Stably tame exponential automorphisms of a polynomial ring

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1 Introduction

Let R be a commutative ring with identity and $R[\boldsymbol{x}] := R[x_1, \ldots, x_n]$ the polynomial ring in $x_1, \ldots, x_n$ over R, where n is a positive integer. We regard an n-tuple $\varphi = (f_1, \ldots, f_n)$ of elements of $R[\boldsymbol{x}]$ as the endomorphism of $R[\boldsymbol{x}]$ over R defined by $x_i \mapsto f_i$ for each $i \in \{1, \ldots, n\}$. We say that φ is *quasi-elementary* if there exist $a \in R \setminus \{0\}$, $i \in \{1, \ldots, n\}$, and $p \in R[\boldsymbol{x}_{(i)}]$ such that

$$\varphi = (x_1, \ldots, x_{i-1}, ax_i + p, x_{i+1}, \ldots, x_n),$$

where $\boldsymbol{x}_{(i)} := \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. If $a = 1$, then φ is an automorphism of $R[\boldsymbol{x}]$ over R. We call such an automorphism an *elementary automorphism.* We denote by $\mathrm{E}_R(R[\boldsymbol{x}])$ the subgroup of the automorphism group $\mathrm{Aut}_R\, R[\boldsymbol{x}]$ of $R[\boldsymbol{x}]$ over R generated by all the elementary automorphisms. We denote by $\mathrm{T}_R(R[\boldsymbol{x}])$ the subgroup of $\mathrm{Aut}_R\, R[\boldsymbol{x}]$ generated by $\mathrm{E}_R(R[\boldsymbol{x}])$ and $GL(n, R)$. Here, we identify A with the automorphism $(x_1, \ldots, x_n)A$ for each $A \in GL(n, R)$. We call an element of $\mathrm{T}_R(R[\boldsymbol{x}])$ a *tame automorphism.*

It is an important problem to decide whether $\mathrm{Aut}_R\, R[\boldsymbol{x}] = \mathrm{T}_R(R[\boldsymbol{x}])$. By Jung [5] and van der Kulk [6], it holds that $\mathrm{Aut}_R\, R[x_1, x_2] = \mathrm{T}_R(R[x_1, x_2])$ if R is a field. There exists an algorithm for deciding whether φ is tame for each $\varphi \in \mathrm{Aut}_R\, R[\boldsymbol{x}]$ when $n = 2$ and R is a domain (cf. [4]), and when $n = 3$ and R is a field of characteristic zero ([8]). By these algorithms, it follows that $\mathrm{Aut}_R\, R[\boldsymbol{x}]$ is not equal to $\mathrm{T}_R(R[\boldsymbol{x}])$ if $n = 2$ and R is a domain which is not a field, or if $n = 3$ and R is a field of characteristic of zero.

For $\varphi = (f_1, \ldots, f_n) \in \operatorname{Aut}_R R[\boldsymbol{x}]$ and $m \in \mathbf{Z}_{\geq 0}$, we define an automorphism $\varphi^{[m]}$ of $R[x_1, \ldots, x_{n+m}]$ over R by

$$\varphi^{[m]} = (f_1, \ldots, f_n, x_{n+1}, \ldots, x_{n+m}),$$

where $\mathbf{Z}_{\geq 0}$ denotes the set of non-negative integers, and $x_{n+1}, \ldots, x_{n+m}$ are new variables. For $\varphi \in \operatorname{Aut}_R R[\boldsymbol{x}]$, we say that φ is *stably tame* if there exists $m \in \mathbf{Z}_{\geq 0}$ such that $\varphi^{[m]}$ belongs to $\mathrm{T}_R(R[x_1, \ldots, x_{n+m}])$. The following is a well-known conjecture with very little progress.

Conjecture. *Every automorphism of $R[\boldsymbol{x}]$ over R is stably tame.*

Recently, Berson-van den Essen-Wright [1] showed that, if R is regular, then there exists $m \geq \max\{2 + \dim R, 3\}$ such that $\varphi^{[m]}$ is tame for any $\varphi \in \operatorname{Aut}_R R[x_1, x_2]$.

Various kinds of automorphisms are obtained from locally nilpotent derivations as follows. A *derivation* D of $R[\boldsymbol{x}]$ over R is by definition an R-linear map $R[\boldsymbol{x}] \to R[\boldsymbol{x}]$ which satisfies $D(fg) = fD(g)+gD(f)$ for any $f, g \in R[\boldsymbol{x}]$. We say that D is *locally nilpotent*, if for any $f \in R[\boldsymbol{x}]$, there exists $l \in \mathbf{Z}_{\geq 0}$ such that $D^l(f) = 0$. We denote by $\operatorname{Der}_R R[\boldsymbol{x}]$ and $\operatorname{LND}_R R[\boldsymbol{x}]$ the sets of derivations of $R[\boldsymbol{x}]$ over R, and locally nilpotent derivations of $R[\boldsymbol{x}]$ over R, respectively. Assume that R is a $\mathbf{Q}$-domain. Then, for each $D \in \operatorname{LND}_R R[\boldsymbol{x}]$, we can define an automorphism $\exp D$, which we call an *exponential automorphism*, of $R[\boldsymbol{x}]$ over R by

$$(\exp D)(f) = \sum_{l \geq 0} \frac{1}{l!} D^l(f)$$

for each $f \in R[\boldsymbol{x}]$. For $m \in \mathbf{Z}_{\geq 0}$, we denote by $D^{[m]}$ the extension of D to $R[x_1, \ldots, x_{n+m}]$ defined by $D^{[m]}(x_{n+j}) = 0$ for each $j \in \{1, \ldots, m\}$. Then, we have $\exp D^{[m]} = (\exp D)^{[m]}$ since $D^{[m]}(x_{n+j}) = 0$ for each $j \in \{1, \ldots, m\}$.

We say that $D \in \operatorname{Der}_R R[\boldsymbol{x}]$ is *triangular* if $D(x_i)$ belongs to $R[x_1, \ldots, x_{i-1}]$ for each $i \in \{1, \ldots, n\}$. It is easy to check that, if D is triangular, then D is locally nilpotent an $\exp D$ belongs to $\mathrm{E}_R(R[\boldsymbol{x}])$. It is known that fD belongs to $\operatorname{LND}_R R[\boldsymbol{x}]$ if and only if $D(f) = 0$ and D belongs to $\operatorname{LND}_R R[\boldsymbol{x}]$ for $f \in R[\boldsymbol{x}]$ and $D \in \operatorname{Der}_R R[\boldsymbol{x}]$ (cf. [3, Corollary 1.3.34]). Even if D is triangular, fD may not be triangular, and hence $\exp fD$ may not be tame. Smith [9] showed that $(\exp fD)^{[1]}$ is tame for any triangular derivation D of $R[\boldsymbol{x}]$ and $f \in \ker D$.

The purpose of this paper is to study stable tameness of the exponential automorphisms for more general locally nilpotent derivations. We denote by $\mathcal{E}(R[\boldsymbol{x}])$ the multiplicative submonoid of the endomorphism ring $\mathrm{End}_R R[\boldsymbol{x}]$ of $R[\boldsymbol{x}]$ over R generated by quasi-elementary endomorphisms of $R[\boldsymbol{x}]$, i.e., the set of the composites of quasi-elementary endomorphisms. For $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$ and $i \in \{1, \dots, n\}$, we define

$$\Delta_{\varphi,i} = D_{(\varphi(x_1),\dots,\varphi(x_{i-1}),\varphi(x_{i+1}),\dots,\varphi(x_n))}.$$

Here, for $\boldsymbol{f} = (f_1, \dots, f_{n-1}) \in R[\boldsymbol{x}]^{n-1}$, we define a derivation $D_{\boldsymbol{f}}$ of $R[\boldsymbol{x}]$ over R by

$$D_{\boldsymbol{f}}(g) = |J_{(f_1,\dots,f_{n-1},g)}|$$

for each $g \in R[\boldsymbol{x}]$, where $J_{(f_1,\dots,f_{n-1},g)}$ denotes the Jacobian matrix of the endomorphism $(f_1, \dots, f_{n-1}, g)$, and $|A|$ denotes the determinant of A for a square matrix A. Then, $\Delta_{\varphi,i}$ is locally nilpotent (see the argument after Lemma 2.1). We can construct various locally nilpotent derivations as $\Delta_{\varphi,i}$. For example, define $\varphi \in \mathcal{E}(R[x_1, x_2])$ by

$$\varphi = (tx_1 + x_2^2, x_2) \circ (x_1, x_2 + x_1^2/4),$$

for $t \in R \setminus \{0\}$. Then, $\Delta_{\varphi,1}$ is a locally nilpotent derivation of $R[x_1, x_2]$ over R such that $\psi \circ \Delta_{\varphi,1} \circ \psi^{-1}$, and t is not a unit of R is not triangular for every $\psi \in \mathrm{Aut}_R R[x_1, x_2]$ if R is a UFD by Daigle ([2]).

Set $\tilde{\boldsymbol{x}} = \{x_1, \dots, x_n, x_{n+1}\}$. Here is the main result of this paper.

Theorem 1.1. *Let R be a $\mathbf{Q}$-domain, and n a positive integer. For any $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$, $i \in \{1, \dots, n\}$, and $f \in \ker \Delta_{\varphi,i}^{[1]}$, it holds that $\exp f\Delta_{\varphi,i}^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$. In particular, we have $(\exp f\Delta_{\varphi,i})^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$ for each $f \in \ker \Delta_{\varphi,i}$.*

We remark that $\mathcal{E}(R[\boldsymbol{x}])$ contains the automorphism of $R[\boldsymbol{x}]$ defined by a permutation of $x_1, \dots, x_n$. Hence, for any $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$ and $i \in \{1, \dots, n\}$, there exists $\varphi' \in \mathcal{E}(R[\boldsymbol{x}])$ such that $\Delta_{\varphi,i} = \Delta_{\varphi',1}$.

Let K be the quotient field of R. We denote by $\bar{D}$ the extension of D to $K[\boldsymbol{x}]$ for each $D \in \mathrm{LND}_R R[\boldsymbol{x}]$. The theorem above implies the following corollary.

Corollary 1.2. *Let n be a positive integer, and $D \in \mathrm{LND}_R R[\boldsymbol{x}]$ be such that $\tau \circ \bar{D} \circ \tau^{-1}$ is triangular for some $\tau \in \mathrm{T}_K(K[\boldsymbol{x}])$. Then, there exists $a \in R \setminus \{0\}$ such that $(\exp aD)^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$.*

Thanks to Rentschler [7], we get the following corollary from Corollary 1.2.

Corollary 1.3. *For any* $D \in \mathrm{LND}_R R[x_1, x_2]$*, there exists* $a \in R \setminus \{0\}$ *such that* $(\exp aD)^{[1]}$ *belongs to* $\mathrm{E}_R(R[x_1, x_2, x_3])$*.*

We prove Theorem 1.1 in Section 2. Note that every element of $\mathcal{E}(R[\boldsymbol{x}])$ extends to a tame automorphism of $K[\boldsymbol{x}]$ over K. In Section 3, we study $\Delta_{\varphi,i}$ from this point of view, and deduce Corollaries 1.2 and 1.3 from Theorem 1.1.

2 Proof of the main result

Note that $|J_{\varphi\circ\psi}| = |\varphi(J_\psi)||J_\varphi|$ for $\varphi, \psi \in \mathrm{End}_R R[\boldsymbol{x}]$. Here, for a matrix $(a_{i,j})_{i,j}$ with $a_{i,j} \in R[\boldsymbol{x}]$, we define $\varphi((a_{i,j})_{i,j}) = (\varphi(a_{i,j}))_{i,j}$.

Lemma 2.1. *For* $\boldsymbol{f} = (f_1, \ldots, f_{n-1}) \in R[\boldsymbol{x}]^{n-1}$ *and* $\varphi \in \mathrm{Aut}_R R[\boldsymbol{x}]$*, we have*

$$\varphi \circ |J_\varphi| D_{\boldsymbol{f}} \circ \varphi^{-1} = D_{\varphi(\boldsymbol{f})},$$

where $\varphi(\boldsymbol{f})$ *denotes* $(\varphi(f_1), \ldots, \varphi(f_{n-1}))$*.*

Proof. Because R is a domain, $|J_\varphi|$ is a unit of R. Take any $g \in R[\boldsymbol{x}]$ and set $\psi = (f_1, \ldots, f_{n-1}, g)$. Then, we have

$$\begin{aligned}(D_{\varphi(\boldsymbol{f})} \circ \varphi)(g) &= D_{(\varphi(f_1),\ldots,\varphi(f_{n-1}))}(\varphi(g)) = |J_{(\varphi(f_1),\ldots,\varphi(f_{n-1}),\varphi(g))}| = |J_{\varphi\circ\psi}| \\ &= |\varphi(J_\psi)||J_\varphi| = \varphi(|J_\psi|)|J_\varphi| = |J_\varphi|\varphi(|J_\psi|) = \varphi(|J_\varphi|)\varphi(D_{\boldsymbol{f}}(g)) \\ &= \varphi(|J_\varphi|D_{\boldsymbol{f}}(g)) = (\varphi \circ |J_\varphi|D_{\boldsymbol{f}})(g).\end{aligned}$$

Therefore, we get $\varphi \circ |J_\varphi| D_{\boldsymbol{f}} \circ \varphi^{-1} = D_{\varphi(\boldsymbol{f})}$. □

Note that φ extends to a tame automorphism $\bar{\varphi}$ of $K[\boldsymbol{x}]$ over K for each $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$. Then, we have $\bar{\Delta}_{\varphi,i} = \bar{\varphi} \circ |J_{\bar{\varphi}}| \bar{D}_{\boldsymbol{x}_{(i)}} \circ \bar{\varphi}^{-1}$ by Lemma 2.1. Since $|J_{\bar{\varphi}}| = |J_\varphi|$ is a constant, and $D_{\boldsymbol{x}_{(i)}} = (-1)^{i+1}\partial_{x_i}$, we know that $\bar{\Delta}_{\varphi,i}$ is locally nilpotent. Here, ∂_{x_i} denotes the partial derivation in x_i. Therefore, $\Delta_{\varphi,i}$ is also locally nilpotent.

It is known that $D + E$ is locally nilpotent, and

$$\exp(D + E) = (\exp D) \circ (\exp E)$$

for $D, E \in \mathrm{LND}_R R[\boldsymbol{x}]$ with $D \circ E = E \circ D$. Hence, we have that

$$\exp(f + g)D = (\exp fD) \circ (\exp gD)$$

for $D \in \mathrm{LND}_R R[\boldsymbol{x}]$ and $f, g \in \ker D$.

Proposition 2.2. *Let* $D \in \mathrm{LND}_R\, R[\boldsymbol{x}]$ *be such that* $\exp a x_{n+1}^r D^{[1]}$ *belongs to* $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$ *for any* $a \in \mathbf{Q}$ *and* $r \in \mathbf{Z}_{\geq 0}$. *Then,* $\exp f D^{[1]}$ *belongs to* $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$ *for any* $f \in \ker D^{[1]}$.

Proof. Take any $f \in \ker D^{[1]}$. Since $\ker D^{[1]} = (\ker D)[x_{n+1}]$, we can write

$$f = \sum_{i=0}^{r} f_i x_{n+1}^i,$$

where $r \in \mathbf{Z}_{\geq 0}$ and $f_0, \ldots, f_r \in \ker D$. Then, we have

$$\exp f D = (\exp f_0 D) \circ \cdots \circ (\exp f_r x_{n+1}^r D).$$

Hence, it suffices to show $\exp f_i x_{n+1}^i D^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$ for each $i \in \{1, \ldots, r\}$. For $g, h \in R[\boldsymbol{x}]$, we can write $g^l h^m$ as a sum of polynomials of the form $a(g + bh)^{l+m}$ with $a, b \in \mathbf{Q}$ and $l, m \in \mathbf{Z}_{\geq 0}$ (cf. [1, Lemma 2.1]). Hence, we may $f_i x_{n+1}^i$ as a sum of the polynomials of the form $a(x_{n+1} + b f_i)^{i+1}$ with $a, b \in \mathbf{Q}$. Thus, we are reduced to proving that $\exp a(x_{n+1} + b f_i)^{i+1} D^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$. Set $\varepsilon = (x_1, \ldots, x_n, x_{n+1} + b f_i)$. Then, we have $\varepsilon \circ D^{[1]} \circ \varepsilon^{-1} = D^{[1]}$. Indeed, since ε fixes any element of $R[\boldsymbol{x}]$ and $D^{[1]}(x_{n+1}) = D^{[1]}(f) = 0$, we get

$$(\varepsilon \circ D^{[1]})(x_i) = \varepsilon(D(x_i)) = D(x_i) = D^{[1]}(x_i) = (D^{[1]} \circ \varepsilon)(x_i)$$

for $i \in \{1, \ldots, n\}$, and $(\varepsilon \circ D^{[1]})(x_{n+1}) = \varepsilon(0) = 0$ is equal to $(D^{[1]} \circ \varepsilon)(x_{n+1}) = D^{[1]}(x_{n+1} + b f_i) = 0$. From this equality, it follows that

$$\exp a(x_{n+1} + b f_i)^{i+1} D^{[1]} = \exp \varepsilon(a x_{n+1}^{i+1}) D^{[1]} = \varepsilon \circ (\exp a x_{n+1}^{i+1} D^{[1]}) \circ \varepsilon^{-1}.$$

This automorphism belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$, since $\exp a x_{n+1}^{i+1} D^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$ by assumption. Therefore, $(\exp f D)^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$. □

Now, let us prove Theorem 1.1. Without loss of generality, we may assume that $i = 1$ by the remark after Theorem 1.1. Take any $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$. Then, we can write $\varphi = \varepsilon_1 \circ \cdots \circ \varepsilon_r$, where $r \in \mathbf{Z}_{\geq 0}$ and ε_i is a quasi-elementary endomorphism of $R[\boldsymbol{x}]$ for each $i \in \{1, \ldots, r\}$. We prove the theorem by induction on r. If $r = 0$, then $\varphi = (x_1, \ldots, x_n)$. Hence, we have $\Delta_{\varphi,1}^{[1]} = \partial_{x_1}$. In this case, $\exp f \Delta_{\varphi,1}$ is an elementary automorphism for any $f \in \ker \Delta_{\varphi,1}^{[1]} = R[x_2, \ldots, x_{n+1}]$. Assume that $r > 0$. By Proposition 2.2, it suffices to prove that $\exp h\Delta_{\varphi,1}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$, where $h = \alpha x_{n+1}^l$ with $\alpha \in \mathbf{Q}$ and $l \in \mathbf{Z}_{\geq 0}$. Put $\psi = \varepsilon_1 \circ \cdots \circ \varepsilon_{r-1} \in \mathcal{E}(R[\boldsymbol{x}])$, and write $\varepsilon_r = (x_1, \ldots, x_{j-1}, ax_j + p, x_{j+1}, \ldots, x_n)$, where $a \in R \setminus \{0\}$, $j \in \{1, \ldots, n\}$, and $p \in R[\boldsymbol{x}_{(j)}]$.

The following is a key proposition.

Proposition 2.3. *There exists* $g \in \ker \Delta_{\psi,j}^{[1]}$ *such that*

$$\exp h\Delta_{\varphi,1}^{[1]} = (\exp g\Delta_{\psi,j}^{[1]}) \circ (\exp ah\Delta_{\psi,1}^{[1]}).$$

Assuming this proposition, the proof of Theorem 1.1 is completed as follows. By induction assumption, $\exp g\Delta_{\psi,j}^{[1]}$ and $\exp ah\Delta_{\psi,1}^{[1]}$ belong to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$. Thus, $\exp h\Delta_{\varphi,1}^{[1]}$ also belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$. Therefore, we obtain Theorem 1.1.

Next, we prove Proposition 2.3. We use the same notation for an automorphism and its extension to $K[\boldsymbol{x}]$. Similarly, we use the same notation for a locally nilpotent derivation of $R[\boldsymbol{x}]$ and its extension to $K[\boldsymbol{x}]$. Note that $\Delta_{\varphi,1}^{[1]} = \Delta_{\varphi^{[1]},1}$, and $D_{\boldsymbol{x}_{(1)}}^{[1]} = D_{\tilde{\boldsymbol{x}}_{(1)}}$, where $\tilde{\boldsymbol{x}}_{(i)}$ denotes

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1})$$

for each $i \in \{1, \ldots, n\}$. Set $b = |J_\psi|$. Since $|J_\varepsilon| = a$, we have $|J_{\varphi^{[1]}}| = |J_\varphi| = |J_{\psi\circ\varepsilon}| = |J_\psi||J_\varepsilon| = ab$. By Lemma 2.1, it follows that

$$\begin{aligned} h\Delta_{\varphi^{[1]},1} &= \varphi^{[1]} \circ |J_{\varphi^{[1]}}|hD_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\varphi^{[1]})^{-1} \\ &= \psi^{[1]} \circ \varepsilon^{[1]} \circ abhD_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\varepsilon^{[1]})^{-1} \circ (\psi^{[1]})^{-1}. \end{aligned}$$

Hence, we get

$$\exp h\Delta_{\varphi,1}^{[1]} = \psi^{[1]} \circ (\exp \varepsilon^{[1]} \circ abhD_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\varepsilon^{[1]})^{-1}) \circ (\psi^{[1]})^{-1}.$$

Then, Proposition 2.3 follows from the following lemma.

Lemma 2.4. *There exists* $q \in R[\tilde{\boldsymbol{x}}_{(j)}]$ *such that*

$$\varepsilon^{[1]} \circ \exp abhD_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\varepsilon^{[1]})^{-1} \circ (\exp abhD_{\tilde{\boldsymbol{x}}_{(1)}})^{-1} = \exp bqD_{\tilde{\boldsymbol{x}}_{(j)}}. \qquad (*)$$

Actually, by this lemma, we get

$$\begin{aligned} \exp h\Delta_{\varphi,1}^{[1]} =& \psi^{[1]} \circ \exp bqD_{\tilde{\boldsymbol{x}}_{(j)}} \circ \exp abhD_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\psi^{[1]})^{-1} \\ =& \psi^{[1]} \circ \exp |J_{\psi^{[1]}}|qD_{\tilde{\boldsymbol{x}}_{(j)}} \circ \exp a|J_{\psi^{[1]}}|hD_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\psi^{[1]})^{-1}. \end{aligned}$$

By Lemma 2.1,

$$\psi^{[1]} \circ \exp |J_{\psi^{[1]}}| q D_{\tilde{\boldsymbol{x}}_{(j)}} \circ (\psi^{[1]})^{-1} = \exp \psi^{[1]}(q) \Delta^{[1]}_{\psi,j},$$
$$\psi^{[1]} \circ \exp |J_{\psi^{[1]}}| ah D_{\tilde{\boldsymbol{x}}_{(1)}} \circ (\psi^{[1]})^{-1} = \exp ah \Delta^{[1]}_{\psi,1}.$$

Note that $\psi^{[1]}(q)$ is killed by $\Delta^{[1]}_{\psi,j} = \psi^{[1]} \circ |J_{\psi^{[1]}}| D_{\tilde{\boldsymbol{x}}_{(j)}} \circ (\psi^{[1]})^{-1}$ since q is an element of $R[\tilde{\boldsymbol{x}}_{(j)}]$. Therefore, $\exp h\Delta^{[1]}_{\varphi,1}$ equals to $\exp \psi^{[1]}(g)\Delta^{[1]}_{\psi,j} \circ \exp ah\Delta^{[1]}_{\psi,1}$.

It remains only to prove Lemma 2.4. First, assume that $j = 1$. Then, we have $\varepsilon^{[1]}(\tilde{\boldsymbol{x}}_{(1)}) = \tilde{\boldsymbol{x}}_{(1)}$. Since $a = |J_\varepsilon|$, we know by Lemma 2.1,

$$\begin{aligned}
&\varepsilon^{[1]} \circ \exp abh D_{\tilde{\boldsymbol{x}}_{(i)}} \circ (\varepsilon^{[1]})^{-1} \circ (\exp abh D_{\tilde{\boldsymbol{x}}_{(1)}})^{-1} \\
&\quad = \exp bh D_{\varepsilon^{[1]}(\tilde{\boldsymbol{x}}_{(1)})} \circ \exp(-abh D_{\tilde{\boldsymbol{x}}_{(1)}}) \\
&\quad = \exp bh D_{\tilde{\boldsymbol{x}}_{(1)}} \circ \exp(-abh D_{\tilde{\boldsymbol{x}}_{(1)}}) \\
&\quad = \exp bh(1-a) D_{\tilde{\boldsymbol{x}}_{(1)}}.
\end{aligned}$$

Therefore, the assertion holds for $q = h(1-a)$. Next, assume that $j \neq 1$. Let $\zeta = \exp abh D_{\tilde{\boldsymbol{x}}_{(1)}}$. Since both sides of $(*)$ fix x_k for $k \in \{2, \ldots, n+1\} \setminus \{j\}$, we show that the images of x_1 and x_j are the same. Note that $\zeta = (x_1 + abh, x_2, \ldots, x_{n+1})$ and $(\varepsilon^{[1]})^{-1}$ fixes $\zeta^{-1}(x_1) = x_1 - abh$. Therefore,

$$(\varepsilon^{[1]} \circ \zeta \circ (\varepsilon^{[1]})^{-1} \circ \zeta^{-1})(x_1) = (\varepsilon^{[1]} \circ \zeta \circ \zeta^{-1})(x_1) = \varepsilon^{[1]}(x_1) = x_1.$$

Then, since $(\varepsilon^{[1]})^{-1} = (x_1, \ldots, x_{j-1}, (x_j - p)/a, x_{j+1}, \ldots, x_{n+1})$, we get

$$\begin{aligned}
&(\varepsilon^{[1]} \circ \zeta \circ (\varepsilon^{[1]})^{-1} \circ \zeta^{-1})(x_j) \\
&\quad = (\varepsilon^{[1]} \circ \zeta \circ (\varepsilon^{[1]})^{-1})(x_j) = (\varepsilon^{[1]} \circ \zeta)((x_j - p)/a) \\
&\quad = \varepsilon^{[1]}((x_j - \zeta(p))/a) = (a x_j + p - \zeta(p))/a \\
&\quad = x_j + (p - \zeta(p))/a.
\end{aligned}$$

Note that $f := (p - \zeta(p))/(ab)$ belongs to $K[\tilde{\boldsymbol{x}}_{(j)}]$, because p and $\zeta(p)$ belong to $R[\boldsymbol{x}_{(j)}]$. We show that f belongs to $R[\tilde{\boldsymbol{x}}]$. Set $E = h D_{\boldsymbol{x}_{(1)}}$. Then, we have

$$\begin{aligned}
f &= \frac{p - \zeta(p)}{ab} = \frac{p - (\exp abE)(p)}{ab} = \frac{1}{ab}\left(p - \sum_{i \geq 0} \frac{(ab)^l}{l!} E^l(p)\right) \\
&= \frac{1}{ab}\left(p - p - \sum_{i \geq 1} \frac{(ab)^l}{l!} E^l(p)\right) = -\sum_{i \geq 1} \frac{(ab)^{l-1}}{l!} E^l(p).
\end{aligned}$$

Hence, $f \in R[\tilde{\boldsymbol{x}}]$. In particular, $bg \in R[\tilde{\boldsymbol{x}}_{(j)}]$, where $q = (-1)^{j+1}f$. Therefore, we have

$$\begin{aligned}\varepsilon^{[1]} \circ \zeta \circ (\varepsilon^{[1]})^{-1} \circ \zeta^{-1} &= (x_1, \ldots, x_{j-1}, x_j + (-1)^{j+1}bq, x_{j+1}, \ldots, x_n, x_{n+1}) \\ &= \exp bqD_{\tilde{\boldsymbol{x}}_{(j)}}.\end{aligned}$$

This proves Lemma 2.4. Thereby, we have completed the proof of Theorem 1.1.

3 Application

Let K be the quotient field of R. An automorphism $\delta \in GL(n, K)$ is *diagonal* if $\delta = (a_1x_1, \ldots, a_nx_n)$ for some $a_1, \ldots, a_n \in K\setminus\{0\}$. We denote by $\mathrm{D}_K(K[\boldsymbol{x}])$ the set of diagonal automorphisms. For each $a \in R \setminus \{0\}$, we denote by R_a the localization of R by $\{a^l \mid l \in \mathbf{Z}_{\geq 0}\}$. Then, we may regard $\mathrm{Aut}_{R_a} R_a[\boldsymbol{x}]$ as a subgroup of $\mathrm{Aut}_K K[\boldsymbol{x}]$.

Let a_i be an element of $R \setminus \{0\}$, and ε_i an elementary automorphism of $R_{a_i}[\boldsymbol{x}]$ over R_{a_i} for $i \in \{1, \ldots, r\}$, where $r \in \mathbf{Z}_{\geq 0}$. Set

$$S = \{a_1^{l_1} \cdots a_r^{l_r} \mid l_1, \ldots, l_r \in \mathbf{Z}_{\geq 0}\}.$$

Then, the following lemma holds for $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_r$.

Lemma 3.1. *There exists $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$, and $c_1, \ldots, c_n \in S$ such that $\varepsilon = \varphi \circ \delta$, where $\delta = (c_1^{-1}x_1, \ldots, c_n^{-1}x_n)$.*

Proof. We prove the lemma by induction on r. When $r = 0$, the assertion is clear. Assume that $r \geq 1$. Then, we have $\varepsilon_1 \circ \cdots \circ \varepsilon_{r-1} = \varphi \circ \delta$ for some $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$ and $\delta = (c_1^{-1}x_1, \ldots, c_n^{-1}x_n)$ with $c_i \in S$ for each $i \in \{1, \ldots, n\}$ by induction assumption. We write

$$\varepsilon_r = (x_1, \ldots, x_{j-1}, x_j + p/a_r^m, x_{j+1}, \ldots, x_n)$$

where $j \in \{1, \ldots, n\}$, $p \in R[\boldsymbol{x}_{(j)}]$, and $m \in \mathbf{Z}_{\geq 0}$. Then, we have $(\delta \circ \varepsilon)(x_i) = c_i^{-1}x_i$ for each $i \in \{1, \ldots, n\} \setminus \{j\}$, and

$$\delta \circ \varepsilon_r(x_j) = (c_j)^{-1}x_j + \delta(p)/a_j^m.$$

Choose $m_1, \dots, m_r \in \mathbf{Z}_{\geq 0}$ such that $c_j d\delta(p)$ belongs to $R[\boldsymbol{x}]$, where $d := a_1^{m_1} \cdots a_r^{m_r}$, and put

$$\begin{aligned} \varepsilon_r' &= (x_1, \dots, x_{j-1}, dx_j + c_j d\delta(p), x_{j+1}, \dots, x_n), \\ \delta' &= \delta \circ (x_1, \dots, x_{j-1}, d^{-1}x_j, x_{j+1}, \dots, x_n). \end{aligned}$$

Then, δ' has the form of $(d_1^{-1}x_1, \dots, d_n^{-1}x_n)$ for some $d_1, \dots, d_n \in S$. Moreover, we have $\varepsilon_r' \circ \delta'(x_i) = c_i^{-1}x_i$ for each $i \in \{1, \dots, n\} \setminus \{j\}$, and

$$(\varepsilon_r' \circ \delta')(x_j) = \varepsilon_r'(c_j^{-1}d^{-1}x_j) = c_j^{-1}x_j + \delta(p).$$

Thus, we get $\delta \circ \varepsilon_r = \varepsilon_r' \circ \delta'$. It follows that

$$\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_r = (\varphi \circ \delta) \circ \varepsilon_r = \varphi \circ \varepsilon_r' \circ \delta'.$$

Because $\varphi \circ \varepsilon_r'$ belongs to $\mathcal{E}(R[\boldsymbol{x}])$ and δ' is a diagonal automorphism as required, we know that the lemma is true. □

Note that every element of $GL(n, K)$ is obtained as a product of elementary matrices and diagonal matrices, and an automorphism defined by an elementary matrix is elementary. Hence, any tame automorphism is obtained by the composition of some elementary automorphisms and diagonal automorphisms.

Proposition 3.2. *For any $\tau \in \mathrm{T}_K(K[\boldsymbol{x}])$, there exist $\varepsilon \in \mathcal{E}(R[\boldsymbol{x}])$ and $\delta \in \mathrm{D}_K(K[\boldsymbol{x}])$ such that $\tau = \varepsilon \circ \delta$.*

Proof. We may write $\tau = \delta_1 \circ \varepsilon_1 \circ \cdots \circ \delta_r \circ \varepsilon_r$, where $r \in \mathbf{Z}_{\geq 0}$, $\delta_i \in \mathrm{D}_K(K[\boldsymbol{x}])$, and ε_i is an elementary automorphism of $K[\boldsymbol{x}]$ over K for each $i \in \{1, \dots, r\}$. Note that $\delta \circ \varepsilon \circ \delta^{-1}$ is an elementary automorphism for any elementary automorphism ε and diagonal automorphism δ. Since

$$\begin{aligned} \tau =& \delta_1 \circ \varepsilon_1 \circ \delta_1^{-1} \circ (\delta_1 \circ \delta_2) \circ \varepsilon_2 \circ (\delta_2^{-1} \circ \delta_1^{-1}) \\ & \circ \cdots \circ (\delta_1 \circ \cdots \circ \delta_r) \circ \varepsilon_r \circ (\delta_r^{-1} \circ \cdots \circ \delta_1^{-1}) \circ \delta_1 \circ \cdots \circ \delta_r, \end{aligned}$$

we may assume that $\tau = \varepsilon_1 \circ \cdots \circ \varepsilon_r \circ \delta$ for some elementary automorphism $\varepsilon_1, \dots, \varepsilon_r$, and a diagonal automorphism δ. For each $i \in \{1, \dots, r\}$, we can find $a_i \in R \setminus \{0\}$ such that ε_i belongs to $\mathrm{Aut}_{R_{a_i}} R_{a_i}[\boldsymbol{x}]$. Then, we have $\varepsilon_1 \circ \cdots \circ \varepsilon_r = \varphi \circ \delta'$ for some $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$ and $\delta' \in \mathrm{D}_K(K[\boldsymbol{x}])$ by Lemma 3.1. Therefore, we get $\tau = \varphi \circ \delta' \circ \delta$ in which $\delta' \circ \delta$ is diagonal. □

Let $\varphi \in \mathrm{Aut}_R R[\boldsymbol{x}]$ be such that $\varphi(x_i) = x_i + g_i$ for each $i \in \{1, \dots, n\}$, where $g_i \in R[x_1, \dots, x_{i-1}]$. Then, we have

$$\varphi = (\exp g_n \partial_{x_n}) \circ \cdots \circ (\exp g_1 \partial_{x_1}).$$

Lemma 3.3. *For a triangular derivation D of $R[\boldsymbol{x}]$ and $b \in R$, we have*

$$\exp bD = (\exp bg_n D_{\boldsymbol{x}_{(n)}}) \circ \cdots \circ (\exp bg_1 D_{\boldsymbol{x}_{(1)}}),$$

for some $g_i \in R[x_1, \dots, x_{i-1}]$ for $i \in \{1, \dots, n\}$.

Proof. Since D is triangular by assumption, $D^l(x_i)$ belongs to $R[x_1, \dots, x_{i-1}]$ for any $l \in \mathbf{Z}_{\geq 0}$ and $i \in \{1, \dots, n\}$. Hence, we may write

$$(\exp bD)(x_i) = \sum_{l \geq 0} \frac{b^l}{l!} D^l(x_i) = x_i + b \sum_{l \geq 1} \frac{b^{l-1}}{l!} D^l(x_i) = x_i + bg_i,$$

where $g_i \in R[x_1, \dots, x_{i-1}]$. Then, we have

$$\exp bD = (\exp bg_n \partial_{x_n}) \circ \cdots \circ (\exp bg_1 \partial_{x_1}).$$

Since $\partial_{x_i} = (-1)^{i+1} D_{\boldsymbol{x}_{(i)}}$, we know that $\exp bD$ is expressed as in the lemma. □

Let us prove Corollary 1.2. Let $D \in \mathrm{LND}_R R[\boldsymbol{x}]$ be such that $D = \tau \circ E \circ \tau^{-1}$ for some $\tau \in \mathrm{T}_K(K[\boldsymbol{x}])$ and a triangular derivation E of $K[\boldsymbol{x}]$. By Lemma 3.2, we have $\tau = \varphi \circ \delta$ for some $\varphi \in \mathcal{E}(R[\boldsymbol{x}])$ and $\delta \in \mathrm{D}_K(K[\boldsymbol{x}])$. Then, $\delta \circ E \circ \delta^{-1}$ is triangular. Choose $a \in R \setminus \{0\}$ such that $a(\delta \circ E \circ \delta^{-1}) = bE'$ for some triangular derivation E' of $R[\boldsymbol{x}]$, where $b = |J_\varphi|$. By Lemma 3.3, we can express

$$\exp bE' = (\exp bg_n D_{\boldsymbol{x}_{(n)}}) \circ \cdots \circ (\exp bg_1 D_{\boldsymbol{x}_{(1)}}),$$

where $g_i \in R[x_1, \dots, x_{i-1}]$ for each $i \in \{1, \dots, n\}$. Hence, we have

$$\begin{aligned}\exp aD = \tau \circ (\exp aE) \circ \tau^{-1} &= \varphi \circ (\exp a(\delta \circ E \circ \delta')) \circ \varphi^{-1} = \varphi \circ (\exp bE') \circ \varphi^{-1} \\ &= \varphi \circ (\exp bg_n D_{\boldsymbol{x}_{(n)}}) \circ \varphi^{-1} \circ \cdots \circ \varphi \circ (\exp bg_1 D_{\boldsymbol{x}_{(1)}}) \circ \varphi^{-1} \\ &= \varphi \circ (\exp |J_\varphi| g_n D_{\boldsymbol{x}_{(n)}}) \circ \varphi^{-1} \circ \cdots \circ \varphi \circ (\exp |J_\varphi| g_1 D_{\boldsymbol{x}_{(1)}}) \circ \varphi^{-1} \\ &= \exp \varphi(g_n) \Delta_{\varphi,n} \circ \cdots \circ \exp \varphi(g_1) \Delta_{\varphi,1}.\end{aligned}$$

By Theorem 1.1, $(\exp \varphi(g_i)\Delta_{\varphi,i})^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$ for each $i \in \{1, \ldots, n\}$. Therefore, $(\exp aD)^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$. This completes the proof of Corollary 1.2.

By Rentschler [7], for any $D \in \mathrm{LND}_R\, R[x_1, x_2]$, there exist $\tau \in \mathrm{T}_K(K[x_1, x_2])$ and $f \in K[x_1]$ such that $D = \varphi \circ f\partial_{x_2} \circ \varphi^{-1}$. Hence, $(\exp aD)^{[1]}$ belongs to $\mathrm{E}_R(R[x_1, x_2, x_3])$ for some $a \in R \setminus \{0\}$ by Corollary 1.2. Therefore, we get Corollary 1.3.

We are interested in the following question.

Question. *Let $D \in \mathrm{LND}_R\, R[\boldsymbol{x}]$. If there exists $a \in R \setminus \{0\}$ such that $(\exp aD)^{[1]}$ belongs to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$, then does $(\exp D)^{[1]}$ belong to $\mathrm{E}_R(R[\tilde{\boldsymbol{x}}])$?*

If this statement is true, then $(\exp D)^{[1]}$ belongs to $\mathrm{E}_R(R[x_1, x_2, x_3])$ for any $D \in \mathrm{LND}_R\, R[x_1, x_2]$ by Corollary 1.3.

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