework of Isakovich's theory, the sound attenuation which occurs in the homogeneous medium is disregarded. This can be easily understood by equalizing α_1 and α_2 in his expression (A·7). Actually, the attenuation occurs even in the homogeneous medium within our formalism. This fact can be realized by (22) or (24). We have included this effect by putting the pressure field from the beginning in the general form $\delta P \propto e^{i(px-\omega t)}$. Spatial variation in the pressure results in the spatial variation in the temperature (see (26)) causing a heat flow even if the medium is homogeneous. This effect is not taken into account in Isakovich's theory.

In the two component medium, however, the dominant contribution to the total attenua-

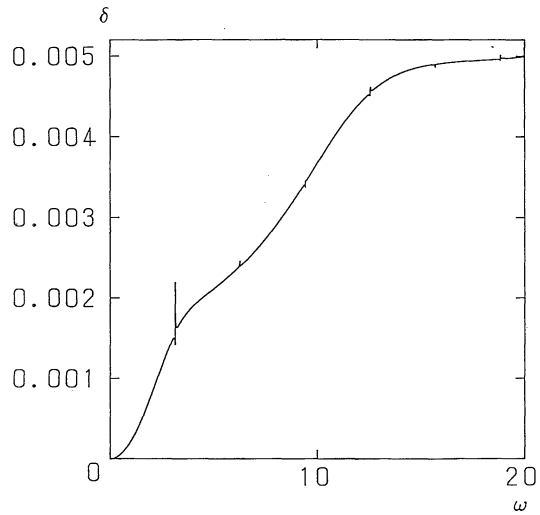

Fig. 4. The attenuation coefficient δ as a function of frequency ω in an inhomogeneous system. The parameters are set as $T=1.0$, $d_1=d_2=0.5$, $\rho_1=\rho_2=2.0$, $c_{P1}=c_{P2}=2.0$, $C_{T1}=C_{T2}=1.0$, $\kappa_1=\kappa_2=1.0$, $\alpha_1=0.1$, $\alpha_2=0.01$. The frequency covers regions 1, 2, 3 and 4 in this graph. This graph is written in arbitrary scales.

tion is attributed to the inhomogeneity of the medium. Therefore, the above criticism against Isakovich's theory is irrelevant in the case of the inhomogeneous system. We can observe this situation from Fig. 3.

Moreover, we would like to emphasize that our general formulation allows us to predict the behavior of the sound attenuation in a wider frequency range. We have divided the frequencies into four different regions by the three critical frequencies χ/d^2, C/d and C^2/χ. We consider the system where the condition (53) is satisfied. The attenuation curve over a wider frequency range is exhibited in Fig. 4 for certain parameters. It is observed that the behavior of the attenuation changes at the critical frequencies.

Let us consider here the physical meaning of each of the four frequency regions. For this purpose we introduce the notation λ_c and λ_t for the wavelengths of the compressional and thermal waves (see Fig. 2).

(1) region 1: $\omega<\chi/d^2$, $d<\lambda_t<\lambda_c$
The frequency is so small that the thermal equilibrium is nearly established in each component at every moment of time. The attenuation coefficient is proportional to ω^2, which agrees with the attenuation behavior in the homogeneous media. However, the physical mechanism of our attenuation is quite different from that in the homogeneous case.

(2) region 2: $\chi/d^2<\omega<C/d$, $\lambda_t<d<\lambda_c$
The temperature between the two components differences does not die out in one period of the pressure oscillation. The fact that the attenuation coefficient is proportional to $\sqrt{\omega}$ in this region was first pointed out by Isakovich and has been confirmed by our general theory.

(3) region 3: $C/d<\omega<C^2/\chi$, $\lambda_t<\lambda_c<d$
The sound is absorbed at each component in the same way as in a homogeneous system. Therefore the attenuation coefficient is again proportional to ω^2.

(4) region 4: $C^2/\chi<\omega$, $\lambda_c<\lambda_t<d$
The wavelength of the compressional wave is smaller than that of the thermal wave. Then the attenuation coefficient does not depend on the frequency as can be noticed from (24).

Isakovich's predictions are valid only in the regions 1 and 2 while ours in all regions.

It is noted that there appear some small dips and cusps which are due to the resonance effect inherent in the one-dimensional periodic system. Since the condition of the resonance effect is given by

$$2D=\mathrm{n}\lambda_c, \tag{55}$$

(n: integer) the dips and cusps will arise near the frequencies

$$\omega=\frac{C}{D}\pi\mathrm{n}. \tag{56}$$

We have set $C/D=1$ in Fig. 4, exhibiting dips and cusps at the frequencies π, 2π, $3\pi\cdots$.

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Appendix A: Review of Isakovich's Theory

In this appendix we present a concise description of Isakovich's theory. Isakovich assumed that the medium is "macroscopically homogeneous." A medium is called "macroscopically homogeneous," if it contains two or more components and can be divided into regions, which are small in comparison with the wavelength of the sound wave but still contain many heterogeneous parts. During the passage of the sound wave, such a region can be regarded as lying in a uniform pressure field. Thus, he solved equation (13) by postulating the following forms of δP and δT:

$$\delta P \propto e^{-i\omega t}, \quad \delta T \propto e^{-i\omega t}\tilde{T} = \delta P\tilde{T}, \tag{A·1}$$

where $\tilde{T}$ has no time dependence. The complex wave number K of the sound wave in this inhomogeneous system is related to the frequency ω by

$$K = \frac{\omega}{C} + i\delta = \omega\sqrt{\bar{\rho}\bar{\zeta}} = \omega\{[\varepsilon\rho_1 + (1-\varepsilon)\rho_2][\varepsilon\zeta_1 + (1-\varepsilon)\zeta_2]\}^{1/2}. \tag{A·2}$$

In the above ε is the volume fraction of the first component; ζ_1 and ζ_2 are the condensations expressed by

$$\zeta_1 = \beta_1 - \alpha_1\tilde{T}_1, \quad \zeta_2 = \beta_2 - \alpha_2\tilde{T}_2, \tag{A·3}$$

where $\tilde{T}_1$ and $\tilde{T}_2$ are the periodic solutions of (13). The mathematical derivation of (A·2) is not given in Isakovich's paper, so we give it in the Appendix B. Assuming that the sound attenuation is small, we find from (A·2) that

$$K = K_{LL} + \frac{(1+i)}{2D} T\rho C_{LL} \left(\frac{\alpha_1}{\rho_1 c_{P1}} - \frac{\alpha_2}{\rho_2 c_{P2}}\right)^2 \frac{\kappa_1\kappa_2 n_1 n_2 \tanh[(1-i)n_1 d_1/2] \tanh[(1-i)n_2 d_2/2]}{\kappa_1 n_1 \tanh[(1-i)n_1 d_1/2] + \kappa_2 n_2 \tanh[(1-i)n_2 d_2/2]}, \tag{A·4}$$

where

$$n_i = \sqrt{\frac{\omega\rho_i c_{Pi}}{2\kappa_i}}, \tag{A·5}$$

$$C_{LL} = \left\{[\varepsilon\rho_1 + (1-\varepsilon)\rho_2]\left[\varepsilon\frac{\beta_1}{\gamma_1} + (1-\varepsilon)\frac{\beta_2}{\gamma_2}\right]\right\}^{-1/2}, \tag{A·6}$$

and $K_{LL} = \omega/C_{LL}$. For simplicity, let us consider the special case where $d_1 = d_2 = d$, $\rho_1 = \rho_2 = \rho$, $c_{P1} = c_{P2} = c_P$ and $\kappa_1 = \kappa_2 = \kappa$. The attenuation coefficient δ, obtained from the imaginary part of K, is

$$\delta = \frac{\omega T C_{LL}(\alpha_1 - \alpha_2)^2}{8ndc_P}(\mathrm{Re} + \mathrm{Im}) \tanh[(1-i)nd/2]. \tag{A·7}$$

In the limiting case where the thermal wavelength is much greater or much less than d, the expression (A·7) is reduced to the following simple forms.

(1) In the case of $nd \ll 1$,

$$\delta = \frac{T(\alpha_1 - \alpha_2)^2 \rho C_{LL} d^2 \omega^2}{96\kappa} \propto \omega^2. \tag{A·8}$$

(2) In the case of $nd \gg 1$,

$$\delta = \frac{T(\alpha_1 - \alpha_2)^2 C_{LL}\kappa n}{4d\rho c_P^2} \propto \sqrt{\omega}. \tag{A·9}$$

The latter result is the characteristic feature of Isakovich's theory.

Appendix B: Derivation of (A·2)

Substituting (A·1) into (14), we obtain

$$-\zeta\frac{\partial^2 \delta P}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \delta P}{\partial x}\right) = 0, \tag{B·1}$$

where

$$\zeta = \beta - \alpha\tilde{T}. \tag{B·2}$$

Let us consider the one-dimensional periodic inhomogeneous medium. At this stage we put $\delta P_i (i=1, 2)$ for each component in the form

$$\delta P_i \propto \exp\{i(k_i x - \omega t)\}. \tag{B·3}$$

Then the dispersion relation for each component is

$$k_i=\omega\sqrt{\rho_i\zeta_i}. \tag{B·4}$$

The boundary condition for the compressional wave at the interfaces are given by (34). Then the implicit dispersion relation of the sound wave reduces to (48). Our next step is to coarse grain the system.[11] To do so, we let $d_1\to 0$, $d_2\to 0$, keeping d_1/D, d_2/D constant. Then we have

$$1-\frac{1}{2}(KD)^2=1-\frac{1}{2}(k_1d_1)^2-\frac{1}{2}(k_2d_2)^2-\frac{1}{2}\rho_1\rho_2\left[\left(\frac{k_1}{\rho_1}\right)^2+\left(\frac{k_2}{\rho_2}\right)^2\right]d_1d_2. \tag{B·5}$$

We finally reach the desired formula

$$K=\omega\sqrt{\bar{\rho}\bar{\zeta}}, \tag{B·6}$$

with

$$\bar{\rho}=\frac{1}{d_1+d_2}(\rho_1d_1+\rho_2d_2),$$
$$\bar{\zeta}=\frac{1}{d_1+d_2}(\zeta_1d_1+\zeta_2d_2). \tag{B·7}$$

Appendix C: Derivation of (43)

The steps for constructing the transfer matrix M is as follows.

$$\phi_1(x_0+d_1-0^+)=E_1\phi_1(x_0+0^+), \tag{C·1}$$
$$\Phi_1(x_0+d_1-0^+)=M_1\phi_1(x_0+d_1-0^+), \tag{C·2}$$
$$\Phi_2(x_0+d_1+0^+)=\Phi_1(x_0+d_1-0^+), \tag{C·3}$$
$$\phi_2(x_0+d_1+0^+)=M_2^{-1}\Phi_2(x_0+d_1+0^+), \tag{C·4}$$
$$\phi_2(x_0+D-0^+)=E_2\phi_2(x_0+d_1+0^+), \tag{C·5}$$
$$\Phi_2(x_0+D-0^+)=M_2\phi_2(x_0+D-0^+), \tag{C·6}$$
$$\Phi_1(x_0+D+0^+)=\Phi_2(x_0+D-0^+), \tag{C·7}$$
$$\phi_1(x_0+D+0^+)=M_1^{-1}\Phi_1(x_0+D+0^+). \tag{C·8}$$

Equations (C·3) and (C·7) are always guaranteed by the boundary conditions (34) and (35). Hence we reach the expression (43).

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