Table 1: Fundamental operations on discrete convex functions

Discrete convexity	Variables: Permut.	Variables: Scaling	Restriction	Projection	Addition: $f + \varphi$	Addition: $f_1 + f_2$	Convolution: $f \Box \varphi$	Convolution: $f_1 \Box f_2$	Reference
Separable conv	Y	Y	Y	Y	Y	Y	Y	Y	
Integrally conv	Y	***N***	Y	Y	Y	***N***	Y	***N***	[10, 11, 19]
L$^\natural$-convex	Y	Y	Y	Y	Y	Y	Y	***N***	[14]
L-convex	Y	Y	***N***	Y	Y	Y	Y	***N***	[14]
M$^\natural$-convex	Y	***N***	Y	Y	Y	***N***	Y	Y	[14]
M-convex	Y	***N***	Y	***N***	Y	***N***	Y	Y	[14]
			Y		Y	Y			[1]
Multimodular	***N***	Y: Prop.7	alt. proof	***N***			***N***	***N***	this paper
	Y*:Prop.6								[6]
	alt. proof			Y*:Prop.10					this paper
Globally d.m.c.	Y	Y	Y	Y	Y	Y	***N***	***N***	[12]
Locally d.m.c.	Y	Y	Y	Y	Y	Y	***N***	***N***	[12]
M-conv (jump)	Y	***N***	Y	Y	Y	***N***	Y	Y	[7, 16]

d.m.c.: discrete midpoint convex, φ: separable convex
Y: Discrete convexity (of that kind) is preserved, ***N***: Not preserved
Y*: Discrete convexity (of that kind) is preserved in some cases

A recurrence formula consists of various kinds of operations, some of which preserve multimodularity and others not. The projection operation (partial minimization) is closely related to the Bellman equation in dynamic programming, and the assumption of U being an interval (consecutive variables) in Proposition 10 is quite natural in this interpretation. The reversal of the ordering of variables in Proposition 6 corresponds to the reversal of "time" in recurrence relations. It is hoped that the results of this paper will find applications in concrete problems in operations research.

The known facts about fundamental operations on discrete convex functions, including those obtained in this paper, are summarized in Table 1.

Acknowledgement

This work was supported by CREST, JST, Grant Number JPMJCR14D2, Japan, and JSPS KAKENHI Grant Numbers 17K00037, 26280004.

References

[1] E. Altman, B. Gaujal, and A. Hordijk: Multimodularity, convexity, and optimization properties. *Mathematics of Operations Research*, **25** (2000), 324–347.

[2] E. Altman, B. Gaujal, and A. Hordijk: *Discrete-Event Control of Stochastic Networks: Multimodularity and Regularity, Lecture Notes in Mathematics* **1829** (Springer, Heidelberg, 2003).

[3] D. Freund, S.G. Henderson, and D.B. Shmoys: Minimizing multimodular functions and allocating capacity in bike-sharing systems. In F. Eisenbrand and J. Koenemann (eds.): *Integer Programming and Combinatorial Optimization Lecture Notes in Computer Science* **10328** (Springer, Berlin, 2017), 186–198.

[4] S. Fujishige: *Submodular Functions and Optimization, Second Edition* (Elsevier, Amsterdam, 2005).