横尾 岩明 — 有限群 [illegible] No1

八ヶ岳代数幾何セミナー

No. 1

Date '03. 7. 7

斎藤政彦氏

Moduli space of stable parabolic connection
Riemann - Hilbert Correspondence and Geometry
Painlevé Ⅳ

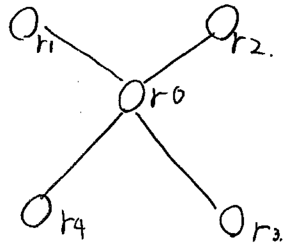
$$(P_{VI}) \quad x'' = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)(x')^2$$
$$- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)x'$$
$$+ \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(x-t)^2}\right]$$

$$P_{VI}(\lambda) \iff (H_{VI}(\lambda))$$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial y}(x, y, t, \lambda) \\ \dfrac{dy}{dt} = -\dfrac{\partial H}{\partial x}(x, y, t, \lambda) \end{cases}, \quad H(x, y, t, \lambda) \in \mathbb{C}(t)[x, y, \lambda]$$

@ パンルヴェ Ⅵ の特徴

- P_{VI} : パンルヴェ性を持つ方程式
- $P_{VI}(\lambda) = H_{VI}(\lambda)$ $\lambda \in \mathbb{C}^4$ ⇒ 4 - parameter に依存
- $P_{VI}(\lambda)$ の一般の解は非常に超越的
- λ が特別な場合 ⟹ ・Riccati 解
  ・有理式解
- ベクルンド変換という双有理変換を許し、その全体は $W(D_4^{(1)})$（$D_4^{(1)}$ 型のワイル群）をなす.



$\langle r_0, \cdots, r_4 \rangle = R$

$\bigoplus_{i=0}^{4} \mathbb{Z} r_i = T$

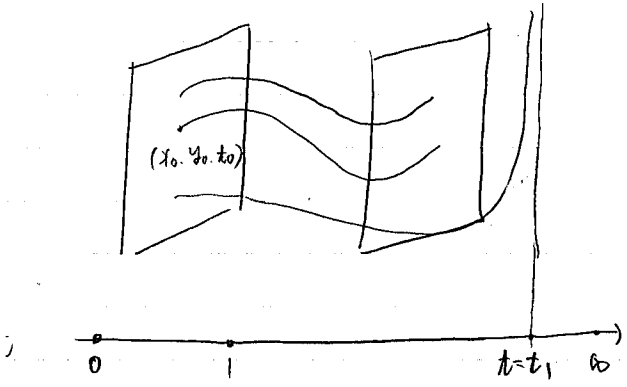
$s_i(x) = x + (r_i . x) r_i$

$s_i : T \xrightarrow{\sim} T$

$W(D_4^{(1)}) = \langle s_0 . s_1, \cdots s_4 \rangle$

○ : $r_i^2 = -2$
$\underset{r_i}{○}-\underset{r_j}{○}$: $(r_i . r_j) = 1$

$T = \mathbb{P}^1 - \{0. 1. \infty\}$



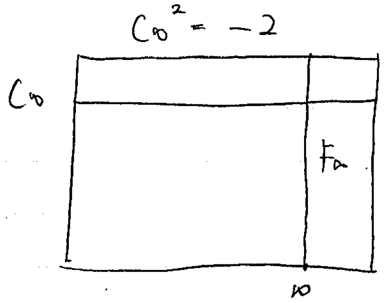
$$\tilde{v} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial y}$$

: $\mathbb{C}^2 \times T$ 上の regular vector field

$$\begin{tikzcd} \mathbb{C}^2 \times T \arrow[d] \\ T \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{S} = \mathbb{C}^2 \times T \times \Lambda_4 \arrow[r, hook] \arrow[d] & \mathbb{F}_2 \times T \times \Lambda_4 = \overline{\mathcal{S}} \arrow[dl] \\ T \times \Lambda_4 & \end{tikzcd}$$

$\Lambda_4 = \mathbb{C}^4 = \mathrm{Spec}\,\mathbb{C}[\lambda_1, \cdots, \lambda_4]$



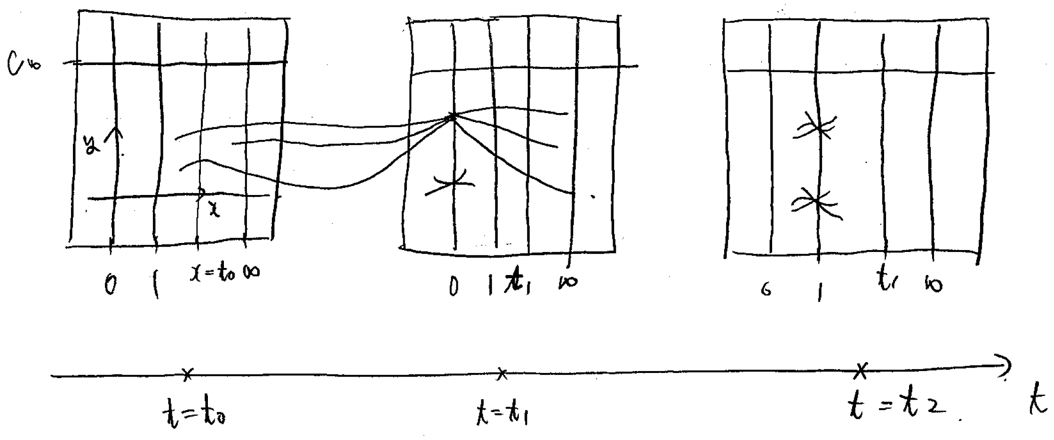
$$\begin{tikzcd} \mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

$\mathbb{F}_2 - C_\infty - F_\infty \underset{\wr}{} \mathbb{C}^2$

$$\begin{tikzcd} \tilde{v} : \mathcal{S} \arrow[d] & \mathcal{D} \arrow[l, hook'] \arrow[dl] \\ T \times \Lambda_4 & \end{tikzcd}$$

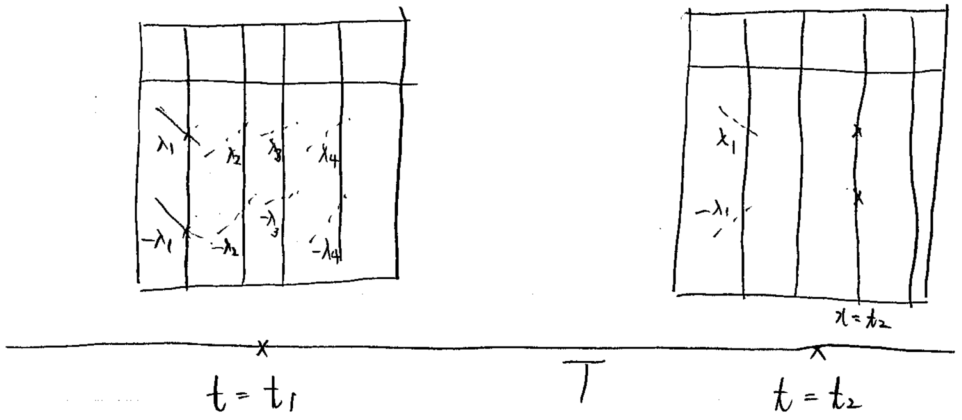
$\mathcal{D}$: $\tilde{v}$ の pole divisor

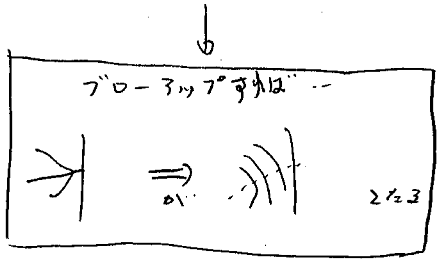
regular vector field



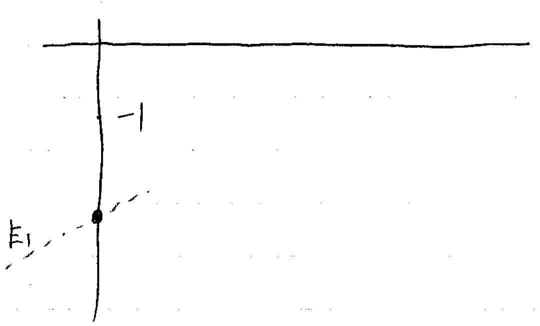
↑

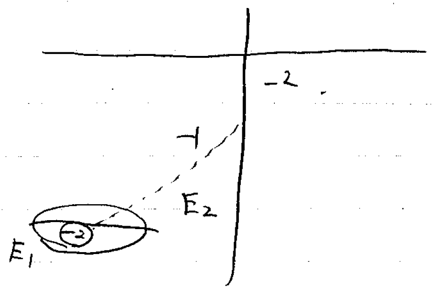
$\tilde{v}$ の到達可能特異点 という





$\lambda_1 = 0$ Resonance





P_{VI} の Riccati 解 の 1-parameter field
$\Leftrightarrow$ $\overline{\mathcal{S}} - \mathcal{D}$ 内の (-2)-$\mathbb{P}^1$

$$\begin{tikzcd}
& \overline{\mathcal{S}'} \arrow[dl] \arrow[ddl] & \\
\overline{\mathcal{S}} \arrow[d] & & \mathcal{D}' \arrow[ul, bend left] \\
T \times \Lambda_4 & &
\end{tikzcd}$$

$\overline{\mathcal{S}'}$ $(= \overline{\mathcal{S}}$ の ブローアップ$)$ ← smooth

$$\overline{\mathcal{S}}' - \mathcal{D}' \longrightarrow T \times \Lambda_4$$ ← smooth

○ ベクルート変換について.

$$\begin{tikzcd}
\overline{\mathcal{S}}' \arrow[r, dashed, "\text{flap}", "= \tilde{s}"'] \arrow[d, "\pi", "i(\tilde{v}) = \frac{\partial}{\partial t}"'] & \overline{\mathcal{S}}' \arrow[d] \\
T \times \Lambda_4 \arrow[r, "\simeq"] & T \times \Lambda_4 \\
(t, \lambda) \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto, "s"] & (t, s(\lambda)) \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

$\tilde{s}$: birarl map $\boxed{\tilde{s}_*(\tilde{v}) = \tilde{v}}$

$\overline{W}(D_4^{(1)})$ を $\Lambda_4 = \mathrm{Spec}\,\mathbb{C}[\lambda_1, \cdots \lambda_4]$ に作用させる.

- $i \neq 0$ $s_i(\lambda_j) = (-1)^{\delta_{ij}} \lambda_j$
- $s_0(\lambda_j) - \lambda_j - \frac{1}{2}\sum_{i=1}^{4} \lambda_4 + \frac{1}{2}$
- $s_i{}^2 = \mathrm{Id}$.
- $s \in \overline{W}(D_4^{(1)}) = \langle s_1, \cdots, s_4 \rangle$

Probrem ベクルット変換の幾何学的背景

- $s_1 \cdots s_4 \Longrightarrow$ とても簡単

  $s_0 \Longrightarrow$ mysterious 多分 Fourier-Mukai 変換であろう

◎ $\overline{\mathcal{S}}'_{t\lambda} - \mathcal{D}_{t,\lambda}$: 正則なシンプレクティック構造 $\Omega_{t\lambda}$ をもつ

$$\begin{tikzcd}
\overline{\mathcal{S}}' \arrow[r] \arrow[d] & \overline{\mathcal{D}'} \\
T \times \Lambda_4 &
\end{tikzcd}$$

$\Omega_{t\lambda} \ni \underset{\text{regular 2-form}}{\tilde{\Omega}_{t\lambda}}$: $\overline{\mathcal{S}}' \longrightarrow \overline{\mathcal{D}'}$, $T \times \Lambda_4$.

$$\left(\begin{array}{l} \Omega_{t\lambda} = dx \wedge dy \\ \Omega_\lambda = dx \wedge dy - dt \wedge dH \end{array} \right.$$

dH ← 小平-Spencer class を bound するもの

$$i_{\tilde{v}}(\tilde{\Omega}_\lambda) = 0 \iff \tilde{v} : \text{Hamiltonian に書ける}$$

目的 これらの事実を全て代数幾何の frame work で示したい

(ex. $\exists i.$ $2\lambda_i \in \mathbb{Z}$ という条件をはずしたい…。)

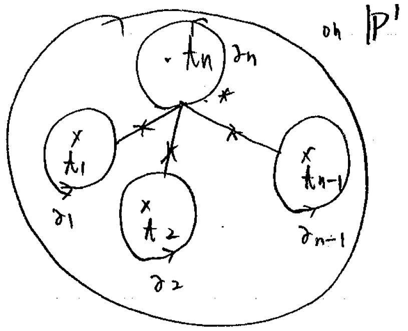
Miwa・Jimbo, Okamoto 等で書かれた

◎ Stable parabolic Connection

- $\mathbb{P}^1$, $n \in \mathbb{N}$, $n \geq 3$.
- $T_n = \{ \mathbf{t} = (t_1, t_2, \cdots t_n) \in (\mathbb{P}^1)^n \mid i \neq j . \ t_i \neq t_j \}$
- $\Lambda_n = \{ \boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n \}$

○ $\mathcal{A}_n = \{ \mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{C}^n \}$

○ $\Gamma_n = \pi_1(\mathbb{P}^1 - D(\mathbf{t}), *)$ $\left\{ \begin{array}{l} \mathbf{t} = (t_1, \cdots t_n) \\ D(\mathbf{t}) = t_1 + \cdots + t_n \end{array} \right.$
divisor on $\mathbb{P}^1$.



$(\boldsymbol{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$: fix

L : $\mathbb{P}^1$ 上の直線束

$\nabla_L : L \longrightarrow L \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$: 接続

$$\left(\nabla_L(f \cdot \sigma) = df \otimes \sigma + f \nabla_L \sigma \quad \overset{\ni}{\underset{\text{rank2}}{\mathcal{O}}} \quad \overset{\ni}{L} \right.$$

Def $(E, \nabla, \varphi, \{l_i\}_{i=1}^n)$: $(\boldsymbol{t}, \boldsymbol{\lambda})$ - parabolic connection

$\Longleftrightarrow$ (1) E : $\mathbb{P}^1$ 上の rank 2 の vector 束

(2) $\nabla : E \longrightarrow E \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ connection

(3) $\varphi : \bigwedge^2 E \xrightarrow{\sim} L$: 同型

(4) $l_i \subset E_{t_i} = E \otimes \mathbb{C}(t_i)$: 1次元部分空間

s.t (a) s_1, s_2 : E の local rection

$$\nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2 = \nabla_L(\varphi(s_1 \wedge s_2))$$

(a) $l_i \subseteq \mathrm{Ker}(\mathrm{res}_{t_i}(\nabla) - \lambda_i)$

i.e. $\mathrm{res}_{t_i}(\nabla) s_i = \lambda_1 s_i$, $s_i \in l_i$.

Def $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \Lambda_n$

$\boldsymbol{\lambda}$ is special $\Longleftrightarrow$ (1) ${}^\exists i$, $1 \leq i \leq n$ $2\lambda_i \in \mathbb{Z}$ ($\boldsymbol{\lambda}$: Resonance)

(2) ${}^\exists \varepsilon_i \in \{\pm 1\}$ $\sum_{i=1}^n \varepsilon_i \lambda_i \in \mathbb{Z}$. ($\boldsymbol{\lambda}$: reducible)

Def $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{2n} < 1$
$\alpha_i \in \mathbb{R}$

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$: weight

$\boxed{\text{Mamford の GIT を使われた}}$

$\Downarrow$

$\boxed{\text{安定性 stability}}$

$(E, \nabla, \varphi, \{l_i\}_{i=1}^n) = E$

$$\mathrm{pardeg}_{\alpha}(E) = \deg E + \sum_{i=1}^n \left(\alpha_{2i-1} \underbrace{(\dim E_{t_i}/l_i)}_{1} + \alpha_{2i} \dim \underbrace{l_i}_{1} \right)$$

$$= \deg E + \sum_{i=1}^n (\alpha_{2i-1} + \alpha_{2i})$$

s.t $F \subset E$ $\nabla F \subset F \otimes \Omega^1_{\mathbb{P}}(D(\boldsymbol{t}))$

$$\mathrm{pardeg}_{\alpha} F := \deg F + \sum_{i=1}^n (\alpha_{2i-1} \dim F_{t_i}/l_i \cap F_{t_i} + \alpha_{2i} \dim (l_i \cap F_{t_i}))$$

Def $(E, \nabla, \varphi, \{l_i\})$ が stable (semistable)

$\Longleftrightarrow$ ${}^\forall F \subsetneq E$ sub line bundle, $\nabla F \subseteq F \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$

$$\mathrm{pardeg}_{\alpha} F \leq \frac{\mathrm{pardeg}_{\alpha} E}{2}$$

$M_n^{\alpha}(\boldsymbol{t}, \boldsymbol{\lambda}, L)$: coarse moduli

$$= \left\{ (E, \nabla, \varphi, \{l_i\}) : \text{stable } (\boldsymbol{t}, \boldsymbol{\lambda})\text{- parabolic connection} \right\} \Big/ \text{isom}$$

Remark Arinkin - Lysonce stack で構成

$\boldsymbol{\lambda}$: non - special を仮定する

$$\begin{cases} L = \mathcal{O}_{\mathbb{P}^1}(-1) \\ \nabla_L (z - t_n) = (z - t_n) \dfrac{dz}{z - t_n} \end{cases}$$

z : $\mathbb{P}^1$ の inhom

$$\begin{tikzcd} \mathcal{M}_n^{\alpha} \arrow[d, "\pi"] \\ T_n \times \Lambda_n \end{tikzcd}$$

$$\pi^{-1}(\boldsymbol{t}, \boldsymbol{\lambda}) = M_n^{\alpha}(\boldsymbol{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbb{P}'}(-1))$$

$$\overline{\mathcal{M}_n^{\alpha}}(\boldsymbol{t}, \boldsymbol{\lambda}, L) = \left\{ (E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \right\}$$

: stable $(\boldsymbol{t}, \boldsymbol{\lambda})$ parabolic ϕ-connect with the director (L, ∇_2) / Isom

$$\begin{tikzcd} \overline{\mathcal{M}_n^{\alpha}}(\boldsymbol{t}, \boldsymbol{\lambda}, L) \\ \mathcal{M}_n^{\alpha}(\boldsymbol{t}, \boldsymbol{\lambda}, L) \arrow[u, hook] \end{tikzcd}$$

$$((E, \nabla, \varphi, \{l_i\}) \longrightarrow (E, E, \mathrm{id}, \nabla, \varphi, \{l_i\}))$$

これら埋め込みをすれば良い.

$$\begin{tikzcd} \mathcal{M}_n^{\alpha} \arrow[r, hook] \arrow[d, "\pi"] & \overline{\mathcal{M}_n^{\alpha}} \arrow[d, "\overline{\pi}"] \\ T_n \times \Lambda_n \arrow[r, equal] & T_n \times \Lambda_n \end{tikzcd}$$

- E_1, E_2 = rank 2 v-b on $\mathbb{P}^1$
- $\phi : E_1 \longrightarrow E_2$ $\mathcal{O}_{\mathbb{P}^1}$-homo
- $\nabla : E_1 \longrightarrow E_2 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$

  $\nabla(fa) = \phi(a) \otimes df + f \nabla(a)$

  $\underset{\mathcal{O}}{\overset{\cap}{f}}\ \underset{E_1}{\overset{\cap}{a}}$
- $\varphi : \Lambda^2 E_2 \xrightarrow{\sim} L$

<u>Theorem</u> (1) $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ $|\alpha_i| \ll 1$

$\overline{\pi} : \overline{\mathcal{M}_n^{\alpha}} \longrightarrow T_n \times \Lambda_n$: projective morphism

$\overline{\mathcal{M}_n^{\alpha}}(\boldsymbol{t}, \boldsymbol{\lambda}, -1)$: projective variety (compact)

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(2) $\alpha = (\alpha_1, \cdots, \alpha_{2n})$, $|\alpha_i| \ll 1$

$\pi : \mathcal{M}_n^{\alpha} \longrightarrow T_n \times \Lambda_n$

smooth morphism of relative dimension $2n-6$

$\forall (\boldsymbol{t}, \boldsymbol{\lambda})$, $\mathcal{M}_n^{\alpha}(\boldsymbol{t}, \boldsymbol{\lambda}, -1)$: smooth algebraic manifold.

<u>Theorem</u> $n = 4$ $|\alpha_i| \ll 1$

(1) $\overline{\pi} : \overline{\mathcal{M}_4^{\alpha}} \longrightarrow T_4 \times \Lambda_4$: projective smooth

$n = 4 \Longrightarrow$ Painlevé

$$\begin{tikzcd} \mathcal{M}_4^{\alpha} \arrow[r, hook] \arrow[d, "\pi"'] & \overline{\mathcal{M}_4^{\alpha}} \arrow[d, "\overline{\pi}"] \\ T_4 \times \Lambda_4 \arrow[r, equal] & T_4 \times \Lambda_4 \end{tikzcd}$$

No.

Date 2003・7・7　19:30 ～ 21:00

小木曽啓示氏　Fermat quartic K3 via finite group symmetries.

$/\mathbb{C}$

Def.　X: 2-dim cpt mfd

X: K3 $\Leftrightarrow$ $\pi_1(X) = \{1\}$　$\exists\, \omega_X$: nowhere vanishing hol. 2-form

($\Leftrightarrow$ $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$)

Ex.　$X_4 = (X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0) \subset \mathbb{P}^3$　(X_4, H_4) pol K3

1つの最もsimpleな例

単純 … 形がきれい. 対称性が高い

テーマ.　Fermat K3 = K3全体の中で. 対称性で特徴付ける。

K3に作用しうる"ある種の"大きな群を逆に特徴付けられるか?

$$\mathrm{Aut}(X_4, H_4) > \widetilde{F}_{384} := (\mu_4^{\oplus 4} : S_4)/\mu_4$$

実は等号になる。

位数 $4^3 \cdot 4! = 2^9 \cdot 3$

solvable

$\cup$

(2-Sylow subgp)

$\widetilde{F}_{128}$　位数 2^9 : nilpotent

$\widetilde{F}_{384} \curvearrowright X_4$

$\widetilde{F}_{128} \curvearrowright X_4$　standard action.

Def.　G : finite gp $\underset{\text{faithful}}{\curvearrowright}$ X : K3 のとき. G: K3 gp (on X)

と呼ぶ。

Main Thm.

(1)　G : finite solvable K3 gp (on X)

$\Rightarrow$ $|G| \leqq 2^9 \cdot 3$　更に $|G| = 2^4 \cdot 3$

$\Rightarrow (X, G) \cong (X_4, \widetilde{F}_{384})$ standard

(2)　G : finite, nilpotent K3 gp (on X)

$\Rightarrow$ $|G| \leqq 2^9$　更に $(X, G) \cong (X_4, \widetilde{F}_{128})$ standard

Remark.　$|\mathrm{Aut}(X_4)| = +\infty$　$(\rho(X_4) = 20)$

Remark　群の階層

{cyclic of prime order} ⋯ {abel} ⊂ {nilpotent} ⊂ {solvable}

{cyclic of prime order} ⋯ {non-comm. simple gp} ⊂ {perfect gp}

$\cap\!\|$　{non-comm. simple gp} ⊇ {g-simple}　　{perfect gp} ⊇ {g-perfect}

{g-simple} ↓ By O-Zhang　$L_2(7) = PSL(2, \mathbb{F}_7) \cong PGL(3, \mathbb{F}_2)$

$L_2(7) \times \mu_4 \Rightarrow X_0^4 + \text{Klein Curveの式} = 0$

{g-perfect} ↓ By Kondo　$M_{20} : \mu_4$

最大位数 $960 \times 4 = 3840$

$\Rightarrow K_m(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$

[K3 gpについて.] (知られていること) ← Nikulin, Mukai

$\boxed{0}$　$G \curvearrowright X$

$H^0(X, \Omega_X^2) = \mathbb{C}\,\omega_X \curvearrowleft G$: 1次表現

$g \mapsto g^*\omega_X = \alpha(g)\,\omega_X$

$$1 \to \underbrace{G_N = \mathrm{Ker}\,\alpha}_{\text{symplectic}} \to G \to \mu_I \to 1 \qquad \mu_I \subset \mathbb{C}^\times$$

$\widetilde{F}_{384} \curvearrowright X_4$

$\widetilde{F}_{128}$ standard action　を考えると. 上の exact seq. は次のようになる。

$$1 \to F_{384} \to \widetilde{F}_{384} \xrightarrow{\alpha} \mu_4 \to 1$$

F_{384}: $2^7 \cdot 3$　$(\zeta_4, 1, 1, 1) \longmapsto \zeta_4$

$$1 \to F_{128} \to \widetilde{F}_{128} \xrightarrow{\alpha} \mu_4 \to 1$$

2^7

$\boxed{1}$ G_N について

(0) 元の位数 (Nikulin)

$\mathrm{ord}(g)$	1	2	3	4	5	6	7	8
X^g	X	8	6	4	4	2	3	2

$$\begin{tikzcd} X \arrow[d] \\ X/\langle g\rangle & Y \arrow[l] \end{tikzcd}$$

包含関係で極大なもの 11 種

1) Thm (Mukai) $G_N \hookrightarrow M_{23}$

$\Omega = \{1-24\}$ $P(\Omega) \supset St(5,8,24)$

8元からなる部分集合の集合で
∀5元を含むものが唯1つある。

$M_{24} := \{g \in S_{24} \mid g(\mathcal{S}) = \mathcal{S}\}$

$\curvearrowright \{1-24\}$ 5 transitive 単純

$\vee$

$M_{23} :=$ 1点 の stabilizer $|M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$

(Xiao) 同型類では 80 コ

By Mukai's thm. $F_{128} \hookrightarrow M_{23}$

$\therefore$ $F_{128} \cong$ (M_{23} の 2-sylow subgp)

$\boxed{2}$ μ_I について. ($I \geq 2 \Rightarrow X$: proj.) $\curvearrowleft X/G$ を考えればよい.

Transcendental lattice

$H^2(X, \mathbb{Z}) \supset NS(X) \oplus T(X)$ $T(X) \otimes \mathbb{C} \supset \mathbb{C}\omega_X$

Nikulin. $\varphi(I)$ $rk\, T(X)$ $2 \leq rk\, T(X) \leq 21$

$\Rightarrow \varphi(\tau) \leq 20,$ $I \leq 66$

$I = 60$ だけは存在しない.

注. $F_{128} > Q_{16} = \langle a, b \mid a^8 = 1,\ b^{-1}ab = a^{-1}, a^4 = b^2 \rangle$

$\boxed{3}$ G: extension [Ivanov - O - Zhang]

完全にはわかっていない。

Outline of proof.

Step 1. $|G| = |G_N| \cdot I$

prop 1. solvable K3 gp $\Rightarrow |G| \leq 2^9 \cdot 3$

"=" $\Rightarrow G_N \cong F_{384}$ (& $I = 4$)

nilpotent K3 gp $\Rightarrow |G| \leq 2^9$

"=" $\Rightarrow G_N \cong F_{128}$ (& $I = 4$)

Step 2. pol. K3 with symplectic Q_{16}-action

prop 2. (X, H) pol K3 $(H^2) = 4$ $Q_{16} \curvearrowright X$ symplectic

$\Rightarrow (X, H) \cong (X_4, H_4)$

Fermat ↗ ↖ standard pol

$\sum x_i^4 = 0 \subset \mathbb{P}^3$
を与えるもの.

Step 3 (KEY)

prop 3. G: K3gp $\curvearrowright X$ $|G| = 2^9$

$\Rightarrow NS(X)^G = \mathbb{Z}H$ $(H^2) = 4$

Prop 1.2.3 $\Rightarrow$ Main Thm.

(1) Y: K3 $\curvearrowleft F$ K3 gp $|F| = 2^4 \cdot 3$ G = F の 2-sylow

$|G| = 2^4$

$\downarrow$

冪零

prop1 $\quad G_N \cong F_{128}$ $\quad$ (& $I=4$) $\hookrightarrow Y$ proj

$$Q_{16} < \underset{\substack{\| \\ G_N}}{F_{128}} < G < F$$

$$\text{☆} - \quad NS(Y)^{F} \subseteq NS(X)^{G} \subseteq NS(X)^{G_N} \subseteq NS(X)^{Q_{16}}$$

Q_{16} 位数	1	2	4	8
元	1	1	10	4

$$rk\, H^*(X, \mathbb{Z})^{Q_{16}} = \frac{1}{|Q_{16}|} \sum_{g \in Q_{16}} Tr(g^* \mid H^*(X, \mathbb{Z}))$$

$$H^* = H^0 + H^1 + H^2$$

$$= \frac{1}{|Q_{16}|} \sum_{g \in Q_{16}} \chi_{top}(X^g) \quad \text{Lefschetz}$$

$$= \frac{1}{16}(24 + 8 + 10 \times 4 + 2 \times 4) = 5$$

$$\therefore rk H^2(X, \mathbb{Z})^{Q_{16}} = 3$$

$$\cup$$

$$T(X) \quad rk T(X) \geq 2$$

従って invariant な polarization の $rk = 1$

$$\therefore NS(X)^{Q_{16}} = \mathbb{Z}H$$

∴ ☆ は全て等号

prop3 より $(H^2) = 4 \overset{\text{prop2}}{+} \Rightarrow X_4 \simeq Y$

以下、$(X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0) \subset \mathbb{P}^3$ としてよい.

$$\mu_4^{\oplus 4} : S_4 / \mu_4$$

$$\mathrm{Aut}(X_4, H_4) \qquad \text{より} \qquad \mathrm{Aut}(X_4, H_4) = (\mu_4^{\oplus 4} : S_4)/\mu_4$$

$$\begin{matrix} \mathrm{Aut}(X_4, H_4) \\ \vee \\ F \\ \text{prop1} \rightarrow \vee \\ F_{384} \end{matrix} \Bigg] \; I \leq 4 \qquad \qquad F = \widetilde{F_{384}}$$

prop1 の idea.

G: solvable $\Rightarrow$ G_N: solvable.

$$1 \longrightarrow G_N \longrightarrow G \longrightarrow \mu_N \longrightarrow 1.$$

構造簡明 $\big\langle$ $5 \mid |G_N|$, $7 \mid |G_N|$, $9 \mid |G_N|$

2-群, $2^n \cdot 3$) 表現論 2 gp $\subset F_{128}$

$5 \mid \|G_N\|$.	$7 \mid \|G_N\|$	$9 \mid \|G_N\|$
≤ 80	C_7	≤ 72 ⟹ 左と同様
$C_2^{\oplus 4} : D_{10}$	$C_7 : C_1$	$A_4 \times A_4 \leftarrow 144$
↑ 160		$A_{4,4} = (S_4 \times S_4) \cap A_8 \leftarrow 288$
⇓ 詳しく評価		↘ 難しい $288 \times 6 = 1728$

$I \leq 6$,

$$rk\, H^2(X, \mathbb{Z})^{G_N} \leq rk\, H^2(X, \mathbb{Z})^{C_3^{\oplus 2}} \quad \text{etc を用いて}$$

$\varphi(I) \leq 12$ が出る。 あとは個別に評価.

$A_{4,4}$ Case より詳しく見ることで $\varphi(I) \leq 2$, $I = 1, 2, 3, 4, 6$ が分かる.

$I = 4$ ならば $288 \cdot 4 < 2^9 \cdot 3$

$I = 6$ を示したいが, その為には $I \neq 3$ を示せばよい。

prop. ([Ivanov - O - Zhang])

$$1 \longrightarrow G_N \longrightarrow G \xrightarrow{\alpha} \mu_3 \longrightarrow 0 \quad \text{とする. このとき.}$$

$$\overset{\rotatebox{90}{$\in$}}{g} \longmapsto \zeta_3$$

(1) $ord(g) = 3k$ $\quad (k, 3) = 1 \quad \Rightarrow G = G_N : \mu_3$

(2) $\mathrm{ord}(g) = 6$ のとき $X^g \; (\subset X^{g^3} = 8pt)$

$p \in X^g$ の型は次の2通り $\begin{pmatrix} \zeta_6^{-1} & 0 \\ 0 & \zeta_6^{3} \end{pmatrix}, \begin{pmatrix} \zeta_6^{-5} & 0 \\ 0 & \zeta_6 \end{pmatrix}$

m_1 コ　m_2コ とする。

$(m_1, m_5) = (2, 0), (4, 1)$ or $(6, 2)$

(2)の場合に hol. Lefschetz を適用すると.

$$1 - \zeta_3 \; \Big(= \underset{(-1)^i}{\Sigma} \mathrm{tr}\, g^* (H^i(\mathcal{O}_X)) \Big) = \frac{m_1}{(1-\zeta_6^{-1})(1-\zeta_6^{3})} + \frac{m_5}{(1-\zeta_6^{5})(1-\zeta_6)}$$

この不定式を解くことで上の組が得られる。

$A_{4,4} \Rightarrow I \neq 3$ の証明のoutline

$I = 3$ とする。　$G \overset{(1)}{=} A_{4.4} : \overset{\mu_3}{\overset{\shortparallel}{\langle g \rangle}}$

交換子群　$[A_{4.4}, A_{4.4}] = A_4 \times A_4$　特性群.

従って. $\underset{\overset{\shortparallel}{(C_2^{\oplus 2} \times C_2^{\oplus 2})}}{(A_4 \times A_4)} : \langle g \rangle$ は K3 gp, 3-sylow of order 3^3

$$(\underbrace{C_2^{\oplus 2} \times C_2^{\oplus 2}}_{2\text{-sylow} = \mathbb{F}_2^{\oplus 4}}) : (C_3 \times C_3)$$

よって　$C_2^{\oplus 2} \times C_2^{\oplus 2} \circlearrowleft C_3 \times C_3$
$\qquad\qquad \curvearrowleft \langle g \rangle$

最初の仮定より g の位数は3であって. formに作用.

$\therefore \; H \xrightarrow{\rho} GL(4, \mathbb{F}_2) \quad |\;| = 3^4.$

$g \in \mathrm{Ker}\,\rho \neq 1.$

$\therefore$ $C_2^{\oplus 4} \times \langle g \rangle$ は K3 gp　これは作用がtrivialであることを示する。

次に $C_2^{\oplus 4} \ni a \neq 0$ を取ってくると. $\mathrm{ord}(ga) = 6$
(2) を用いて
$(ga)^* \omega_X = \zeta_3 \omega_X$
$\Rightarrow$ $X^{ga} = M_1 \amalg M_5$ かつ M_1 or M_5 は 少なくとも片方 2点以下

ga と $C_2^{\oplus 4}$ の各元は可換.

$$\begin{tikzcd} C_2^{\oplus 4} \arrow[r, phantom, "\curvearrowright"] \arrow[dr, phantom, "\curvearrowright"] & M_1 \\ & M_5 \end{tikzcd}$$

$|M_i| \leq 2$ であったので

$C_2^{\oplus 4} \xrightarrow{\rho} \mathrm{Aut}(M_i) \subset S_2 \simeq C_2$　$\therefore \mathrm{Ker}\,\rho \geqq C_2^{\oplus 3}$

$\underset{\overset{\uparrow}{\text{symplectic}}}{C_2^{\oplus 3}} \hookrightarrow SL(T_X, p) = SL(2, \mathbb{C})$　$p \in M_i$

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Ⅱ 斎藤 政彦

- $T_n = \{ \mathbb{t} = (t_1, \dots, t_n) \in (\mathbb{P}^1)^n \mid \overset{i \neq j}{t_i \neq t_j} \} \ni \mathbb{t} \to \underset{\{\mathbb{P}^1\text{上の Divisor}\}}{\underset{\cap}{D(\mathbb{t})}} = t_1 + \dots + t_n$

- $\Lambda_n = \operatorname{Spec} \mathbb{C}[\lambda] = \operatorname{Spec} \mathbb{C}[\lambda_1, \dots, \lambda_k] \simeq \mathbb{C}^n$

$\alpha \quad 0 < \alpha_1 < \dots < \alpha_{2n} < 1 \qquad \nabla_L : \overset{\text{line bundle}}{\overset{\downarrow}{L}} \to L \otimes \Omega^1_{\mathbb{P}^1}(D(\mathbb{t}))$

$M_n^\alpha(\mathbb{t}, \lambda, L) = \{ (E, \nabla, \varphi, \{l_i\}_{i=1}^n \} :$ stable rank2 $(\mathbb{t}, \lambda)$ parabolic connection $\det \simeq (\underset{\text{rank2}}{L}, \nabla_L)$ $\} / \simeq$

$M_n^\alpha(\mathbb{t}, \lambda, L) = \{ (E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n) :$ stable $(\mathbb{t}, \lambda)$-parabolic ϕ connection $\det = (L, \nabla_L)$

thm. $\forall i \quad |\alpha_i| << 1 \qquad n \geq 4$

$(L, \nabla_L) = (\mathcal{O}_{\mathbb{P}^1}(-t), d)$

$$\begin{tikzcd}
\exists \quad M_n^\alpha \arrow[r, hook] \arrow[d, "\pi_t"'] & \overline{M}_n^\alpha \arrow[d, "\overline{\pi}_t"'] \\
T_n \times \Lambda_n \arrow[r, phantom, "="] & T_n \times \Lambda_n
\end{tikzcd}$$

$\forall (t, \lambda) \in T_n \times \Lambda_n$

$\pi_t^{-1}((t, \lambda)) = M_n^\alpha(t, \lambda, \mathcal{O}_{\mathbb{P}}(-t))$

$\cap$

$\pi_t^{-1}(t, \lambda) = \overline{M}_n^\alpha(t, \lambda, \mathcal{O}_{\mathbb{P}}(-t)) \}$

(1) $\overline{\pi}_n$: projective morphism

(2) π_n : smooth morphism of ______ $2n-6$

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thm $n = 4 \qquad |\alpha_i| << 1$

$$\begin{tikzcd}
M_4^\alpha \arrow[r, hook] \arrow[dr, "\pi_4"'] & \overline{M}_4^\alpha \arrow[d, "\overline{\pi}_4"] \arrow[r] & \mathbb{F}_2 \times T_4 \times \Lambda_4 \arrow[dl] \\
& T_4 \times \Lambda_4 &
\end{tikzcd}$$

$\ni$ ______

$\mathbb{F}_2 = \mathbb{P}(\Omega^1_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$

$\downarrow$

$\mathbb{P}^0$ ______

(1) $\overline{\pi}_4$: smooth proj

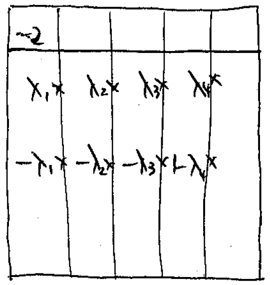
$(t, \lambda) \in T_4 \times \Lambda_4$ fix

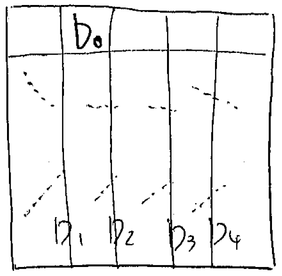
$S = M_4^\alpha \qquad \exists D = S - M_4^\alpha$

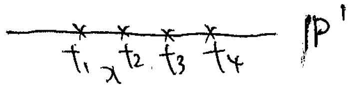
$-K_S = Y \qquad Y = D. \qquad (S, Y)$ Okamoto Painleve Pair of type D_4''

D^0

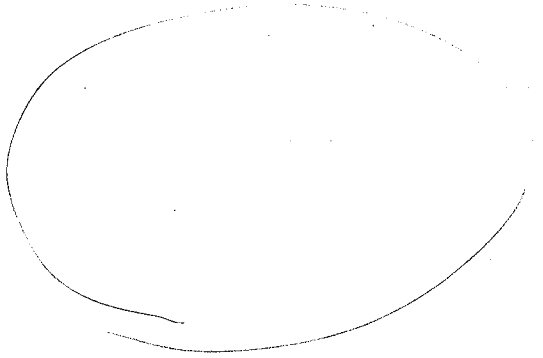
$D_0^2 = -2$







λ = non-special



Riemann - Hilbert 対応

$$= \mathbb{E}$$

$$(E, \nabla, \varphi\{l_i\}_{i=1}^{n}) \longmapsto \ker \nabla|_{\mathbb{P}^1 - D(t)}$$ 局所系.

$$\downarrow$$

$$\nabla = d + \sum \frac{A(z_i)}{(z-t_i)} \qquad \xrightarrow{R.H} \qquad \rho : \pi_1(\mathbb{P}^1 - D(t), *) \longrightarrow \mathrm{Aut}(\mathbb{E}_*, \det)$$

$$\mathrm{Aut}(\mathbb{E}_*, \det) \cong SL_2(\mathbb{C})$$

$$\nabla \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

の同値類

$[\rho]$ $\quad \rho_1 \sim \rho_2 \quad \exists P \in SL_2(\mathbb{C})$

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = - \sum \frac{A(z)}{(z-t_i)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$\forall \gamma \in \pi_1 \quad \rho_1(\gamma) = P^{-1} \rho_1(\gamma) \cdot P$

$$\Gamma_{n,t} = \pi_1(\mathbb{P}^1 - D(t), *)$$

$$= \langle \gamma_1, \gamma_2, \cdots, \gamma_n, \ \gamma_1 \cdots \gamma_n = 1 \rangle$$

$$\mathrm{Hom}(\Gamma_{n,t}, SL_2(\mathbb{C})) \simeq SL_2(\mathbb{C})^{n-1}$$

$$R_{0,t} = SL(2, \mathbb{C})^{n-1} / \mathrm{Ad}(SL_2(\mathbb{C}) \qquad (M_1, \cdots, M_n)$$

$\uparrow$ 商をとりたい

$$\mapsto (P^{-1}M_1P, \cdots P^{-1}M_nP)$$

(Categorical ______ を取る)

$SL_2(\mathbb{C})^{n-1}$ の affine coordinate ring

$$R_{n-1} = \bigotimes_{i=1}^{n-1} \mathbb{C}[a_i, b_i, c_i, d_i] / (a_i d_i - b_i c_i - 1)$$

$\nwarrow$ $\mathrm{Ad}(SL_2(\mathbb{C}))$

$$R_{n-1}^{\mathrm{Ad}(SL_2(\mathbb{C})} : \text{不変式環} \hookrightarrow R_{n-1}$$

$\downarrow$ Hilbert

有限生成

$$R_{0,t} = SL_2(\mathbb{C})^{n-1} / \mathrm{Ad}(SL_2(\mathbb{C}))$$

$$= \mathrm{Spec}\ R_{n-1}^{\mathrm{Ad}(SL_2(\mathbb{C}))}$$

Lemma

$$\Phi_n : SL_2(\mathbb{C})^{n-1} \longrightarrow R_{n,t} \qquad \begin{pmatrix} \alpha_i & c_i \\ 0 & \alpha_i^{-1} \end{pmatrix} \qquad \begin{pmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{-1} \end{pmatrix}$$

$$\ni x \text{ closed point} \qquad \searrow x \swarrow$$

$\Phi_n^{-1}(x)$: Jordan equivalence class のなす閉軌道

Prop $\quad M_1, \cdots, M_n \in SL_2(\mathbb{C})$

$$R_{n-1}^{\mathrm{Ad}(SL_2(\mathbb{C}))} = \mathbb{C}[\mathrm{Tr}(M_i), \mathrm{Tr}(M_iM_j), \mathrm{Tr}(M_iM_jM_k) \mid 1 \leq i,j,k \leq$$

$\mathrm{Tr}(M_i) = \alpha_i + \alpha_i^{-1} = a_i$ とおく

$$a_i = \mathrm{Tr}(\_\_\_\_\_\_)^{(f)}.$$

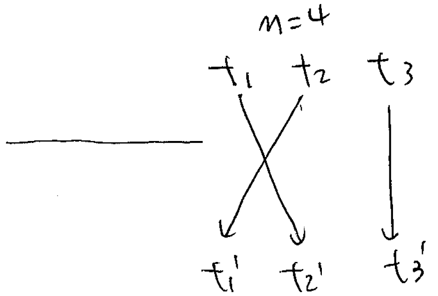
$$\mathbb{C}[a_1, \cdots, a_n] \hookrightarrow R_{n-1} \curvearrowleft \mathrm{Aut}(SL_2(\mathbb{C}) \ \mathfrak{a} = \mathrm{Tr}(\_\_\_\_\_\_)$$

$$R_{n,t} \rightarrow \mathbb{A}_n = \mathrm{Spec}\ \mathbb{C}[a_1, \cdots, a_n]$$

$\pi_1(T_n, \_) = PB_{n-1}$ $\qquad$ pure braid group of ______

$$m_n : PB_{n-1} \longrightarrow \mathrm{Aut}(R_{n,t})$$

Non-linear monodromy



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$$\begin{tikzcd}
R_{n,t_0} \times \tilde{T}_n \arrow[rr] \arrow[dr] \arrow[dd] & & R_n \arrow[dd] \\
 & R'_n \arrow[ur] \arrow[d] & \\
\tilde{T}_n \arrow[r] \arrow[rr, bend right] & T'_n \arrow[r] & T_n
\end{tikzcd}$$

$R_n = R_{n,t_0} \times \tilde{T}_n / PB_{n-1}$

universal cover

$$\begin{cases} R'_n = R_{n,t_0} \times \tilde{T}_n / T_{n-1} \\ T'_n = \tilde{T}_n / T_{n-1} \end{cases}$$

$\Gamma_{n-1} = \ker (PB_{n-1} \to \mathrm{Aut}(\mathbb{C}[a_1, \cdots, a_n]))$

$n=4$ Γ PB_{n-1}

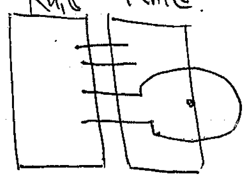
$PB_3 \simeq SL_2(\mathbb{Z})$

$\Gamma_3 = \Gamma(2)$

$$\begin{tikzcd}
R'_n \arrow[d, "\varphi'_n"] \\
T'_n \times A_n
\end{tikzcd}$$

(t, a)

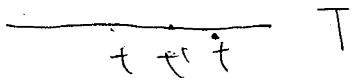
$R_{n,t}$ $R_{n,t}$



$a \in$ ○ fix

$$\begin{tikzcd}
R_{n,a} \arrow[d] & R_{n,t_0} \arrow[l, hook'] \arrow[d] \\
T'_n \times a & t_n \times a \arrow[l, phantom, "\ni"]
\end{tikzcd}$$

fiber bundle.



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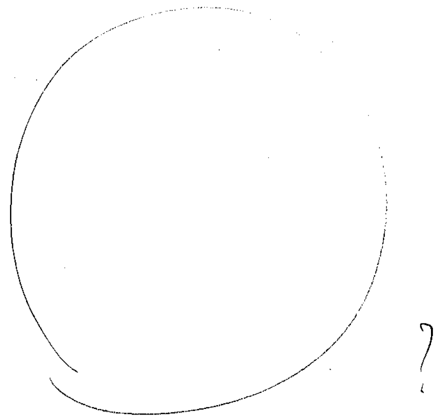
$$\begin{tikzcd}
\mathcal{M}^{\alpha}_n \arrow[r, "R.H"'] \arrow[d] & R'_n \arrow[d] \\
T'_n \times a \arrow[r] & T'_n \times A_n
\end{tikzcd}$$

(t, λ) (t, a)

$(E, \nabla, \varphi, \{l_i\}) \to \mathbb{E} = \ker \nabla|_{\mathbb{P}^1 - D(t)}$
$\to [\rho_E]$



?

$\mathcal{M}^{\alpha}_{n(t,\lambda)} \xrightarrow{RH_{(t,\lambda)}} R'_{n(t,a)}$

thm (1) $^{\forall}(t, \lambda) \in T'_n \times \Lambda_n$

$RH_{(t,\lambda)}$: surjective bimero___ ~~map~~ morphism

(2) (t, λ) : non special.

$RH_{(t,\lambda)}$: analytic

(3) (t, λ) : special

$\mathcal{M}^{\alpha}_{n(t,\lambda)} - RH(S) \simeq R_{n(t,\lambda)} - (\overset{S}{\underline{\ ?\ }})$

thm

(1) $^\forall(t,\lambda)$ $M_n^\alpha(t,\lambda)$: 標準的シンプレクティック形式 $\Omega_{t\lambda}$ をもつ

(2) $^\forall(t,a)$ $R^{1\,\mathrm{reg}}_{n(t,a)}$ は シンプレクティック構造をもち $\Omega^1_{(t,a)}$

$$\Omega^1_{t,\lambda}|_{RH^1(R^{\mathrm{reg}})} = RH^*_{t,\lambda}(\Omega^1_{(t,a)})$$

⇒ RHは analytic simultaneous synplectic resolution を与える。

$N = \bigcirc - 3$

$\Omega(t,\lambda) = \Omega|_{M_n^\alpha(t,\lambda)}$ $\quad\Omega \in \Gamma(M_n^\alpha, \Omega^2_{M_n^\alpha/T_n\times\Lambda_n})$

$\Omega|(t,\lambda) = \sum_{i=1}^{N} dp_i\wedge dq_i$ $\quad{}^\exists\tilde{\Omega} \in \Gamma(M_n^\alpha, \Omega^2_{M_n^\alpha/\Lambda_n})$

$$\tilde{\Omega} = \sum_{i=1}^{N} dp_i\wedge dq_i - \sum_{i=1}^{n} dt_i\wedge dH_i(p,q,t,\lambda)$$

Tangent Speaces to $M_n^\alpha(t,\lambda)$ $R_n(t,a)$

$(\tilde{E}, \tilde{\nabla}, \tilde{\varphi}, \{\tilde{l}_i\})$ universal family

$$\begin{tikzcd}
(\tilde{E}, \tilde{\nabla}, \tilde{\varphi}, \{\tilde{l}_i\}) \arrow[d] & \\
\mathbb{P}^1\times M_n^\alpha \arrow[dd, "\tilde{\pi}_1"'] \arrow[dr] & \\
& \mathbb{P}^1\times T_n\times\Lambda_n \arrow[dl] \\
\mathbb{P}^1\times T_n \arrow[d] & \\
T_{..} &
\end{tikzcd}$$

fiber $M_n^\alpha(t,\lambda)$ $\quad x \in M_n^\alpha$ fix $\quad \pi(x) = (t,\lambda)$

$$\begin{tikzcd}
0 \arrow[r] & \Theta_{M_n^\alpha(t,\lambda)\,x} \arrow[r] \arrow[d, equal] & \Theta_{M_n^\alpha,x} \arrow[r] \arrow[d, equal] & \pi^+(\Theta_{T_n\times\Lambda_n/T_n}(t. \\
\cdots \arrow[r] 0 \,/\!/ & \mathbb{H}^1(F_x^0) \arrow[r] & \mathbb{H}^1(F_x^+) \arrow[r] & \mathbb{H}^1(F_x^+/F_x^0) \arrow[r] & 0
\end{tikzcd}$$

$\mathbb{P}^1\times M_n^\alpha$ 上の 3つの sheaf $\quad$ ⓪ $\mathbb{H}^2(F_x^0) = \{0\}$

$F^0 := \{ s\in \mathcal{E}nd(\tilde{\mathcal{E}}) \mid \underline{\mathrm{Tr}(s)=0 \qquad\qquad}_{\,^\forall x\in M_n^\alpha} \}$

$F^1 := \{ s\in \mathcal{E}nd(\tilde{\mathcal{E}})\otimes \pi(\underline{\qquad}) \mid \mathrm{tr}(s)=0 \;\; {}^\forall x\in M_n^\alpha \}$

$F^{1+} = \{ s\in \quad : \quad \mid \underline{\mathrm{tr}(s)=0 \qquad} \}$

$\nabla_1 : \mathcal{E}nd(\tilde{\mathcal{E}}) \to \mathcal{E}nd(\tilde{\mathcal{E}})\otimes \underline{\qquad\qquad}$

$\qquad\qquad \overset{\in}{s}$

$\nabla_1 s = s\tilde{\nabla} - \tilde{\nabla}s$

$$\begin{tikzcd}
\mathbb{P}^1\times M_x^\alpha & \\
\mathbb{P}^1\times x \arrow[u, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[d] \arrow[r, phantom, "="] & {[}F^0 \arrow[r, "\nabla"] & F^1{]} \arrow[d] \\
F^{1+} \arrow[d] \arrow[r, phantom, "="] & {[}F^0 \arrow[r] & F^{1+}{]} \arrow[d] \\
F^{1+}/F \arrow[r, phantom, "="] & {[}0 \arrow[r] & F^{1+}/F{]}
\end{tikzcd}$$

Prop $^{\exists}\, \mathbb{H}^1(\mathbb{P}^1, F^0_x) \otimes \mathbb{H}^1(\mathbb{P}^1, F^1_x) \to H^2(\Omega^1_{\mathbb{P}})$

$$\begin{tikzcd} H^2(\Omega^1_{\mathbb{P}}) \arrow[d, phantom, "\shortparallel"] \\ H^2(\mathbb{P}^1, \mathbb{C}) \end{tikzcd}$$

$\Rightarrow\ ^{\exists}\Omega \in \Gamma(\mathcal{M}^{\alpha}_n, \Omega^2_{\mathcal{M}^{\alpha}_n / T_n \times A_n})$

$t \in T_n'$: fix $\quad x \in R_n'$ fix $\quad \pi(x) = (t, a)$

$$\begin{tikzcd} R_n' \arrow[d, "\pi"] \\ T_n' \times A_n \end{tikzcd}$$

$0 \to \Theta_{\bigcirc} \to \Theta_{\bigcirc} \to \pi^0(\Theta_0) \to 0$

$x \in R_{\bigcirc}^{\bigcirc}$

Prop $\mathbb{E}$ xに対応する $\mathbb{P}^1 - D(t)$上の local system
$\pi(x) = 0$

(1) $\mathbb{E}$: Irreducible $\Rightarrow$ $\mathcal{R}_n'$は xで非特異

(2) (1)かつ $M_1, \cdots, M_n$: monodromy Matrix

$\forall i,\ M_i \neq \pm I_2 \Rightarrow \mathcal{R}'_{n,a} = \pi^{-1}(a)$は
xで非特異.

(1), (2) を仮定する

$\mathbb{V} = \mathrm{End}(E)'' = \{ s \in \mathrm{End}(E) \mid \mathrm{Tr}(s) = 0 \}$
$\mathbb{P}^1 - D(t)$ 上の局所系、 $j : \mathbb{P}^1 - D(t) \subset \mathbb{P}^1$

$$\begin{tikzcd} 0 \arrow[r] & \Theta_{\bigcirc} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \Theta_{\bigcirc} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi^0(\Theta_{\bigcirc}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\ 0 \arrow[r] & H^1(\mathbb{P}^1, j_*\mathbb{V}) \arrow[r] & H^1(\mathbb{P}^1 - D(t), \mathbb{V}) \arrow[r] & H^0(\mathbb{P}^1, \bigcirc) \arrow[r] & 0 \end{tikzcd}$$

$\mathbb{E}$: ____ $\Rightarrow H^2(\mathbb{P}^1 - D(t), \mathbb{V}) = (0)$.

$$\begin{tikzcd} H^1(\mathbb{P}^1, j_*\mathbb{V}) \otimes H^1(\mathbb{P}^1, j_*\mathbb{V}) \arrow[r] \arrow[dr] & H^2(\mathbb{P}^1, \quad) \arrow[d] \\ & H^2(\mathbb{P}^1, \quad) \end{tikzcd}$$

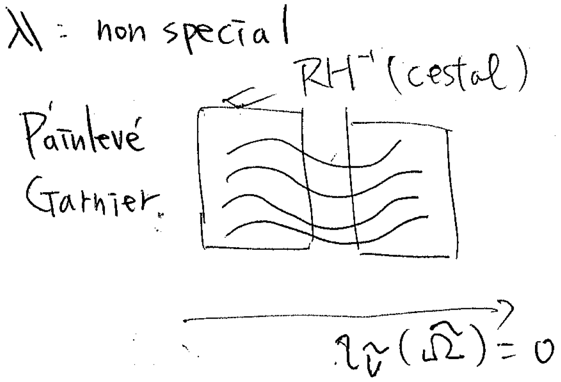
$j_*\mathbb{V} \otimes j_*\mathbb{V} \to$ ________

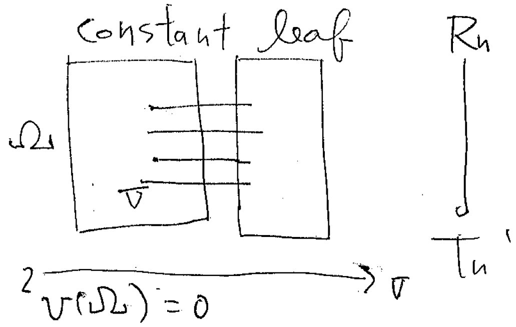
(2) ならば

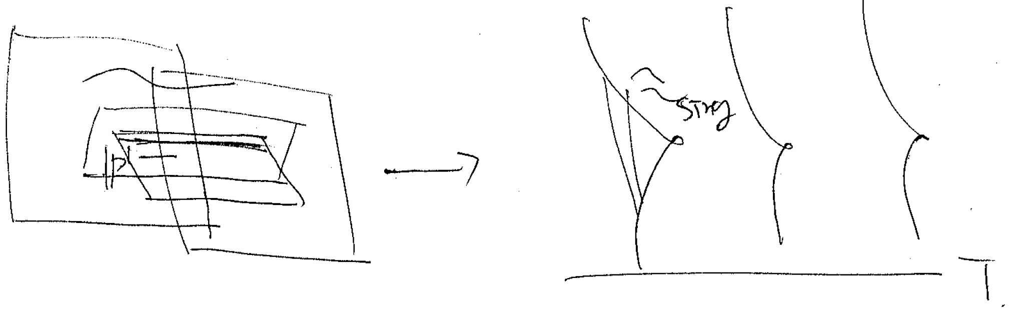
$j_*\mathbb{V} = \ker(F^0 \xrightarrow{\nabla_1} F^1)$

$H^1(\mathbb{P}^1, j_*\mathbb{V}) \simeq H^1(F^0 \xrightarrow{\nabla_1} F^1)$

λ : non special







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Date 2003: 7. 8. 14:00 ~ 15:30

稲場道明氏

Joint work with K. Iwasaki and M-H. Saito

§ Definition of the moduli space.

$T_n := \{ (t_1, \cdots, t_n) \in (\mathbb{P}^1)^n \mid t_i \neq t_j \text{ for } i \neq j \}$

$\Lambda_n = \mathbb{C}^n \quad (n \geqq 4)$

$T_n \times \Lambda_n \ni (t, \lambda) = (t_1, \cdots, t_n, \lambda_1, \cdots, \lambda_n)$

L : line bundle on $\mathbb{P}^1$

$\nabla_L : L \to L \otimes \Omega_{\mathbb{P}^1}(t_1 + \cdots + t_n)$: connection

$\mathrm{Res}_{t_i}(\nabla_L) \in \mathbb{Z} \quad (1 \leqq i \leqq n)$

$(L, \nabla_L)|_{\mathbb{P}^1 - \{t_1, \cdots, t_n\}}$: trivial. $\quad \sum_{i=1}^{n} \mathrm{Res}_{t_i}(\nabla_L) = \deg L$

Def. $(E, \nabla_E, \{l_i\}_{i=1}^{n}, \psi)$ is a (t, λ) parabolic connection

$\overset{\text{def}}{\Longleftrightarrow}$

(1) E : rk 2. vector bundle on $\mathbb{P}^1$

(2) $\nabla_E : E \to E \otimes \Omega_{\mathbb{P}^1}(t_1 + \cdots + t_n)$

connection (algebraic)

(3) $\psi : \det E \xrightarrow{\sim} L$ horizontal

(4) $E|_{t_i} \supset l_i$: line $\quad (\mathrm{Res}_{t_i}(\nabla_E) - \lambda_i)|_{l_i} = 0$

Remark. If $\lambda_i \neq \mathrm{Res}_{t_i}(\nabla_L) - \lambda_i \qquad l_i = \ker(\mathrm{Res}_{t_i}(\nabla_E) - \lambda_i)$

$\mathrm{Res}_{t_i}(\nabla_E)$ の2つの固有値

If $\lambda_i = \mathrm{Res}_{t_i}(\nabla_L) - \lambda_i$ and $\mathrm{Res}_{t_i}(\nabla_E) = \lambda_i \mathrm{id}_{E|_{t_i}}$

$\{ l_i : \text{satisfying } (4) \} \simeq \mathbb{P}(E|_{t_i}) \cong \mathbb{P}^1 \quad \leftrightarrow$ Riccati solution

$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{2n} < 1$ rational numbers (generic. fix)

Def. $(E, \nabla_E, \{l_i\}, \psi)$ is stable.

$\overset{\text{def}}{\Longleftrightarrow} \quad 0 \neq \forall (F, \nabla_F) \subsetneqq (E, \nabla_E)$

$$\frac{1}{\mathrm{rk} F}\left(\deg F + \sum_{i=1}^{n} \left(\alpha_{2i-1} \dim\left(F|_{t_i} / l_i \cap F|_{t_i}\right) + \alpha_{2i} \dim(F|_{t_i} \cap l_i)\right)\right)$$

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$$< \frac{1}{\mathrm{rk} E}\left(\deg E + \sum_{i=1}^{n} \left(\alpha_{2i-1} \dim\left(E|_{t_i} / l_i\right) + \alpha_{2i} \dim(E|_{t_i} \cap l_i)\right)\right)$$

Th 1. $\exists M^{\alpha}(t, \lambda, L)$: moduli space of stable parabolic connections

$M^{\alpha}(t, \lambda, L)$ is smooth and quasi projective.

Remark $M^{\alpha}(t, \lambda, L)$ was constructed analytically by H. Nakajima

$M^{\alpha}(t, \lambda, L)$ is not compact

構成では

$\mathbb{P}^1 \rightsquigarrow X/S$: flat family of smooth curves

rank 2 $\rightsquigarrow$ rank r

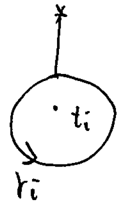
$t_1, \cdots, t_n \rightsquigarrow D \subset X$: effective divisor flat over S

$\cdots$, etc

(GIT construction)

$$R(t, \lambda) := \left\{ \rho : \pi_1(\mathbb{P}^1 \setminus \{t_1, \cdots, t_n\}, *) \to SL_2(\mathbb{C}) \;\middle|\; \begin{array}{l} \mathrm{Tr}(\rho(\gamma_i)) \\ = 2\cos 2\pi\lambda_i \end{array} \right\} \Big/ \sim$$

representation



$\rho_1 \sim \rho_2 \overset{\text{def}}{\Longleftrightarrow}$ the Jordan-Hölder filtrations of ρ_1 and ρ_2 have the same composition factors.

$R(t, \lambda)$ becomes an affine variety

§ Riemann-Hilbert morphism

$RH : M^{\alpha}(t, \lambda, L) \longrightarrow R(t, \lambda)$: holomorphism

$(E, \nabla_E, \{l_i\}, \psi) \longmapsto \underbrace{\ker(\nabla_E^{hol}|_{\mathbb{P}^1 \setminus \{t_1, \cdots, t_n\}})}_{\text{locally constant sheaf}} \leftrightarrow \pi_1(\mathbb{P}^1 - \{t_1, \cdots, t_n\}, *)$-rep

Th 2. Assume $n \geqq 2$ $\deg L$ is odd and $|\alpha_i| \ll 1$ $(1 \leqq i \leqq n)$

$\Rightarrow$ $RH : M^{\alpha}(t, \lambda, L) \longrightarrow R(t, \lambda)$

is a bimeromorphic surjective morphism

Sketch of proof

Th′ (Deligne, Man. ?)

$$\left\{ \begin{array}{l} (E, \nabla_E) : \text{rank2 connection} \quad \det E \overset{\text{horizontal}}{\cong} L \\ \text{the eigenvalues of } \mathrm{Res}_{t_i}(\nabla_E) \in \{ \tau \in \mathbb{C} \mid 0 \leqq \mathrm{Re}\,\tau < 1 \} \end{array} \right\}$$

$$\Big\updownarrow \text{bijective}$$

$$\{ \rho : \pi_1(\mathbb{P}^1 \setminus \{t_1 \cdots t_n\}, *) \longrightarrow SL_2(\mathbb{C}) \ \text{representation} \}$$

$M(t, \lambda, L) \supset M^{irr}(\lambda, L)$: irreducible connections

$Elm_i : M^{irr}(\lambda, L) \xrightarrow{\sim} M^{irr}(\lambda', \underset{\underset{L(-t_i)}{\shortparallel}}{L'})$

$(E, \nabla_E, \{l_j\}, \varphi) \longmapsto (E', \nabla_{E'}, \{l_j'\}, \varphi')$

$E' := \ker(E \longrightarrow E|_{t_i}/l_i)$

elementary transform

$\lambda_i' = 1 + \mathrm{Res}_{t_i}(\nabla_L) - \lambda_i$

$\otimes \mathcal{O}(t_i)$: $M^{irr}(\lambda, L) \xrightarrow{\sim} M^{irr}(\lambda', L(2t_i))$

$(E, \cdots) \longmapsto (E \otimes \mathcal{O}_{\mathbb{P}^1}(t_i), \cdots)$

$\lambda_i' = \lambda_i - 1$

$s_i : M^{irr}(\lambda, L) \xrightarrow{\sim} M^{irr}(\lambda', L)$

$(2\lambda_i \neq \mathrm{Res}_{t_i}(\nabla_L))$ change the eigenspace l_i of $\mathrm{Res}_{t_i}(\nabla_E)$

composition of Elm_i $\otimes \mathcal{O}(t_i)$ s_i $(1 \leqq i \leqq n)$

$$\begin{tikzcd} M^{irr}(\lambda, L) \arrow[r, "\exists \tau", "\sim"'] \arrow[dr, "\cong"] & M^{irr}(\lambda', L') \arrow[d, "RH"] \\ & R^{irr}(t, \lambda) \end{tikzcd}$$

$$\left(\begin{array}{l} 0 \leqq \mathrm{Re}(\lambda_i') < 1 \\ 0 \leqq \mathrm{Re}(\mathrm{Res}_{t_i}(\nabla_L) - \lambda_i') < 1 \end{array} \right)$$

$M_\lambda = \coprod_{t \in T_n} M(t, \lambda, L) \longrightarrow T_n$

relative moduli space.

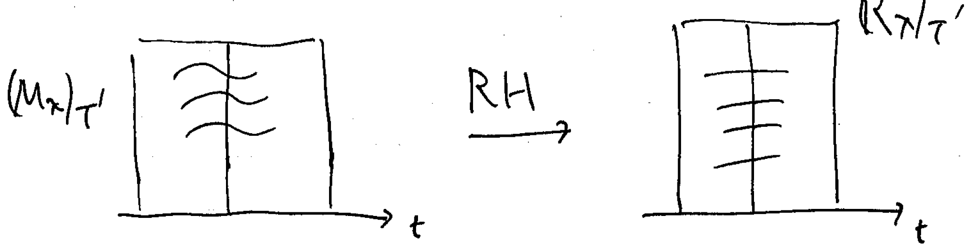
$T_n \supset T'$: open

$(R_\lambda)_{T'} = \coprod_{t \in T'} R(t, \lambda)$

$$\begin{tikzcd} (M_\lambda)_{T'} \arrow[r, "RH"] \arrow[dr, "\pi_1"'] & (R_\lambda)_{T'} \cong R(t_0, \lambda) \times T' \arrow[d, "\pi_2"'] \arrow[dl, phantom, "\cong"] \\ & T' \arrow[u, bend right, "\gamma"'] \end{tikzcd}$$

γ : section

sings を通らない…



$\{ RH^{-1}(\gamma) \}$ the solutions of Garnier equation (Painleve VI equation $n=4$)

$$\Theta_{R_\lambda \setminus \{sings\}} \longrightarrow \pi_2^* \Theta_{T_n} \longrightarrow 0 \quad \text{splitting}$$

$$\Theta_{M_\lambda} \longrightarrow \pi_1^* \Theta_{T_n} \longrightarrow 0$$

splitting ← Garnier equation (algebraic 相談中) (Painleve VI equation $n=4$

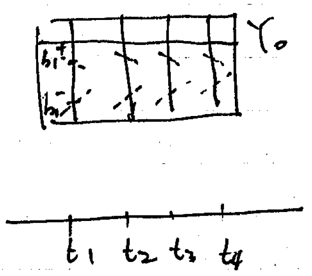
§ The case of $n=4$

From now on, we assume $n=4$

$L=\mathcal{O}(-t_4)$

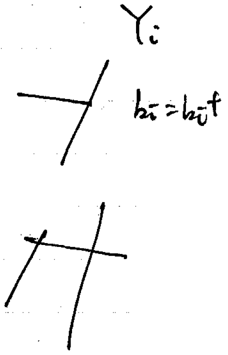
$\lambda_i^+=\lambda_i \qquad \lambda_i^- := \mathrm{Res}_{t_i}(\nabla_L)-\lambda_i$

$$\begin{tikzcd} Y_i \arrow[d] & \subset & \mathbb{P}(\Omega_{\mathbb{P}^1}(t_1+\cdots+t_n)\oplus\mathcal{O}_{\mathbb{P}^1}) \arrow[d, "\pi"] & \cong \mathcal{O}_{\mathbb{P}^1}(2) \\ t_i & \in & \mathbb{P}^1 & \end{tikzcd}$$



$$\begin{tikzcd} Y_i \arrow[r, "\sim"] \arrow[d, mapsto] & \mathbb{P}^1 \arrow[d, mapsto] \\ b_i^{\pm} \arrow[r, leftrightarrow] & {[\lambda_i^{\pm}:1]} \end{tikzcd} \qquad (1\leqq i\leqq 4)$$

Y_0 : section of π s.t. $Y_0^2=-2$



$\bar{S}(t.\lambda) \longrightarrow \mathbb{P}(\Omega_{\mathbb{P}^1}(t_1+\cdots+t_n)\oplus\mathcal{O}_{\mathbb{P}^1})$

blow-up at 8-points $\{b_i^{\pm}\}_{1\leqq i\leqq 4}$

$Y :=$ the proper transform of $2Y_0+\sum_{i=1}^{4} Y_i$

$\bar{S}(t.\lambda)\setminus Y$ is called the space of initial condition of Okamoto

compactification of $M^{\alpha}(t.\lambda.L)$

Def $(E_1.E_2.\phi.\nabla.\{l_i\}_{i=1}^{4}.\psi)$ is $(t.\lambda)$-parabolic ϕ-connection

$\overset{\text{def}}{\Leftrightarrow}$ (1) $E_1.E_2$, rk 2. v.b. on $\mathbb{P}^1$ $\quad \deg E_1=\deg E_2$

(2) $\phi : E_1\to E_2$ $\quad\mathcal{O}_{\mathbb{P}^1}$-hom.

---

(3) $\nabla : E_1 \to E_2\otimes\Omega_{\mathbb{P}^1}(t_1+\cdots+t_n)$

$\nabla(fs)=\phi(s)\otimes df+f\nabla(s)$

for $f\in\mathcal{O}_{\mathbb{P}^1}$ $\quad s\in E_1$

(4) $\psi :$ $\det E_2 \xrightarrow{\sim} L$ $\qquad \forall s_1.s_2\in E_1.$

$(\psi\otimes 1)(\nabla s_1\wedge\phi s_2+\phi s_1\wedge\nabla s_2)$

$=\nabla_L(\psi(\phi(s_1)\wedge\phi(s_2)))$

(5) $E_1|t_i\supset l_i$: line s.t.

$(\mathrm{Res}_{t_i}(\nabla)-\lambda_i\phi(t_i))|_{l_i}=0 \qquad 1\leqq i\leqq 4.$

We can define stability for ϕ-connection

Th 1′ $\exists\, \bar{M}_4^{\alpha}(t.\lambda.L)$: moduli space of stable parabolic ϕ-connections

$\bar{M}_4^{\alpha}(t.\lambda.L)$ is smooth and projective / $\mathbb{C}$

$\bar{M}_4(t.\lambda.L)\supset \mathcal{Y}$: closed subscheme defined by $\overset{2}{\wedge}\phi=0$

$M_4^{\alpha}(t.\lambda.L)=\bar{M}_4^{\alpha}(t.\lambda.L)\setminus\mathcal{Y}$

Th 3
$$\begin{tikzcd} \bar{M}_4(t.\lambda.\mathcal{O}_{\mathbb{P}^1}(-t_4)) \arrow[r, "\sim"] & \bar{S}(t.\lambda) \\ \mathcal{Y} \arrow[u, phantom, "\cup"] \arrow[r] & Y \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Outline of pf of Th 3

$\forall(E_1.E_2.\phi.\nabla.\{l_i\}.\psi)\in\bar{M}_4(t.\lambda)$

$E_1\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\cong E_2$

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$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^1} \arrow[r, hook] & E_1 \arrow[r, "\nabla"] \arrow[dr, "u"] & E_2 \oplus \Omega(t_1+\cdots+t_4) \arrow[d, "p_2\otimes 1"] \\
& & \mathcal{O}(-1)\otimes\Omega(t_1+\cdots+t_4) \\
& & \mathcal{O}(1) \arrow[u, phantom, "\cup S"]
\end{tikzcd}$$

$\mathcal{O}_{\mathbb{P}^1}$-hom

$\exists! z \in \mathbb{P}^1$ s.t. $u(z)=0$

$B : E_1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}\text{-hom}} \mathcal{O}_{\mathbb{P}^1}(-1)\otimes\Omega(t_1+\cdots+t_4)$

$B(A) := (p_2\otimes 1)\nabla(A) - \nabla_L(p_2\phi(A))$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}|_z \arrow[r] \arrow[dr, "u(z)" ', "0 \,\|"] & E_1|_z \arrow[r] \arrow[d, "B(v)"] & \mathcal{O}(-1)|_z \arrow[r] \arrow[dl, dashed, "\exists! h"] & 0 \\
& & \mathcal{O}(-1)\otimes\Omega(t_1+\cdots+t_4) & &
\end{tikzcd}$$

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d, "\phi" '] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}(-1) \arrow[r] \arrow[d, "\exists\phi_2"] & 0 \\
E_2 \arrow[r] & \mathcal{O}(-1) \arrow[r] & 0
\end{tikzcd}$$

$(h,\phi_2(z)) : \mathcal{O}(-1)|_z \hookrightarrow (\mathcal{O}(-1)\otimes\Omega(t_1+\cdots+t_4)\oplus\mathcal{O}(-1))|_z$

determines a point $p(E_1, E_2, \cdots)$ of $\mathbb{P}(\mathcal{O}(2)\oplus\mathcal{O})$

$p : \overline{M}_4(t,\lambda L) \longrightarrow \mathbb{P}(\Omega(t_1+\cdots+t_4)\oplus\mathcal{O})$

is the desired blow-up.

No.

16:00 - 17:30 Date . .

並河 良典 氏 Stratified Mukai flops and Derived categories

Introduction

Definition (1) X, Y birational equivalent smooth quasi-projective varieties

$$X \underset{\text{K-equivalent}}{\sim} Y \overset{\text{def}}{\iff} \exists Z \quad \begin{tikzcd} & Z \arrow[dl, "f" ', "\text{proper birational}"] \arrow[dr, "g", "\text{proper birational}" '] & \\ X & & Y \end{tikzcd}$$

$$f^*K_X = g^*K_Y$$

(2) X, Y smooth quasi-projective varieties

$$X \underset{\text{D-equivalent}}{\sim} Y \overset{\text{def}}{\iff} D(X) \underset{\exists\text{ equivalence}}{\xrightarrow{\sim}} D(Y)$$

$D(X)$: bounded derived categories of coherent sheaves on X

Problem 1 X, Y. birat. equ. quasi-proj. varieties

$$X \underset{K}{\sim} Y \Longrightarrow X \underset{D}{\sim} Y \ ?$$

Definition A flop is a ~~sequence~~ diagram

$$\begin{tikzcd}
\underset{\text{smooth}}{D} \subset X \arrow[rr, dashed] \arrow[dr, "s" ', "\text{proj birat}"] & & X^+ \supset \underset{\text{smooth}}{D^+} \text{ strict transform} \arrow[dl, "s^+", "\text{proj birat}" '] \\
& \underset{\text{normal}}{\overline{X}} &
\end{tikzcd}$$

$\rho(X/\overline{X}) = \rho(X^+/\overline{X}) = 1$

s, s^+ : $\left[\begin{array}{l}\text{crepant resol.}\\ \text{small.}\end{array}\right.$

D : s-negative

$\Rightarrow D^+$: s^+-ample

Problem 2 flop $\begin{tikzcd}[cramped] X \arrow[rd] & & X^+ \arrow[ld] \\ & \overline{X} & \end{tikzcd}$ に対して， $X \underset{D}{\sim} X^+$?

Examples

(1) (Bondal-Orlov) $\widetilde{X} = X \times_{\overline{X}} X^+$

$$\begin{tikzcd}
 & \widetilde{X} \arrow[ld, "\mu"'] \arrow[rd, "\mu^+"] & \\
\mathbb{P}^{h-1} \cong M^{h-1} \subset X^{2h-1} \arrow[rr, dashed, "flop"] \arrow[rd, "blowing up"'] & & X^+ \supset M^+ \cong \mathbb{P}^{h-1} \arrow[ld] \\
 & \overline{X} &
\end{tikzcd}$$

$N_{\mathbb{P}^{h-1}/X} = \mathcal{O}(-1)^{\oplus h}$ $\qquad$ $N_{\mathbb{P}^{h-1}/X^+} = \mathcal{O}(-1)^{\oplus h}$

$$\mu_* \circ \mu^{+*} : D(X^+) \xrightarrow{\sim} D(X)$$

(2) ~~多~~Mukai flop (川又, —)

$$\begin{tikzcd}
 & \widetilde{X} \neq X \times_{\overline{X}} X^+ \arrow[ld, "\nu"'] \arrow[rd, "\nu^+"] & \\
\mathbb{P}^{h-1} \cong M^{h-1} \subset X^{2h-2} \arrow[rd] & & X^+ \supset M^+ \cong \mathbb{P}^{h-1} \arrow[ld] \\
 & \overline{X} &
\end{tikzcd}$$

$N_{\mathbb{P}^{h-1}/X} = \Omega^1_{\mathbb{P}^{h-1}}$

$$\begin{tikzcd}
\nu_* \circ \nu^{+*} : D(X^+) \arrow[r] \arrow[d] & D(X) \arrow[d] \\
K(X^+) \arrow[r, "同型でない"'] & K(X)
\end{tikzcd}$$

not equivelence

$$\begin{tikzcd}
 & X \times_{\overline{X}} X^+ \arrow[ld, "\mu"'] \arrow[rd, "\mu^+"] & \\
X & & X^+
\end{tikzcd}$$

$$\mu_* \circ \mu^{+*} : D(X^+) \xrightarrow{\sim} D(X)$$

(3) 3-dim (Bridgeland, ⋯)

3-dim flop

$$\begin{tikzcd}
 & X \times_{\overline{X}} X^+ \arrow[ld, "\mu"'] \arrow[rd, "\mu^+"] & \\
X \arrow[rd] & & X^+ \arrow[ld] \\
 & \overline{X} &
\end{tikzcd}$$

$\mu_* \circ \mu^{+*} : D(X^+) \xrightarrow{\sim} D(X)$ equivalence

§1. Stratified Mukai flop. (Markman)

H : h-dim $\mathbb{C}$-vector space

$t \leqq \frac{h}{2}$

$G := G(t, H) = \{ \underset{\substack{\text{subspace} \\ \text{of dim } t}}{V} \subset H \}$

T^*G cotangent bundle of G.

$\overline{N}^t(H) = \{ A \in \mathrm{End}(H) \mid A^2 = 0,\ \mathrm{rank}\, A \leqq t \}$

$T^*G \xrightarrow{s} \overline{N}^t(H)$
birat morphism

<u>sの構成</u> $\quad 0 \to \underset{\text{rank } t}{\tau} \to H \underset{\mathbb{C}}{\otimes} \mathcal{O}_G \to \underset{\text{rank } h-t}{q} \to 0$

$$\begin{tikzcd}
T^*G \arrow[rr, "\sim"] \arrow[rd] & & \underline{Hom}(q, \tau) \supset \mathrm{Hom}(q(p), \tau(p)) \ni A \arrow[ld] \\
 & G \ni p &
\end{tikzcd}$$

$$H \twoheadrightarrow q(p) \xrightarrow{A} \tau(p) \hookrightarrow H$$

$$\begin{tikzcd}
T^*G \ni (p, A) \arrow[d, "s"'] \arrow[d, phantom, shift left=8ex] & \\
\overline{N}^t(H) \ni A &
\end{tikzcd}$$

$$A = \begin{array}{c|c|c} & \overbrace{\quad}^{t} & \overbrace{\quad}^{h-t} \\ \hline & 0 & * \\ & 0 & 0 \end{array} \Big\}t$$

$$s^{-1}(A) = \{ p \in G \mid \underbrace{\mathrm{Im}(A)}_{i\text{-dim}} \leq \underbrace{\tau(p)}_{t \text{ dim}} \leq \underbrace{\mathrm{Ker}(A)}_{h-i \text{ dim}} \}$$

$i = \mathrm{rank}\, A$

$\simeq G(t-i, h-2i)$

$\mathrm{rank}(A) = t \Rightarrow s^{-1}(A) = \text{unique}$ $(\tau(p) = \mathrm{Im}\, A)$

$s : T^*\mathbb{G} \longrightarrow \bar{N}^t(H)$ の 1-parameter deformation とする

$$0 \to \underline{\mathrm{Hom}}(q,\tau) \to \overset{\exists}{E}(H) \to \Theta_{\mathbb{G}} \to 0$$

$\mathbb{G} \ni p$

$$\mathrm{Hom}(q(p),\tau(p)) = \left\{ \left(\begin{array}{c|c} \overset{t}{\overbrace{0}} & \overset{n-t}{\overbrace{*}} \\ \hline 0 & 0 \end{array} \right) \right\}^{t} \Big\}$$

$$\downarrow$$

$$E(H)(p) = \left\{ \left(\begin{array}{c|c} \alpha I & * \\ \hline 0 & 0 \end{array} \right) \;\middle|\; \alpha \in \mathbb{C} \right\}$$

$$\tfrac{1}{t}\,\mathrm{trace} \downarrow \qquad \qquad \downarrow$$

$$\mathbb{C} = \Theta_{\mathbb{G}}(p) \ni \alpha$$

$$E(H) = \bigcup_{p \in \mathbb{G}} E(H)(p)$$

$$n^t(H) = \{ A \in \mathrm{End}(H) \mid A^2 = \alpha A,\ \underset{(\exists \alpha \in \mathbb{C})}{} \mathrm{rank}\, A = t \} \subset \mathrm{End}(H)$$

$\bar{n}^t(H)$ closure of $n^t(H)$ in $\mathrm{End}(H)$

$$A \sim \left(\begin{array}{c|c} \alpha I & * \\ \hline 0 & 0 \end{array} \right) \qquad \bar{n}^t(H) \xrightarrow{\frac{1}{t}\mathrm{trace}} \mathbb{C}^1$$

$$\begin{tikzcd}
E(H) \arrow[r, "\widehat{s}", "\text{resol.}"'] & \bar{n}^t(H) \arrow[r] & \mathbb{C}^1 \\
(p,A) \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & A \arrow[u, phantom, "\in" rotate=90] & \\
T^*\mathbb{G} \arrow[u, phantom, "\cup" rotate=180] \arrow[r, "s"] & \bar{N}^t(H) \arrow[u, phantom, "\cup" rotate=180] \arrow[r] & \{0\} \arrow[uu, phantom, "\bigcup" rotate=180]
\end{tikzcd}$$

<u>Dual objects</u> H^* H の dual space

$$\begin{tikzcd}
E(H^*) \arrow[d, "\tilde{s}^t"] & \supset T^*\mathbb{G}^t \arrow[d] \\
\bar{n}^t(H^*) \arrow[d] & \supset \bar{N}^t(H^*) \arrow[d] \\
\mathbb{C}^1 & \ni 0
\end{tikzcd}$$

$\mathbb{G}^t := \mathbb{G}(t, H^*)$

<u>$t=1$</u>

Bondal-Orlov $\leftrightarrow$

$$\begin{tikzcd}
E(H) \arrow[rr, dashed, "\text{birat map}"] \arrow[d, "\tilde{s}"] & & E(H^*) \arrow[d, "s"] \\
\bar{n}^t(H) \arrow[rr, "\sim"] & & \bar{n}^t(H^*) \\
\mathrm{End}(H) \arrow[u, hook] \arrow[rr, "\sim", "\text{can}"'] & & \mathrm{End}(H^*) \arrow[u, hook]
\end{tikzcd}$$

Mukai flop $\leftarrow$

$$\begin{tikzcd}
T^*\mathbb{G} \arrow[r, dashed] \arrow[d, "s"] & T^*\mathbb{G}^t \arrow[d, "s^t"] \\
\bar{N}^t(H) \arrow[r, "\sim"] & \bar{N}^t(H^*)
\end{tikzcd}$$

stratified Mukai flop

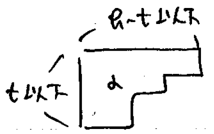
$$\begin{tikzcd}
& T^*\mathbb{G} \times_{\bar{N}^t(H)} T^*\mathbb{G} \arrow[dl, "\mu_0"'] \arrow[dr, "\mu_0^t"] & \\
T^*\mathbb{G} & & T^*\mathbb{G}^t
\end{tikzcd}
\qquad
\begin{tikzcd}
& E(H) \times_{\bar{n}^t(H)} E(H^*) \arrow[dl, "\mu"'] \arrow[dr, "\mu^t"] & \\
E(H) & & E(H^*)
\end{tikzcd}$$

$$\begin{tikzcd}
\psi_0 := \mu_{0*} \circ \mu_0^{t*} : D(T^*\mathbb{G}^t) \arrow[r] \arrow[d] & D(T^*\mathbb{G}) \arrow[d] \\
\overline{\psi_0} : K(T^*\mathbb{G}^t) \arrow[r] & K(T^*\mathbb{G})
\end{tikzcd}$$

$$\begin{tikzcd}
\psi := \mu_* \circ \mu^* : D(E(H^*)) \arrow[r] \arrow[d] & D(E(H)) \arrow[d] \\
\overline{\psi} : K(E(H^*)) \arrow[r] & K(E(H))
\end{tikzcd}$$

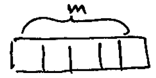
- $\overline{\psi_0}, \overline{\psi}$: isomorphisms
- $t=1$. ψ, ψ_0 isomorphisms
- $t=2, n=4$ $\psi, \underset{\substack{\uparrow \\ (-2)\text{ reflexion みたいなもの}}}{\psi_0}$: 両方とも equivalence ではない

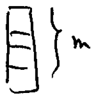
§ K - theory

$\Lambda = \{$ Young diagram  $\} \cup \{\phi\}$

τ : $\mathbb{G}$ 上の univ. bundle

$\Sigma^\alpha\tau$ vector bundle on $\mathbb{G}$

<u>例</u> $\alpha =$  $\Sigma^\alpha\tau = \mathrm{Sym}^m(\tau)$

$\alpha =$  $\Sigma^\alpha\tau = \Lambda^m(\tau)$

<u>Fact</u> (1) $K(\mathbb{G}) = \bigoplus_{\alpha\in\Lambda} \mathbb{Z}[\Sigma^\alpha\tau]$

(2) $K(\mathbb{G}) \xrightarrow[\pi^*]{\sim} K(T^*\mathbb{G})$

$$\begin{tikzcd}
T^*\mathbb{G} \arrow[r, hook] \arrow[dr, "\pi"'] & E(H) \arrow[d, "\tilde{\pi}"] \\
 & \mathbb{G}
\end{tikzcd}$$

$K(\mathbb{G}) \xrightarrow[\tilde{\pi}^*]{\sim} K(E(H))$

<u>Theorem</u>

$$\begin{tikzcd}
\bar{\Psi} : K(E(H^*)) \arrow[r, "\sim"] & K(E(H)) \\
{[\Sigma^\alpha\tau^+]} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & {[\Sigma^\alpha(\tau^*)]} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$\alpha \in \Lambda$

$\underset{\text{univ}}{\tau^+}$ on $\mathbb{G}^+$

<u>Theorem</u>

$\bar{\Psi}_0 : K(T^*\mathbb{G}^+) \xrightarrow{\sim} K(T^*\mathbb{G})$

$$\begin{tikzcd}
K(E(H^*)) \arrow[r] \arrow[d] & K(T^*\mathbb{G}^+) \arrow[d] \\
K(E(H) \times_{\bar{n}} E(H^*)) \arrow[r] \arrow[d] \arrow[ur, phantom, "\circlearrowright"] & K(T^*\mathbb{G}^+ \times_N T^*\mathbb{G}) \arrow[d] \\
K(E(H)) \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & K(T^*\mathbb{G})
\end{tikzcd}$$

∵)

$E(H) \times_{\bar{n}} E(H^*)$ integral scheme

作る

§ Derived Categories

$\mathbb{G} = \mathbb{G}(2,4)$ $t=2$, $\ell=4$

$$\begin{tikzcd}
\mathfrak{X}^9 \arrow[d, phantom, "\shortparallel"] & X^8 \arrow[d, phantom, "\shortparallel"] & Z^7 \arrow[d, phantom, "\shortparallel"] & M^4 \arrow[d, phantom, "\shortparallel"] \\
E(H) \arrow[r, phantom, "\supset"] \arrow[ddr, bend right] & T^*\mathbb{G} \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\shortparallel"] & \underline{Hom}^1(q,\tau) \arrow[r, phantom, "\supset"] \arrow[ddl] & \underline{Hom}^0(q,\tau) \arrow[ddll, bend left, "\sim"'] \\
 & \underline{Hom}(q,\tau) \arrow[d] & & \\
 & \mathbb{G}^4 & &
\end{tikzcd}$$

rank 1 以下

0-section

$$\begin{tikzcd}
\mathfrak{X}^+ \arrow[r, phantom, "\supset"] \arrow[dr] & X^+ \arrow[r, phantom, "\supset"] \arrow[d] & Z^+ \arrow[r, phantom, "\supset"] \arrow[dl] & M^+ \arrow[dll] \\
 & \mathbb{G}^+ & &
\end{tikzcd}$$

dual.

$Z \supset M$
family of ODP

$$\begin{tikzcd}
 & \tilde{\mathfrak{X}} \simeq \tilde{\mathfrak{X}}^+ \arrow[dl, "Z\text{ で blowup}"'] \arrow[dr, "Z^+"] \arrow[ddl, bend right, "\nu"] \arrow[ddr, bend left, "\nu^+"'] & \\
\mathfrak{X}_1 \arrow[d, "M\text{ で blow-up}"'] & & \mathfrak{X}_1^+ \arrow[d, "M^+"] \\
\mathfrak{X} \arrow[rr, dashed] \arrow[dr] & & \mathfrak{X}^+ \arrow[dl] \\
 & \bar{n} = \bar{n}^+ &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \tilde{\mathfrak{X}} \arrow[ddl, bend right, "\nu"'] \arrow[d, "\alpha"] \arrow[ddr, bend left, "\nu^+"] & \\
 & \mathfrak{X} \times_{\bar{n}} \mathfrak{X}^+ \arrow[dl, "\mu"] \arrow[dr, "\mu^+"'] & \\
\mathfrak{X} & & \mathfrak{X}^+
\end{tikzcd}$$

$\Phi := \nu_* \circ \nu^{+*} : D(\mathfrak{X}^+) \to D(\mathfrak{X})$

① Φ は fully faithful ではない

$$\begin{tikzcd}
\mathfrak{X}^+ = E(H^*) \arrow[d, "\tilde{\pi}^+"'] \\
\mathbb{G}^+
\end{tikzcd}$$

$\tilde{\pi}^{+*} =: \mathcal{O}^+(1)$

$\mathcal{O}_{\mathbb{G}^+}(1) = (\Lambda^2\tau^+)^\vee$

$$\begin{tikzcd}
\mathfrak{X} = E(H) \arrow[d, "\tilde{\pi}"'] \\
\mathbb{G}
\end{tikzcd}$$

$\mathcal{O}(1) = \tilde{\pi}^*\mathcal{O}_{\mathbb{G}}(1)$

$\mathcal{O}_{\mathbb{G}}(1)$

$\Phi(\mathcal{O}^+(1)) = \mathcal{O}(-1) \otimes I_{Z'}$

$\underset{\text{subscheme}}{Z' \subset \mathfrak{X}}$ $\mathrm{supp}\, Z' = Z$ $0 \to I_{Z'} \to I_Z \to \mathcal{O}_M \to 0$

$\mathrm{Ext}^i_{\mathfrak{X}^+}(\mathcal{O}^+(1), \mathcal{O}^+(1)) = 0 \qquad (i>0)$

$\mathrm{Ext}^5(\mathcal{O}(-1)\otimes I_{Z'}, \mathcal{O}(-1)\otimes I_{Z'}) = \mathrm{Ext}^5_{\mathcal{O}_{\mathfrak{X}}}(I_{Z'}, I_{Z'})$

$\cong \mathrm{Ext}^5_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_M, \mathcal{O}_M) \neq 0$

Koszul resol
を使って計算
+ $G = G(2,4)$

② $\psi = \Phi$ ∵) $\mathfrak{X} \underset{n}{\times} \mathfrak{X}^+$ rational sing $\Rightarrow d_* \cdot d^* = id.$

③ ψ は fully faithful でない $\Rightarrow$ ψ_0 : equivalence でない

∵) $\psi \xleftrightarrow[\text{adjoint}]{} \psi'$

$\psi_0 \qquad \psi_0'$

ψ_0 : equivalence $\Rightarrow$ $\psi_0 \cdot \psi_0' \simeq id$, $\psi_0' \cdot \psi_0 \simeq id$

$$\begin{tikzcd}
D(X^+) \arrow[r, "\psi_0' \cdot \psi_0"] \arrow[d, "i_*^*"'] & D(X^+) \arrow[d, "i_*"] \\
D(\mathfrak{X}^+) \arrow[r, "\psi' \circ \psi"] & D(\mathfrak{X}^+) \\
\mathcal{O}_p \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \psi' \circ \psi(\mathcal{O}_p) = \mathcal{O}_p
\end{tikzcd}$$

いつでも

$p \in \mathfrak{X}^+$
closed pt
Spanning class $\Rightarrow \psi$: fully faithful $\Rightarrow$ 矛盾

□

No.

Date 03 . 7 . 9　　9:45 - 11:45

岡本和夫氏 : パンルヴェ方程式のハミルトン構造

$$Dq = \frac{\partial H}{\partial p}, \quad Dp = -\frac{\partial H}{\partial q}, \quad H(t,q,p) : \text{ハミルトニアン}$$

(t,q,p) の多項式

$D = \frac{d}{dt}$　for $P_{I,II,IV}$

$D = t\frac{d}{dt}$　for $P_{III,V}$

$D = t(t-1)\frac{d}{dt}$　for P_{VI}.

- P_{VI} について考える.

$B = \mathbb{P}^1 \setminus \{0,1,\infty\}$

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt}$$
$$+ \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right\}.$$

q を掛けると

$$q\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 + \frac{(q-1)(q-t)}{t^2(t-1)^2}\beta + q(\ \cdots\ )$$

$t \to t_0 \neq 0,1,\infty$,　$q \to 0$

$$\frac{1}{2}\left(\frac{dq}{dt}\right)^2(t_0) + \frac{\beta}{(t_0-1)^2} = 0 \qquad \beta = -\frac{1}{2}\kappa_0^2.$$

$$q(t) = \pm \quad (t-t_0) + \cdots$$

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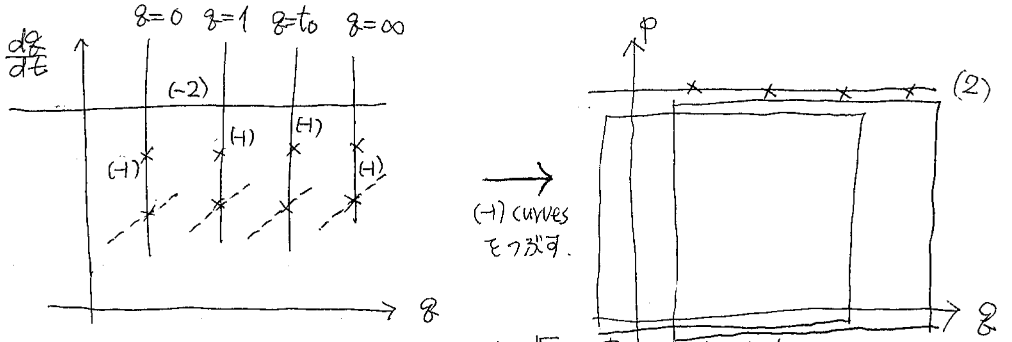


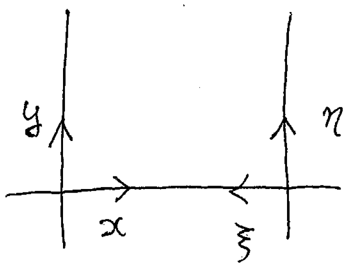
上は $\mathbb{F}_2$ をちょっとひねったものである.

$$\rho = \varepsilon_0\kappa_0 + \varepsilon_1\kappa_1 + \varepsilon_t\theta + \varepsilon_\infty\kappa_\infty - 1$$

↑ ひねり具合をあらわす量.

- $\rho = 0$ だと $\mathbb{F}_2$ になる.　0-section あり
  - ⇝ $\mathbb{P}^1$
  - ⇝ リッカチ解

はり合わせ.



$$\begin{cases} x\xi = 1 \\ \eta = \rho x - x^2 y \end{cases} \qquad (\mathbb{F}_2 \text{の変形})$$

上の $(x,y) \mapsto (\xi,\eta)$ は正準変換である.

i.e.　$dy \wedge dx = d\eta \wedge d\xi$

cf) Weierstrass の $\wp$-ftn. : $\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2 y - g_3$.

2階の方程式としては，$\frac{d^2y}{dx^2} = 6y^2 - \frac{g_2}{2}$ → 初期値の空間は有理楕円曲面.

$$\wp(x) = -\frac{d^2}{dx^2}\log\sigma(x)$$

$$\parallel$$

$$\frac{1}{(x-x_0)^2} + \cdots \qquad \sigma(x) = (x-x_0)(\cdots). \quad \text{entire fn.}$$

$$\frac{d^2}{dx^2}\log f = \left(\frac{f'}{f}\right)' = \frac{ff'' - (f')^2}{f^2}$$

□

(P_{I}) $\frac{d^2q}{dt^2} = 6q^2 + t$

$$q = \frac{1}{(t-t_0)^2} + \cdots$$

$$q = -\frac{d^2}{dt^2}\log\tau$$

（この関数 τ 自体は，Painlevé 自身も考察している.）

タウ関数の定義（データ関数の類似）

$$\boxed{D\log\tau = H}$$

← $D\tau = H\tau$ という線形方程式の解である．という見方もできる.

$$H = \frac{1}{2}p^2 - 2q^3 - tq$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

$$\frac{dH}{dt} = \underbrace{\frac{dq}{dt}\frac{\partial H}{\partial q} + \frac{dp}{dt}\frac{\partial H}{\partial p}}_{=0} + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

このように τ を定義すると，すべてのパンルヴェ方程式に対して，τ は正則関数になる.

タウ関数まで考えると，パンルヴェ方程式が Gauss の超幾何関数等の‘古典超越関数’の族を（解として）含んでいることが良く分かる.

タウ関数の満たす微分方程式

$H =$ q, p, t の多項式

$DH =$ 〃

$D^2H =$ 〃

q, p 消去して H の微分方程式を得る．（$H \mapsto h$ てきとうな変数変換の下で）

$$\frac{dh}{dt}\left(t(t-1)\frac{d^2h}{dt^2}\right)^2 + \left(\frac{dh}{dt}\left(2h - (2t-1)\frac{dh}{dt}\right) + b_1b_2b_3b_4\right)^2$$

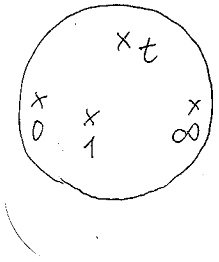
$$= \prod_{i=1}^{4}\left(\frac{dh}{dt} + b_i^2\right).$$

↖ D_4 型ワイル群の不変式を与える（有限の）

b_1, b_2, b_3, b_4 : 定数

$$\begin{tikzcd} \overset{\text{パンルベ方程式}}{E(v)} \arrow[rr, dashed, "\text{ベックルント変換}"] \arrow[dr] & & E(v^g) \arrow[dl] \\ & H(v) = H(v^g) & \end{tikzcd}$$

$\mathbb{P}^1$ 上 4点の入れかえ … $\mathfrak{S}_4$



Klein の4元群 … t をうごかさない

$$q :\Rightarrow \frac{t(q-1)}{q-t}$$

$$p :\Rightarrow \frac{1}{t(t-1)}\left[-(q-t)^2 + (b_1+b_3)(q-t)\right]$$

$H(t; q, p; b_1, b_2, b_3, b_4)$

$:\Rightarrow H(t; q, p; b_1, -b_2, b_3, -b_4 - 1)$

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ここから 平行移動 が得られる.

↓

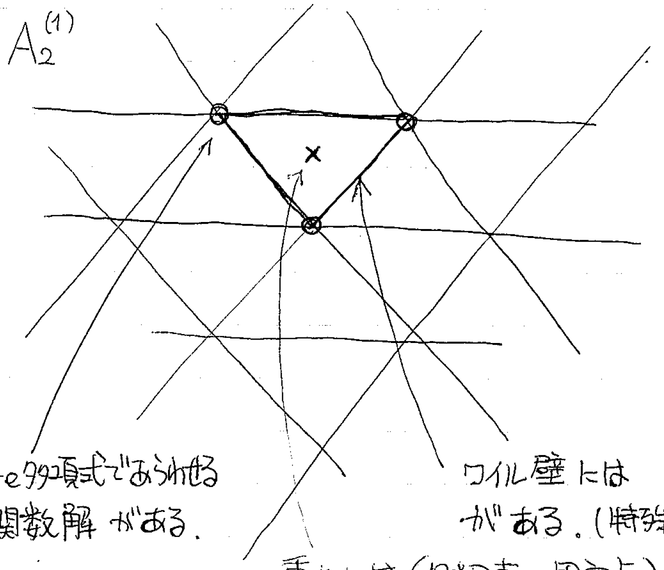
$D_4^{(1)}$ 型の アファイン・ワイル群 が P_{VI} に双有理変換群として作用する.（tを動かさない変換）

注 tを動かすことを許すと F_4 のワイル群も作用している.

P_{IV} を例に考える.（特殊解）

P_{IV} のパラメタ空間（定数）

$A_2^{(1)}$



Hermite多項式であらわせる
有理関数解がある.

ワイル壁には リッカチ解（P_{IV} の場合 Hermite方程式）
がある.（特殊解）

重心には（120°回転の固定点）
有理関数解がある.
↑これは 非古典 特殊多項式 とでも
いうべき 多項式の族を 用いて書かれる.

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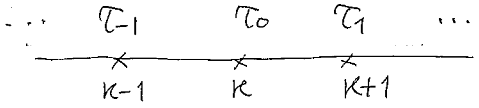
$$H(k-1) = H(k) + q(k), \qquad \frac{d}{dt}\log\tau(k) = H(k)$$

$$q(k) = \frac{d}{dt}\log\frac{\tau(k-1)}{\tau(k)}$$



タウ函数の列
$\{\tau_n\}_{n\in\mathbb{Z}}$
は 戸田方程式 を満たす:

$$\frac{d^2}{dt^2}\log\tau_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

P_{II} の場合

$$H = \frac{1}{2}p^2 - \left(q^2+\frac{t}{2}\right)p - kq$$

$$\frac{d^2}{dt^2}\log\tau = \frac{dH}{dt} = -\frac{1}{2}p$$

$$\{\tau_n\} \qquad \begin{array}{c}[\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5]\\ \rotatebox{90}{$\in$}\\ \mathbb{P}^5\end{array}$$ 6つとる.

τの列の満たす代数関係式 $\Phi(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = 0$ Φはtを含まない。
kは含んでいる.

$$(\Phi = 0) \subset \mathbb{P}^5$$

$\boxed{4次元}$ ← ・t がかくれている
・τの方程式は 3階.

$$f(k) - f(k-1) = t$$

広田の双線形形式（P_{II}の場合）　$g=\tau(k-1)$, $f=\tau(k)$

$$(*)\begin{cases}\mathcal{D}^2 g\cdot f + \frac{t}{2} g\cdot f = 0\\ \mathcal{D}^3 g\cdot f + \frac{t}{2}\mathcal{D} g\cdot f = \alpha g\cdot f\end{cases}$$

但し，広田微分　$(g,f)\mapsto \mathcal{D}^n g\cdot f$

$$\mathcal{D}\, g\cdot f = (Dg)f - g\, Df$$

$$\cdots$$

$$\mathcal{D}^n g\cdot f = (D^n g)f - n(D^{n-1}g)\,Df + \cdots$$

（つまり 積の微分のライプニッツ則 の符号を交代的に ± を入れたもの）

(*)において　$f=1$ とおくと，

$$\begin{cases}g'' + \frac{t}{2} g = 0\\ g''' + \frac{t}{2} g' = \alpha g\end{cases}$$

$\alpha = -\frac{1}{2}$ とすれば　g は

$$g'' + \frac{t}{2} g = 0$$

（Airy 方程式）。　リッカチ解に対応。

P_{I} 以外の全てのパンルヴェ方程式に対して，

$$\begin{cases}L(\mathcal{D})\, g\cdot f = 0\\ (L(\mathcal{D})\mathcal{D})\, g\cdot f = (\text{おつりの項})\end{cases}$$

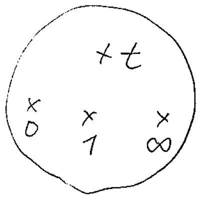
という広田の双線形形式がある．　$L(D)$：線形作用素。

線形方程式との関係.

$$\frac{d^2y}{dx^2} + P_1(x,t)\frac{dy}{dx} + P_2(x,t)y = 0$$

P_{VI} の場合　$x=0,1,\infty,t$ ：確定特異点.

$x=q$ ：見かけの特異点.



モノドロミー保存変形

$$\begin{cases}\frac{\partial^2 y}{\partial x^2} + P_1(x,t)\frac{\partial y}{\partial x} + P_2(x,t)y = 0,\\ \frac{\partial y}{\partial t} = A(x,t)\frac{\partial y}{\partial x} + B(x,t)y.\end{cases}$$

（P_1, P_2, A, B : xの有理関

$$\operatorname*{Res}_{x=q} P_2 = p$$

$$-\operatorname*{Res}_{x=t} P_2 = \frac{H}{t(t-1)}$$

上の system の両立条件から　P_{VI} の Hamilton系がえられる。

- パンルヴェ方程式 の 特殊解 と 線形方程式のモノドロミー の関係．（cf. 大山）
  - リッカチ解　⇢　モノドロミーが reducible
    （つまり，2階の線形 op が 1階 × 1階 となっている）
  - 有理解・代数解　⇢　いくつかの例に対して モノドロミーが計算できる。

No.

Date 03 · 7 · 9

今野宏 Variation of toric hyperkähler manifolds.

1. Introduction

Toric varieties in symplectic geom.

$$(\mathbb{C}^N, \omega) \curvearrowleft T^N$$

$$\begin{array}{ccc} \mathbb{C}^N \ni z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} & T^N \ni \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} & z\xi = \begin{pmatrix} z_1\xi_1 \\ \vdots \\ z_N\xi_N \end{pmatrix} \end{array}$$

$K \subset T^N$ subtorus. $T^n = T^N/K$

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathfrak{t}^N \to \mathfrak{t}^n \to 0$$

$\mathfrak{t}^N \ni X_1, \cdots, X_N$

$$0 \leftarrow \mathfrak{k}^* \xleftarrow{\iota^*} (\mathfrak{t}^N)^* \leftarrow (\mathfrak{t}^n)^* \leftarrow 0$$

$(\mathfrak{t}^N)^* \ni u_1, \cdots, u_N$

$$\nu : \mathbb{C}^N \to \mathfrak{k}^*$$

$$\mathbb{C}^N \ni z \mapsto \sum_i |z_i|^2 \iota^* u_i$$

moment map $f_n (\mathbb{C}^N, \omega) \curvearrowleft K$

K-equivalent

$\forall X \in \mathfrak{k}$ に対し

$$i(X^*)\omega = -d\langle \nu(\ ), X \rangle \qquad \nu(\ ) \in \mathfrak{k}^*,\ X \in \mathfrak{k}$$

Def $\nu^{-1}(\alpha) \curvearrowleft K$ free

$$\Rightarrow \left(M(\alpha) = \nu^{-1}(\alpha)/K, \ \dot{\omega}\right) \curvearrowleft T^n = T^N/K$$

$2n$-$\dim_{\mathbb{R}}$ toric mfd.

Thm. (Kirwan) $M(\alpha)$ smooth.

(1) $\nu^{-1}(\alpha) \subset (\mathbb{C}^N)^{\alpha\text{-st}} \curvearrowleft K_{\mathbb{C}}$.

(2) $\underset{\omega}{\nu^{-1}(\alpha)/K} \xrightarrow{\cong} \underset{I}{(\mathbb{C}^N)^{\alpha\text{-st}}/K_{\mathbb{C}}}$

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@ toric h.K. mfd

$$\mathbb{H}^N = T^*\mathbb{C}^N \simeq \mathbb{C}^N \times \mathbb{C}^N \curvearrowleft T^N \supset K.$$

$\mathbb{H}^N \curvearrowleft (g, I_1, I_2, I_3)$ $\quad \mathbb{C}^N \times \mathbb{C}^N \ni (z, w) \leftrightarrow z + wj$ $\quad T^N \ni \xi$

$\omega_1, \omega_2, \omega_3$

$\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$

$$(z, w)\xi = (z\xi, w\xi^{-1})$$

$$\mu(\mu_1, \mu_2, \mu_3) : \mathbb{H}^N \to \mathfrak{k}^* \otimes \mathbb{R}^3$$

$$\mu_j : \mathbb{H}^N \to \mathfrak{k}^* : \underline{(\mathbb{H}^N, \omega_j)} \curvearrowleft K$$

に対する moment map

$$\mu = (\mu_1, \mu_{\mathbb{C}}) : \mathbb{H}^N \to \mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}} \ni (\alpha, \beta)$$

$\mu_{\mathbb{C}} = \mu_2 + \sqrt{-1}\mu_3$.

$$\mu_1(z, w) = \pi \sum_{i=1}^N (|z_i|^2 - |w_i|^2)\iota^* u_i \in \mathfrak{k}^*$$

$$\mu_{\mathbb{C}}(z, w) = -2\pi\sqrt{-1} \sum_{i=1}^N z_i w_i \iota^* u_i \in \mathfrak{k}^*_{\mathbb{C}}.$$

Def. $\mu^{-1}(\alpha, \beta) \curvearrowleft K$, free

$$\Rightarrow \left(X(\alpha, \beta) = \mu^{-1}(\alpha, \beta)/K \quad g, I_1, I_2, I_3 \right) \curvearrowleft T^n = T^N/K$$

$4n$-$\dim_{\mathbb{R}}$.

$$\circ\ \mathbb{C}^N \hookrightarrow \mathbb{H}^N = \mathbb{C}^N \times \mathbb{C}^N$$

$$z \longmapsto (z, c)$$

$\varepsilon \in \mathbb{Z}^n$

$$\left.\begin{array}{l} \mu_1|_{\mathbb{C}^N} = \nu \\ \mu_{\mathbb{C}}|_{\mathbb{C}^N} \equiv 0 \end{array}\right\} \quad M_{\varepsilon}(\alpha) \subset X(\alpha, a)$$

cpx lag.

2. topology

Lemma. $\mu^{-1}(\alpha,\beta) \curvearrowleft K$ free

$\Longleftrightarrow$

(C1) Kに関する ある条件

(R). $(\alpha,\beta) \in \underline{(\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}})_{reg}}$: μ の regular value 全体の集合

$$= \mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}} \setminus \bigcup_{s=1}^{\ell} P_s \times P_{s\mathbb{C}}$$

$P_1, \cdots, P_\ell$: $\mathfrak{k}^*$ の wall 全体の集合

- codim 1 subsp.
- $\{\iota^* u_1, \cdots, \iota^* u_N\}$ の subset ではられる

Prop. $X(\alpha,\beta)$ の topology は $(\alpha,\beta) \in (\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}})_{reg}$ によらない

Remark. $M(\alpha)$. の topology は α による

$$\begin{tikzcd}
H^*_{T^N}(\mathbb{H}^N : \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\cong"] & \underline{H^*_K(\mathbb{H}^N, \mathbb{Z})} \arrow[r, "r"] & H_K^*(\mu^{-1}(\alpha,\beta) : \mathbb{Z}) \arrow[d, "\cong"] \\
\mathbb{Z}[u_1, \cdots, u_N] \arrow[rr, bend right, "\Phi"'] & & H^*(X(\alpha,\beta) : \mathbb{Z})
\end{tikzcd}$$

$\otimes\mathbb{R}$ すると $\cong S^*(\mathfrak{k}^*)$

$\cup$ $\mathfrak{k}^*$

thm. (1) Φ surjective

$H^*(X(\alpha,\beta) : \mathbb{Z}) \cong \mathbb{Z}[u_1, \cdots, u_N] / \sim$

$\sim$ は次で生成される

- $\mathrm{Ker}\{\iota^* : (\mathfrak{t}^N)^* \to \mathfrak{k}^*\} \cap (\mathfrak{t}^N)^*_{\mathbb{Z}}$
- $\prod_{i \in \theta_s} u_i \quad (s = 1, \cdots, \ell)$

$$\begin{cases} P_s = \{ v \in \mathfrak{k}^* \mid \langle {}^{\exists} Y_s \overset{\in \mathfrak{k}}{}, v \rangle = 0 \} \\ Y_s = \sum_i a_i^{(s)} X_i \qquad X_1, \cdots, X_N \in \mathfrak{t}^N \\ \theta_s = \{ i \mid a_i^{(s)} \neq 0 \}. \end{cases}$$

Prop. (1). (C1), (C2) の下で

$$K : \mathfrak{k}^* \cong H^2(X(\alpha,\beta) : \mathbb{R})$$

marking.

(2). $K^*(\alpha) = [\omega_1]. \in H^2(X(\alpha,\beta) : \mathbb{R})$

$K^*(\beta) = [\omega_{\mathbb{C}}] \in H^2(X(\alpha,\beta) : \mathbb{C})$

3. 例

$K = \{ \xi \in T^N \mid \xi_1 = \cdots = \xi_N \} \cong S^1$, $\mathfrak{k}^* \cong \mathbb{R}$.

- $\mathbb{C}^N \curvearrowleft K$

$\alpha > 0$ $\quad \nu^{-1}(\alpha) \subset (\mathbb{C}^N)^{\alpha\text{-}st}$.

$$\overset{\nu^{-1}(\alpha)}{\parallel} \qquad \overset{(\mathbb{C}^N)^{\alpha\text{-}st}}{\cong}$$

$$S^{2N-1}\left(\frac{\alpha}{\pi}\right) \Big/ S^1 \qquad \mathbb{C}^N \setminus \{0\} \Big/ \mathbb{C}^*$$

$$M(\alpha) \cong \mathbb{P}^{N-1}.$$

$\mathfrak{k} = \mathrm{span}\{\underbrace{X_1 + \cdots + X_N}_{= Y_1}\}$

$\mathfrak{k}^*$ の wall は $\mathrm{Ker}\, Y_1 \cong \{0\}$ のみ.

$(\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}})_{reg} = \mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}} \setminus \{(0,0)\}$

$\mathfrak{k}^* \cong \mathbb{R}$

$\ni v$

$\langle Y_1, v \rangle = 1$

$\iota^* : (t^N)^* \to \mathfrak{k}^*$

$\iota^* u_1 = \cdots = \iota^* u_N = v.$

$(\alpha,\beta) \neq (0,0)$

$$H^*(X(\alpha,\beta)) \cong \mathbb{Z}[u_1, \cdots, u_N] \Big/ \Big(\underset{u_i - u_j}{\ker \iota^*}, \underset{u_1 \cdots u_N}{\prod_{i \in \theta_1} u_i}\Big)$$

$$\cong \mathbb{Z}[v] \big/ (v^N)$$

$$\cong H^*(M(\alpha)) \quad \alpha > 0.$$

$X(\alpha, 0) \cong \underline{T^* \mathbb{P}^{N-1}}$

4. h.K. 構造の変化.

Recall $(\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}})_{reg} = \mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}} \setminus \bigcup_{s=1}^{l} P_s \times P_{s\mathbb{C}}$

$\beta \in \mathfrak{k}^*_{\mathbb{C}}$ fix

$\Lambda_\beta = \{ s \in \{1, \cdots, l\} \mid \beta \in P_{s\mathbb{C}} \}$

$(\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbb{C}})_{reg} \cap \mathfrak{k}^* \times \{\beta\} = \mathfrak{k}^* \setminus \bigcup_{s \in \Lambda_\beta} P_s$

この connected component を chamber とよぶ.

Thm. $\alpha, \alpha' \in \mathcal{C} \leftarrow$ 1つの chamber

$(X(\alpha,\beta), I_1) \cong (X(\alpha',\beta) : I_1)$

canonical biho l.

(∵)

$$X(\alpha,\beta) = (\mu_1^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(\beta)) \big/ K.$$

$= (\mu_{\mathbb{C}}^{-1}(\beta), \omega)$ の K に関する symp. quat

$= \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha\text{-st.}} \big/ K_{\mathbb{C}}.$

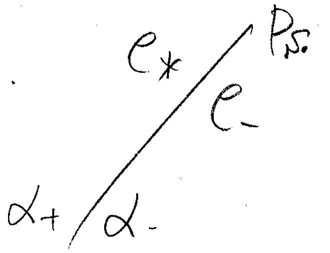
Lemma $(z, w) \in \mu_{\mathbb{C}}^{-1}(\beta).$

(z, w) が α-st. $\iff \alpha \in \sum_i \mathbb{R}_{>0} |z_i|^2 \iota^* u_i$

$+ \sum_i \mathbb{R}_{>0} |w_i|^2 (-\iota^* u_i)$

$\alpha, \alpha' \in \mathcal{C} \Rightarrow \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha\text{-st.}} = \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha'\text{-st.}}$ ┘

Thm. 2.



$\beta \in P_{s\mathbb{C}}$

$\exists f : (X(\alpha_+, \beta), I_1) \dashrightarrow (X(\alpha_-, \beta), I_1)$

Mukai flop.

すなわち.

$V = (\mu_{\mathbb{C}}^{-1}(\beta)^{\alpha_+\text{-st}} \cap \mu_{\mathbb{C}}^{-1}(\beta)^{\alpha_-\text{-st}}) \big/ K_{\mathbb{C}}$

$S_+ = X(\alpha_+, \beta) \setminus V$

$S_- = X(\alpha_-, \beta) \setminus V$ とする.

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\text{blow up}"'] \arrow[dr, "\text{blow up}"] & \\ \overset{\text{codim } r}{S_+} \subset X(\alpha_+,\beta) \arrow[dr, "\mathbb{P}^r\text{-bdle}"'] & & X(\alpha_-,\beta) \supset \overset{\text{codim } r}{S_-} \arrow[dl, "\mathbb{P}^r\text{-bdle}"] \\ & {}^{\exists}\overline{X}(\alpha,\beta) & \end{tikzcd}$$

別の toric h.K. mfd

Remark. $P_{s_0} = \{ \nu \in \mathfrak{k}^* \mid \langle {}^{\exists}Y_{s_0}, \nu \rangle = 0 \}$
$\uparrow$
$\mathfrak{k}$

さらに $Y_{s_0} \in \mathfrak{k}_{\mathbb{Z}}^*$ primitive, $\langle Y_{s_0}, \alpha_+ \rangle > 0$

とすると $Y_{s_0} \in \mathfrak{k}$ は一意的に決まる. $Y_{s_0} = \sum a_i^{(s_0)} X_i$

$\bar{K} := K / \{ \mathrm{Exp}\, tY_{s_0} \mid t \in \mathbb{R} \}$ $\quad O_{s_0} = \{ i \mid a_i^{(s_0)} \neq 0 \}$

このとき $\bar{\mathfrak{k}}^* \cong P_{s_0}$ と思える. $\beta \in P_{s_0\mathbb{C}} \cong \bar{\mathfrak{k}}_{\mathbb{C}}^*$

$\bar{X}(\alpha,\beta) = \mathbb{H}^{N'} /\!/\!/ \bar{K}$ $\quad N - \# O_{s_0} = r+1$

Cor. 1. $(X(\alpha_+,\beta), I_1) \supset S_+ \supset \mathbb{P}^r \supset C_+ (\cong \mathbb{P}^1)$

$K^{(*)} : \mathfrak{k}^* \cong H^2(X(\alpha,\beta) : \mathbb{R})$

$$\begin{tikzcd} K_* : H_2(X(\alpha,\beta) : \mathbb{R}) \arrow[r, "\sim"] & \mathfrak{k} \\ {[C_+]} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & Y_{s_0} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

$\therefore \mathrm{vol}(C_+)_{\mathrm{in}\, X(\alpha_+,\beta)} = \int_{C_+} \omega_1 = \langle [\omega_1], [C_+] \rangle$

$\|$

$\therefore \langle \alpha_+, Y_{s_0} \rangle$

$= \langle K^*(\alpha_+), [C_+] \rangle$

$= \langle \alpha_+, K_*([C_+]) \rangle$

Cor. 2. $\alpha \in \mathcal{C}$

このとき

$(X(\alpha,\beta), I_1)$ の Kähler cone $= K^*(\mathcal{C})$

$CH^2(X(\alpha,\beta))$

Cor 3. $(X(\alpha,\beta), I_1)$ affine

$\Longleftrightarrow \beta \in \mathfrak{k}_{\mathbb{C}}^* \setminus \bigcup_{s=1}^{\ell} P_{s\mathbb{C}}$

$\Longleftrightarrow \Delta_\beta = \phi$

Remark Thm. (Bielawski)

X^{4n} 完備 h.K. T^n-mfd
Euclidean volume growth をもつ

$\Longrightarrow X^{4n} \cong {}^{\exists}X(\alpha,\beta)$
h.K. T^n-mfd として isom

"Conj." T^n-mfd でなくても成り立つ ?
Euclidean は外せる。

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山田 紀美子氏

A sequence of morphisms connecting moduli of sheaves and the Donaldson polynomials under change of polarization

X : nonsingular proj. surface $/\mathbb{C}$

H : ample line bundle $/X$

$(c_1, c_2) \in \mathrm{Pic}(X) \times \mathbb{Z} \Rightarrow$

$M_H(c_1, c_2)$: coarse moduli of H-stable sheaves on X with Chern class $(2, c_1, c_2)$ (proj. $/\mathbb{C}$)

<u>Def</u> $\mathcal{E}$ is H-semistable $\Longleftrightarrow$

$$\mathcal{E} \twoheadrightarrow {}^\forall \mathcal{G} \quad 0 < \mathrm{rk}\,\mathcal{G} < \mathrm{rk}.\mathcal{E}$$

$$\frac{\chi(\mathcal{G}(nH))}{\mathrm{rk}\,\mathcal{G}} \geqslant \frac{\chi(\mathcal{E}(nH))}{\mathrm{rk}\,\mathcal{E}} \quad (n \gg 0)$$

"Donaldson polynomial defined using $M_H(0, c_2)$"

Suppose

- $\mathcal{U} \in \mathrm{Coh}(X_{M_H})$: universal sheaf of $M_H(0, c_2)$
- $\dim M_H(0, c_2) = 4c_2 - 3\chi(\mathcal{O}_X) \ni d$

<u>Def.</u> $\mu_H : NS(X) \longrightarrow NS(M_H)$

defined by $C \subset X$ nonsing. curve

$$\mu_H([C]) \in [\det R p_{2*}(\mathcal{U}|_C \otimes L_C)]$$

$(p_2 : C_{M_H(0, c_2)} \longrightarrow M_H,\ L_C \in \mathrm{Pic}^{g-1}(C))$

- extend it linearly
- $\gamma_H(c_2) : \mathrm{Sym}^d NS(X) \longrightarrow \mathbb{Z}$

$$(\alpha_1, \ldots, \alpha_d) \longmapsto \deg(\textstyle\prod_i \mu_H(\alpha_i))$$

✿ $\gamma_H(2)$ is related with the Donaldson polynomial of X

— · —— · ——

(Differential-Geometric) theory about Donaldson invariants deduces :

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$\boxed{\text{Fact}}$ Suppose X : 単連結, $p_g(X) > 0$

H_+, H_- : ample l. b.

$\Rightarrow$ Then $\gamma_{H_+}(c_2) = \gamma_{H_-}(c_2)$ if $c_2 \gg 0$ w.r.t. $H_\pm$. //

Observe $\boxed{\text{Fact}}$ from. alg-geom.

— · —— · —— · ——

【Some Concepts】 $(c_2 : \text{fix})$

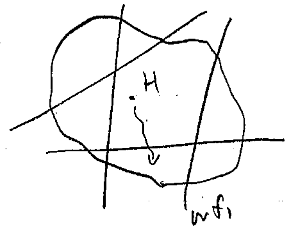
- Walls

$$\underset{\text{ample cone}}{\mathrm{Amp}(X)} \supset \bigcup_{f \in A(c_2)} W^f$$

$$A(c_2) = \left\{ f \in \mathrm{Num}(X) \,\middle|\, \begin{matrix} f \in 2\mathrm{Num}(X) \\ 0 < -f^2 \leq 4c_2 \end{matrix} \right\}$$

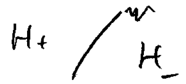
$f \in \mathrm{Num}(X) \Rightarrow$

$$W^f = \{ x \in \mathrm{Amp}(X) \mid x f = 0 \}$$



choose $H_\pm$ so that

$H_\pm$: contained in no wall, separated by only one wall of type c_2, say W



$M_- = M_{H_-}(0, c_2) \supset P_- = \{ [E] \mid E : H_+\text{-semistable} \text{ではない} \}$

$M_+ = M_{H_+}(0, c_2) \supset P_+ =$ 同じように

《with natural closed subsch. structure》

E : not H_+-semistable $\Rightarrow$

$$0 \longrightarrow L \otimes I_{Z_1} \longrightarrow E \longrightarrow L^{-1} \otimes I_{Z_2} \longrightarrow 0 \quad {}^\exists\text{ex.}$$

s.t. $([2L], l(Z_1), l(Z_2)) \in A^+(W)$

(HN filtration of E).

where

$$A^+(W) = \left\{ \begin{matrix} f_1 = (f, m, n) \\ \cap \\ \mathrm{Num}(X) \times \mathbb{Z}_{\geq 0}^2 \end{matrix} \,\middle|\, \begin{matrix} W = W^f,\ f \cdot H_+ > 0 \\ n + m - \frac{1}{4} f^2 = c_2 \end{matrix} \right\}$$

$f_1 \in A^+(W) \Rightarrow$

$$P_- \underset{\substack{\uparrow \\ \text{open closed}}}{\supset} P_-^{f_1} = \{ E \mid ([2L], l(Z_1), l(Z_2)) = f_1 \}$$

$$P_- = \coprod_{f_1 \in A^+(W)} P_-^{f_1}$$

Similarly $P_+ = \coprod_{f_1 \in A^+(W)} P_+^{f_1}$

$f_1 \in A^+(W) \Rightarrow \exists\, C(c_2, f_1) : \mathrm{Sym}^d NS(X) \to \mathbb{Z}$

s.t. $\gamma_+(c_2) - \gamma_-(c_2) = \underbrace{\sum_{f_1 \in A^+(W)} C(c_2, f_1)}$

$(P_\pm \neq \phi)$ contribution of $P_\pm^{f_1}$ to $\gamma_+(c_2) - \gamma_-(c_2)$

$\boxed{\text{Main}}$ X : 単連結, $p_g > 0$.

$S \subset \mathrm{Amp}(X)$: compact subset

$d_1(S), d_2(S)$: some constant depending on S.

Suppose the following about $f_1 \in A^+(W)$

(0) $H_\pm \in \mathbb{R}_{>0} S$

(1) $-f^2 > \frac{4}{3} c_2 + d_1(S)\sqrt{c_2} + d_2(S)$

(2) $T^{f_1} = \mathrm{Pic}^{f/2}(X) \times \mathrm{Hilb}^m(X) \times \mathrm{Hilb}^n(X)$

$(\mathrm{Pic}^{f/2} = \{L \in \mathrm{Pic}(X) \mid [2L] = f\})$

Then two functions

$$\begin{array}{ccc} T^{f_1} & \longrightarrow & \mathbb{Z} \\ \rotatebox{90}{$\in$} & & \\ (L, Z_1, Z_2) & \longmapsto & \dim \mathrm{Ext}^1(L \otimes I_{Z_1}, L^{-1} \otimes I_{Z_2}) \\ = & \longmapsto & \dim \mathrm{Ext}^1(L^{-1} \otimes I_{Z_2}, L \otimes I_{Z_1}) \end{array}$$

are locally constant $\Longrightarrow$ $C(c_2, f_1) = 0$.

⊗ meaning / weakness of (1), (2)

(1) : $\dim T^{f_1} = c_2 + \frac{1}{4} f^2$ ☹ (1) asserts that $\dim T^{f_1}$ is not very large.

(Recall) $f \in A^+(c_2) \Longleftrightarrow f \in 2\mathrm{Num}(X),\ 0 < -f^2 \leq 4c_2$

☺ (1) is reasonably weak.

(2) asserts $P_\pm^{f_1}$ is nonsingular $\Longleftarrow$ X : K3 ならOK.

(2) is strict.

To prove $\boxed{\text{Main}}$, we use

$$\begin{tikzcd}[column sep=small] & \bigcirc \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\ M_- & & \bigcirc & & M_+ \end{tikzcd}$$

connect by sequence of bir. morphism

【Another works】

- × Ellingsrud-Göttsche (95) : K3 のとき
- × Mochizuki (math. AG/0210211)
  observed this fact in all case
  (i.e. he does not require $P_\pm$ to be nonsingular)
  - × Mochizuki's proof is very different from ours

【Outline of the proof】

① Structure of $P_-^{f_1} \subset P_- \subset M_{H_-}(0, c_2)$ $f_1 = (f, m, n)$

$$\begin{array}{cccc} \tau : & P_-^{f_1} & \longrightarrow & T^{f_1} = \mathrm{Pic}^{f/2} \times \mathrm{Hilb}^m \times \mathrm{Hilb}^n \\ & [E] & \longmapsto & (L, Z_1, Z_2) \end{array}$$

$(0 \to L \otimes I_{Z_1} \to E \to L^{-1} \otimes I_{Z_2} \to 0)$

$\mathcal{J}_0 = p_{12}^*(\underset{\substack{\uparrow \\ \text{universal} \\ \text{l.b. of Pic}(X)}}{\mathcal{P}}) \otimes p_{13}^*(\underset{\substack{\uparrow \\ \text{univ.} \\ \text{l.b. of Hilb}^m}}{\mathcal{I}_{Z_1}}) \in \mathrm{Coh}(X_T)$

$\mathcal{G}_0 = p_{12}^*(\mathcal{P}^\vee) \otimes p_{14}^*(\mathcal{I}_{Z_2})$, univ. family of Hilb^n

$\mathcal{A}_- := \mathcal{E}xt^1_{\pi_T}(\mathcal{J}_0, \mathcal{G}_0(K_X))$

$\boxed{\text{Claim}}$

$$\begin{array}{ccc} P_-^{f_1} & \xrightarrow{\sim} & \mathbb{P}(\mathcal{A}_-) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [E] & \longmapsto & (\text{HN filt の ext. class}) \end{array}$$

Similarly $\mathcal{A}_+ = \mathcal{E}xt^1_{X_T/T}(\mathcal{G}_0, \mathcal{J}_0(K_X))$

$P_+^{f_1} = \mathbb{P}(\mathcal{A}_+)$

② $$\begin{array}{cccc} \phi_- : & \widehat{M}_- & \longrightarrow & M_{H_-}(0, c_2) \\ & E_- & \longmapsto & P_- \end{array}$$ blow-up along P_-

(Roughly) One constructs a map

$$\begin{array}{cccc} \Phi_+ : & \widehat{M}_- & \longrightarrow & M_{H_+}(0, c_2) \\ & E_- & \longrightarrow & P_+ \end{array}$$

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$\hat{v}$: pull back of univ. family of $M_{H_+}(0, c_2)$ to $X_{\hat{M}}$

$\downarrow$ elementary transform

w : new flat family

※ some structure of

$\phi_- \times_T \Phi_+ : E_- \longrightarrow P_- \times_T P_+$

$\hat{v}|_{X_{E_-}}$: flat family of not H_+-s. stable sheaves

$\Longrightarrow\ \exists\ 0 \to \mathcal{J} \to \boxed{\hat{v}|_{X_{E_-}} \to g \to 0}$

relative HN filtration

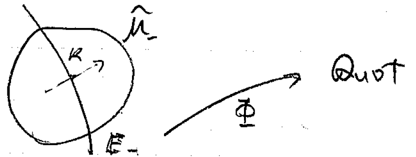
$\Phi : E_- \longrightarrow \mathrm{Quot}_{v_-/X_{M_T}/M_-}$

$$\begin{tikzcd} E \arrow[r] & \mathcal{J} \\ R \arrow[u, phantom, "\cup"] & t \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\begin{tikzcd} & P_- = \mathbb{P}(A_-) \supset P_- \times_T k(t) = \mathbb{P}(A_- \otimes k(T)) \\ & \phi_-(R) \in\ = \mathbb{P}(\mathrm{Ext}^1(J_0 \otimes k(t), g_0 \otimes k(t)(K_X))) \\ E \arrow[uur, "\phi_-"] \arrow[dr, "\overline{\Phi}_+"'] & \Phi_+(R) \in\ = \mathbb{P}(\mathrm{Ext}^1(g_0 \otimes k(T), J_0 \otimes k(t))^\vee) \\ & P_+ = \mathbb{P}(A_+) = \mathbb{P}(\mathrm{Ext}^1(J_0 \otimes k(T), g_0 \otimes k(t))^\vee) \end{tikzcd}$$

$\phi_-(R)$: extension class of HN filtration $\phi_-(R) \in P_-$

$\Phi_+(R)$: obstruction to extend $\Phi : E_- \longrightarrow \mathrm{Quot}$ to the normal direction to E in $\widehat{M}_-$



One can study $\mathrm{Im}(\phi_- \times_T \overline{\Phi}_+ : E \longrightarrow P_- \times_T P_+)$ using obstruction theory.

③ Calculation of $\gamma_+ - \gamma_-$ (Suppose ext^1 is locally constant)

$\Rightarrow A_\pm$: locally free $/T$

$$\begin{tikzcd} & \widetilde{M}_- \arrow[dl, "\phi_-"'] \arrow[dr, "\Phi_+"] & \\ M_- \arrow[dd] & E_- \arrow[u, phantom, "\cup"] \arrow[ddl] \arrow[ddr] & M_+ \arrow[dd] \\ & & \\ \mathbb{P}(A_+) & & \mathbb{P}(A_-) \end{tikzcd}$$

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$\alpha \in NS(X)$

$M_\pm : NS(X) \to NS(M_\pm)$

$\gamma_+(\alpha, \ldots, \alpha) - \gamma_-(\alpha, \ldots, \alpha) =$

$= [M_-(\alpha)]^d - [M_+(\alpha)]^d$

$= [\phi_-^* M_-(\alpha)]^d - [\Phi_+^* M_+(\alpha)]^d$

$\boxed{\text{Fact}}$ ∗ $\phi_-^* M_-(\alpha) - \Phi_+^* M_+(\alpha) = \lambda E_-$ $(\lambda \in \mathbb{Z})$ in $NS(\widetilde{M}_-)$

∗ $M_-(\alpha)|_{P_- = \mathbb{P}(A_-)} = \beta + \mathcal{O}_-(-N)$ in $NS(P_-)$

$(\beta \in \mathrm{Pic}(T),\ \mathcal{O}_-(1) = \mathcal{O}_{\mathbb{P}(A_-)}(1))$

$M_+(\alpha)|_{P_+ = \mathbb{P}(A_+)} = \beta + \mathcal{O}_+(N)$

$(\mathcal{O}_+(1) = \mathcal{O}_{\mathbb{P}(A_+)}(1))$. //

$\gamma_+(\alpha, \ldots, \alpha) - \gamma_-(\alpha, \ldots, \alpha)$

$$= \sum_{t=0}^{d-1} N_t^\alpha \sum_{s=0}^{d-1-t} [\beta^t \cdot \mathcal{O}_+(1)^s \cdot \mathcal{O}_-(-1)^{d-1-t-s}]_{\mathrm{Im}(\phi_- \times_T \Phi_+)}$$

$(N_t^\alpha \in \mathbb{Z})$ $\quad \mathrm{Im}(\phi_- \times_T \overline{\Phi}_+ : E_- \longrightarrow P_- \times_T P_+) \subset P_- \times_T P_+$

☺ related with intersection theory on $P_- \times_T P_+$

We want to relate $\gamma_+ - \gamma_-$ with the intersection theory on $D_\pm \subset \mathbb{P}(A_\pm) \times_T \mathbb{P}(A_\pm^\vee)$.

($A_- \otimes \mathcal{O}_{\mathbb{P}(A_-)} \twoheadrightarrow \mathcal{O}_-(1)$

$0 \longrightarrow \mathcal{O}_-(-1) \longrightarrow A_-^\vee \otimes \mathcal{O}_{\mathbb{P}(A_-)} \longrightarrow \mathrm{Cok} \longrightarrow 0$

$D_- = \mathbb{P}(\mathrm{Cok}) \subset \mathbb{P}(A_-^\vee \otimes \mathcal{O}_{\mathbb{P}(A_-)}) = \mathbb{P}(A_-^\vee) \otimes_T \mathbb{P}(A_-)$)

∘ Why on $D_\pm$?

On $D_- \subset \mathbb{P}(A_-) \times_T \mathbb{P}(A_-^\vee)$ $\qquad$ (☆)

$\Sigma_{s=0}^{e-t} [\beta^t \cdot \mathcal{O}_-^\vee(1)^s \cdot \mathcal{O}_-(-1)^{e-t-s}]_{D_-}$

$(e = \dim D_-,\ \mathcal{O}_-^\vee(1) = \mathcal{O}_{\mathbb{P}(A_-^\vee)}(1))$

(Here, $s_i(A_- \otimes \mathcal{O}_{\mathbb{P}(A_-)}) \in A^i(\mathbb{P}(A_-))$)

i-th Segre class. $\longleftarrow$ Chern 類から求まる.

A is vector bundle on T

$\Rightarrow s_i(A_- \otimes \mathcal{O}_{\mathbb{P}(A_-)}) \in \mathrm{Im}(A^i(T) \to A^i(\mathbb{P}(A_-)))$

$\beta \in \mathrm{Pic}(T)$

☺ (☆) = 0 if $\dim \mathbb{P}(A_-) > \dim T$ $\quad \mathrm{rk}\, A_- > 1$

ⓐ connect $\mathbb{P}(A_-) \times_T \mathbb{P}(A_+)$ with $\mathbb{P}(A_\pm) \times_T \mathbb{P}(A_\pm^\vee)$

$p_g > 0 \Rightarrow \exists \kappa \in H^0(K_X) \setminus (0)$

$$\begin{tikzcd}
\otimes\kappa : \mathcal{E}xt^1_{X_T/T}(J_0, g_0) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{E}xt^1_{X_T/T}(J_0, g_0(K_X)) \arrow[d, phantom, "\shortparallel"] \\
\mathcal{E}xt^1_{X_T/T}(g_0, J_0(K_X))^\vee \arrow[d, phantom, "\shortparallel"] & A_- \\
A_+^\vee &
\end{tikzcd}$$

ⓑ $$\begin{tikzcd}
\mathbb{P}(\otimes\kappa) : \mathbb{P}(A_-) \arrow[r] & \mathbb{P}(A_+^\vee) \\
\mathcal{O}_-(1) & \mathcal{O}_+^\vee(1) \arrow[l, squiggly]
\end{tikzcd}$$

ⓒ connect $\mathrm{Im}(\phi_- \times_T \Phi_+) \subset \mathbb{P}(A_-) \times_T \mathbb{P}(A_+)$
with $D_\pm$ $\subset \mathbb{P}(A_\pm) \times_T \mathbb{P}(A_\pm^\vee)$

Recall that
One can study $\mathrm{Im}(\phi_- \times_T \Phi_+ : E_- \longrightarrow P_- \times_T P_+)$
using obstruction theory, and also about

$$\begin{tikzcd}
\mathrm{Im}(E_- \arrow[r, "\phi_- \times_T \Phi_+"'] & P_- \times_T P_+ \arrow[r, dashed] \arrow[d, phantom, "\shortparallel"] & \mathbb{P}(A_+^\vee) \times_T \mathbb{P}(A_+)) \\
& \mathbb{P}(A_-) \times_T \mathbb{P}(A_+) &
\end{tikzcd}$$

実は $\subset D_\pm \subset P_\pm \times_T P_\pm^\vee$.

◎ 以後やりたいこと

① $\phi_- \times_T \Phi_+ : E_- \longrightarrow \mathbb{P}(A_-) \times_T \mathbb{P}(A_+)$
は $A_\pm$ が locally free でないと分かりにくい。
（ベクトル束の道具が使えない）
$p_g > 0$ のとき何が起こるか？

② $\dim M_\pm(0, c_2) \boxed{\geq} d = 4c_2 - 3\chi(\mathcal{O}_X)$
c_2 が小さければ $\boxed{>}$ かもしれない。

$\boxed{>}$ のときに $\gamma_H(c_2)$ を定義 ⇒ virtual fundamental class
$\in A^d(M_H(0, c_2))$

定義 — $M_\pm$ の global な obstruction theory がいる。

$M_\pm$ — VFC の理論がある。
$\hat{M}_-$ — 〃 〃 は未整備 ⟸ これも

$$\begin{tikzcd}
& \tilde{M}_- \arrow[dl] \arrow[dr, hook] & \\
M_- & & M_+
\end{tikzcd}$$

No.

Date 03・7・10

坂井 秀隆 I （有理曲面とパンルヴェ方程式の幾何）

I. 微分方程式 ⟶ 曲面

1. Painlevé 方程式の位置づけ

"可積分" ということ

- 古典力学 ⟵ 十分多くの保存量 ⟶ Abel 函数

  ⇓

- ソリトン理論など もう少し広い範囲で

Painlevé 性 （可積分系の ひとつの いい 方程式の性質）

動く特異点は極のみ

問題 与えられた非線形微分方程式 が Painlevé 性をみたすか 判定せよ

- Painlevé 性をみたさない ⟵ Painlevé の α – method

  比較的容易

- 逆 (Painlevé 性をみたすことの証明) は難しい

Painlevé 方程式 の Painlevé 性証明

1. Painlevé – 福原 ⟵ 個別の方程式に増大度評価
2. Malgrange
3. Miwa

（2., 3.） } ⟶ 線形方程式 の 変形理論 （cf 斉藤政彦の今回の講演）

線形方程式のモノドロミー保存変形からくる非線形微分方程式は P 性をもつ

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2.3 をみるために

問題 2 非線形の微分方程式があたとき、それが線形方程式の変形理論からくるものか判定し、もしそうなら具体的にそれを記述せよ

2. 岡本初期値空間

(P_I) : $y'' = 6y^2 + x$

$y'' = 6y^2 + \frac{1}{2}$ ⟵ 楕円関数

$y'' = 6y^2 + x^2$ ⟵ Painlevé 性をみたさない

$$\begin{cases} y' = 2z \\ z' = 3y^2 + \frac{x}{2} \end{cases}$$

初期値 $(x_0, y_0, z_0) \in \mathbb{C}^3$ で解析的な解が一意に存在、

y や z が ∞ のときも考えたい

P^2

$$\begin{cases} y' = 2z_1 \\ z_1' = 3y_1^2 + \frac{x}{2} \end{cases}$$

$$\begin{cases} y_2' = \frac{3}{z_2} - 2y_2^2 + \frac{x}{2}z_2 \\ z_2' = -2y_2z_2 \end{cases}$$

$$\begin{cases} y_3' = -3z_3^2 - \frac{x}{2}y_3^2 \\ z_3' = -3\cdot\frac{z_3^3}{y_3} - 2 - \frac{x}{2}y_3z_3 \end{cases}$$

$$\mathbb{P}^2 = U_1 \cup U_2 \cup U_3$$

$\{(x:y:z)\}$ $\{(\frac{y}{x}, \frac{z}{x})\}$ $\{(\frac{x}{z}, \frac{y}{z})\}$

$\{(z\ \ x)\}$

vertical leaf

$z_2 = 0$ $y_3 = 0$

$\Rightarrow x = 0$

$dy = \left(-3z^2 - \frac{t}{2}y^2\right)dt$

$ydz = \left(-3z^2 - 2y - \frac{t}{2}y^2 z\right)dt$

$y_3 = 0 \longrightarrow dt = 0$

t を局所座標にとれない ⟶ vertical leaf

$\left\{dy - \left(-3z^2 - \frac{t}{2}y^2\right)dt\right\} \wedge \left\{ydz - \left(-3z^2 - 2y - \frac{t}{2}y^2 z\right)dt\right\}$

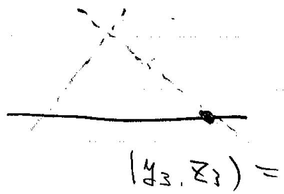
特異点

致達可能特異点 ←------ その初期値をもつ正則解が存在する.

accessible singularity (a.s)

$(P_3) \cdots (y_3, z_3) = (0.0)$ a.s ×

$y'' = 6y^2 + t^2$



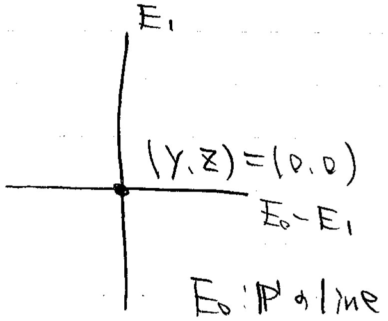
$(y, z) = \left(\frac{y_3}{z_3}, z_3\right)$

$(y, Z) = \left(y_3, \frac{z_3}{y_3}\right)$

と変数変換（解析学でいうブローアップ）

$$\begin{cases} y' = -3Z^2 - \frac{t}{2}y^2 \\ Z' = \frac{2}{y} \end{cases}$$

$$\begin{cases} Y^2 = -2\frac{Y}{z} \\ z' = -3\frac{z^2}{Y} + \underset{(2?)}{z} - \frac{t}{2}z^2 Y \end{cases}$$



E_0 : $\mathbb{P}$ の line

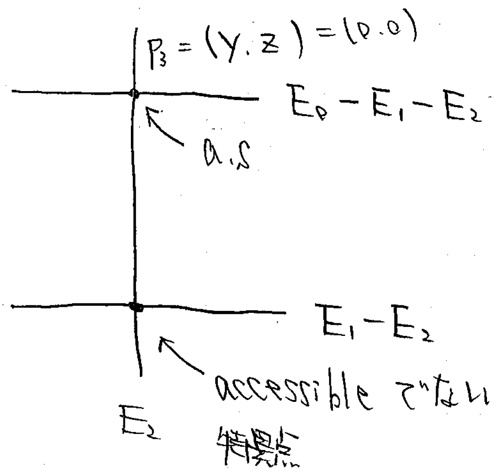
E_1 : 1回目の blow-up でできる例外曲線

$(Y, z) = \left(\frac{y_3}{z_3}z, z_3\right)$

$(y, Z) = \left(\frac{y_3}{z_3}, \frac{z_3^2}{y_3}\right)$

$$\begin{cases} y' = -\frac{2}{Z} \\ Z' = \frac{4}{y} - \left(3 + \frac{t}{2}y^2\right)Z^2 \end{cases}$$

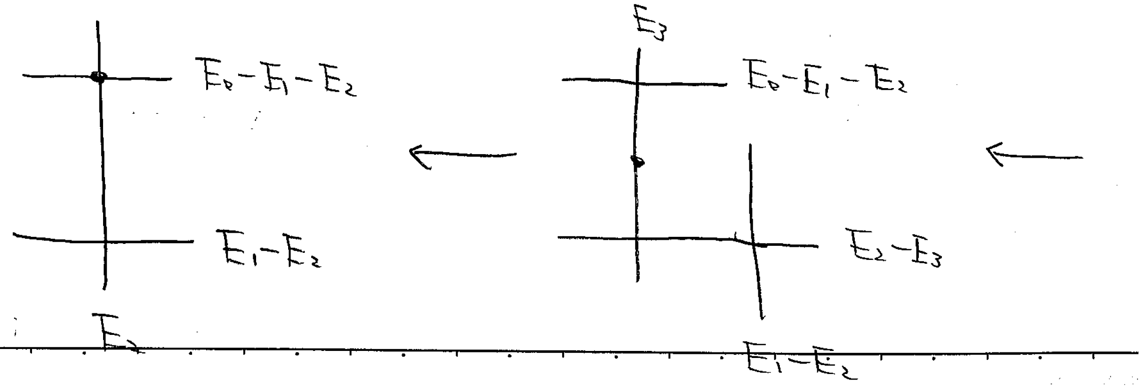
$$\begin{cases} Y' = -4\frac{Y}{z} + 3 + \frac{t}{2}z^2Y^2 \\ z' = -3\cdot\frac{z}{Y} + 2 - \frac{t}{2}z^3 Y \end{cases}$$

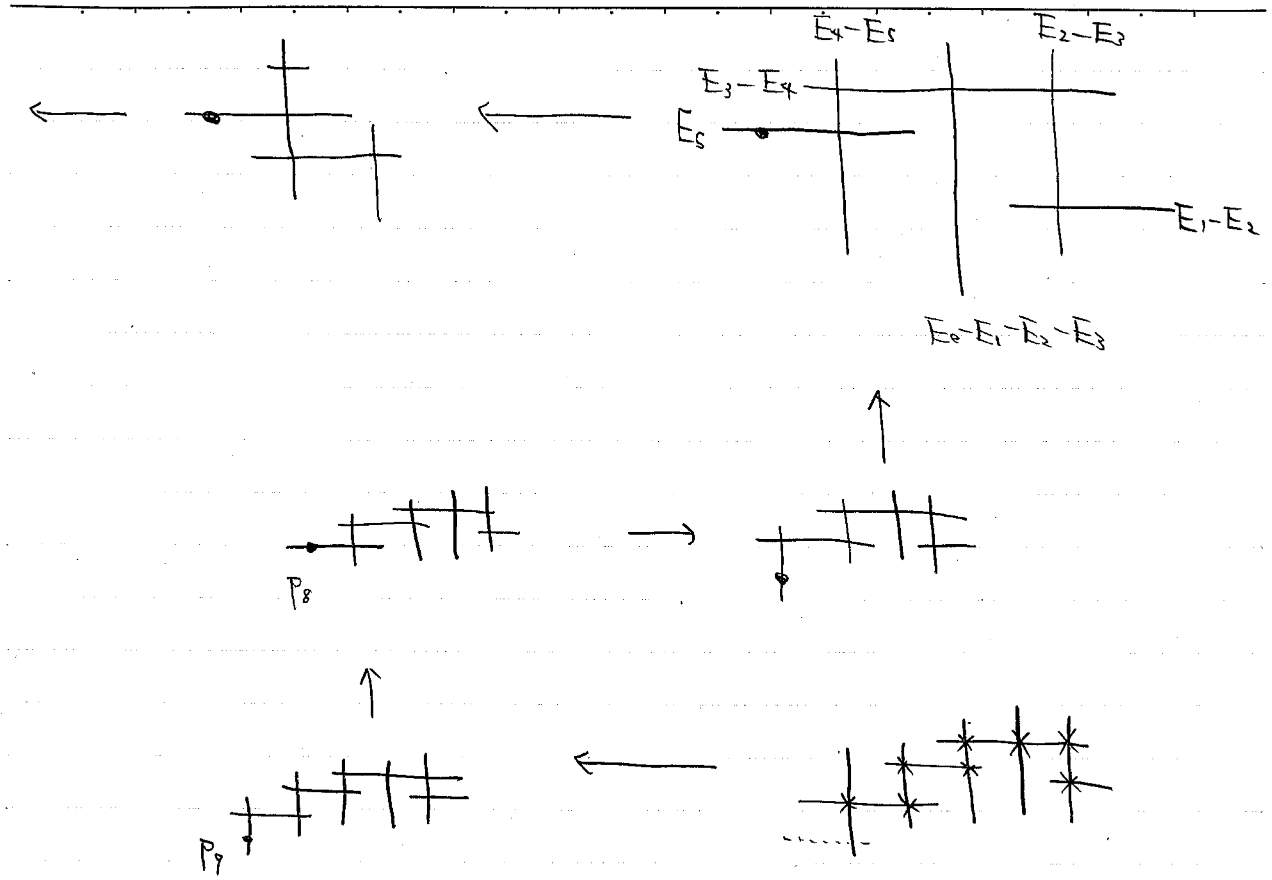


$$\left.\begin{matrix} y' = \frac{\lambda}{z} + \cdots \\ z' = \frac{\mu}{y} + \cdots \end{matrix}\right\}$$ a.s でない

$$\begin{cases} y' = \frac{\lambda y}{z} + \cdots \\ z' = \frac{\mu z}{y} + \cdots \end{cases}$$ ⟶ a.s かもしれない

9点の blow-up で a.s のない曲面がつくれる.





$$\begin{cases} y' = \dfrac{\lambda}{z} \\ z' = -\dfrac{\mu}{y} \end{cases}$$

これをとく

$$zy' + yz' = \lambda - \mu$$

$$(yz)' = \lambda - \mu$$

$$yz = (\lambda - \mu)t + C$$

$$\frac{zy' - yz'}{z^2} = \frac{\lambda + \mu}{z^2}$$

$$\left(\frac{y}{z}\right)' = \frac{\lambda + \mu}{z^2}$$

$$\uparrow$$

$$\frac{(\lambda-\mu)t + C}{z^2}$$

$$\frac{\lambda-\mu}{z^2} - 2\frac{(\lambda-\mu)t+C}{z^3} z' = \frac{\lambda+\mu}{z^2}$$

$$\frac{z'}{z} = \frac{-\mu}{(\lambda-\mu)t+C}$$

$$\log z = \frac{-\mu}{\lambda-\mu}\log\left(t + \frac{C}{\lambda-\mu}\right) + C_1$$

$$z = \tilde{C}\left(t + \frac{C}{\lambda-\mu}\right)^{-\frac{\mu}{\lambda-\mu}}$$

$\lambda-\mu = \pm 1$ でないと Painlevé 性をみたさない.

もし 微分方程式 が コンパクト空間上有理的に定義されていて何本かの vertical leaves と singularities 以外で正則かつ singularities のタイプが of 1st class のとき Painlevé 性をもつ⁇

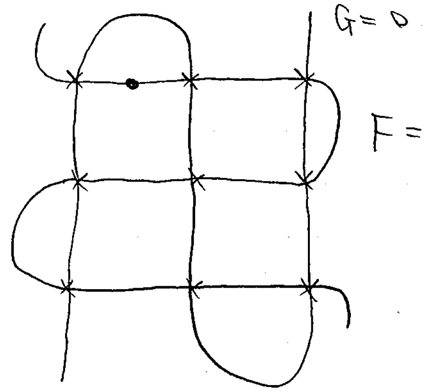
II. 曲面 → 微分方程式, 差分方程式

楕円関数の$\underbrace{\text{みたす}}_{\text{2階}}$微分方程式

→ 初期値空間 ≃ 楕円曲面

有理楕円曲面の構成

$\mathbb{P}^2$



F = 0 ‥ 3次曲線

$\lambda_0 F + \lambda_1 G = 0$

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^1 \ni [\lambda_0 : \lambda_1]$$

$\mathbb{P}^2$ をこの9点で blow-up する

$$X \to \mathbb{P}^2$$

$$X \longrightarrow \mathbb{P}^1 \ni [\lambda_0 : \lambda_1]$$

楕円ファイブレーションが入る

定義 X : 非特異射影的有理曲面

$\exists D \in |-K_X|$　Dは標準的 $\left(D = \sum m_i D_i, \ \forall D_i \cdot K_X = 0 \right)$

となるとき X を一般化 Halphen 曲面という

Remark この条件は accessible singularity が存在しないための十分条件になっている (斉藤)

· Okamoto-Painlevé pair (X, D)

$$\dim|-K_X| = \begin{cases} 1 & \cdots \text{Halphen 曲面} \longrightarrow \text{楕円函数} \\ 0 & \longrightarrow \text{Painlevé} \end{cases}$$

このとき

$\exists D \in |-K_X|$ の形で細かい分類をする

$\| $

$\sum m_i D_i$

命題 X : 有理曲面 $(K_X)^2 = 0$ のとき

1) $\operatorname{rank} \operatorname{Pic}(X) = 10$

2) $|-K_X| \neq \phi$

3) $\exists D \in |-K|$: 標準的 $\Rightarrow \exists \rho$: 双有理射 $X \to \mathbb{P}^2$

$\mathbb{P}^2$ の9点ブローアップ

$$\operatorname{Pic}(X) = \mathbb{Z}\varepsilon_0 + \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \cdots + \mathbb{Z}\varepsilon_9$$

↑ lineの引き戻し　↑ 例外曲線

$$K_X = -3\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_9$$

$D \in |-K_X|$

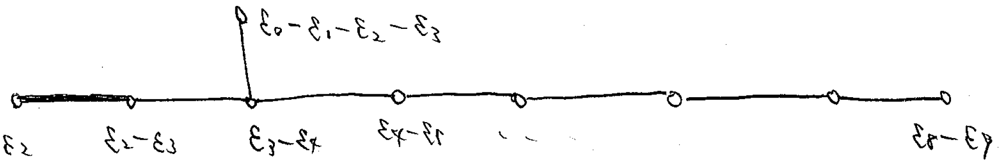
命題 Dは連結

$D = \sum m_i D_i$　　$D_i \cdot K_X = 0$

$$(K_X^{\perp}) := \{ F \in \operatorname{Pic}(X) \mid F \cdot K_X = 0 \}$$

$$= Q(E_8^{(1)})$$

↑ $E_8^{(1)}$ 型のルート格子



$Q(R) = \sum \mathbb{Z} D_i \subset \operatorname{Pic}(X)$ を考えると

これは $Q(E_8^{(1)})$ の部分格子になる

既約ルート (affine type)

$Q(E_8^{(1)})$ の部分格子の分類から

R : $A_0^{(1)}$, $A_1^{(1)}$, $A_2^{(1)}$, - - - , $A_8^{(1)}$

$D_4^{(1)}$, - - $D_8^{(1)}$

↑

$Q(A_0^{(1)}) = \mathbb{Z}\delta$ ← null root

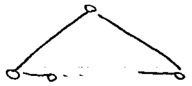
$E_6^{(1)}$, $E_7^{(1)}$ $E_8^{(1)}$

$= \mathbb{Z}(K_X)$

$A_8^{(1)} =$ $\varepsilon_0-\varepsilon_1-\varepsilon_7-\varepsilon_8$ $\varepsilon_1-\varepsilon_2$ $\varepsilon_2-\varepsilon_6$

$\varepsilon_0-\varepsilon_1-\varepsilon_2-\varepsilon_3$ $\varepsilon_3-\varepsilon_4$ $\varepsilon_4-\varepsilon_5$

$\varepsilon_0-\varepsilon_3-\varepsilon_4-\varepsilon_7$ $\varepsilon_7-\varepsilon_8$ $\varepsilon_8-\varepsilon_9$

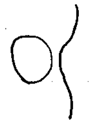


いくつかさらに細かい分類が必要で

① $A_7^{(1)}$ 型の格子は2通りの仕方で $Q(E_8^{(1)})$ にうめ込める.

$A_7^{(1)}$ $A_7^{(1)\prime}$

② $A_0^{(1)}$ 非特異3次曲線 二重特異点をもつ カスプをもつ

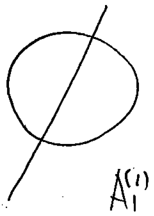


$A_0^{(1)}$

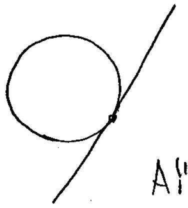
$A_0^{(1)*}$

$A_0^{(1)**}$

$A_1^{(1)}$

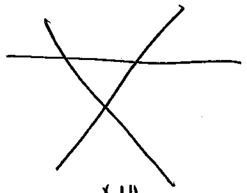


$A_1^{(1)}$

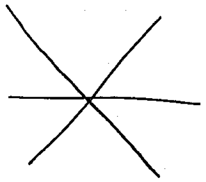


$A_1^{(1)*}$

$A_2^{(1)}$



$A_2^{(1)}$



$A_2^{(1)}$ *

22種類に分類

ちなみに $D_\ell^{(1)}$, $E_\ell^{(1)}$ 型に Painlevé 微分方程式がでてくる.

No. 1

Date H15 · 7 · 10

寺島ひとみ氏「Painlevé VI 方程式の Monodromy space と Cayley の3次曲面族」

のべたいこと

1. $\pi_1(\mathbb{P}^1 - \{0, 1, \infty, t\})$ の $SL_2(\mathbb{C})$ 表現の空間

$(SL_2(\mathbb{C})^3 /\!/ SL_2(\mathbb{C}))$

2. Naruki, Sekiguchi's modified Cayley-family との関係

§1. Monodromy 表現の空間

$SL_2(\mathbb{C})^3 / SL_2(\mathbb{C})$ = $\mathbb{P}^1$ 上 4点確定特異点をもつ 2階 Fuchs 型方程式の monodromy の空間

$\downarrow$

$SL_2(\mathbb{C})^3 /\!/ SL_2(\mathbb{C})$: $\mathbb{C}^4$ 上の3次曲面族 $\mathcal{F}$

$$\begin{tikzcd} \mathcal{F} \arrow[d] \\ A \end{tikzcd}$$

$$(x, a) \in \mathcal{F} := \{(x, a) \in \mathbb{C}^7 \mid f(x, \theta(a)) = 0\}$$

$x = (x_1, x_2, x_3)$　$a = (a_1, a_2, a_3, a_4)$

$$a \in A \simeq \mathbb{C}^4$$

ただし $f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4$

$$※ \begin{cases} \theta_i(a) = a_i a_4 + a_j a_k \quad (i = 1, 2, 3) \\ \theta_4(a) = a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 \end{cases}$$

(i, j, k) は $(1, 2, 3)$ の巡回置換

$\widetilde{W}(D_4)$　birational

$W(D_4)$

$(M_1, M_2, M_3) \in SL_2(\mathbb{C})^3$

No. 2

Date　・　・

$x_i = Tr(M_j M_k)$　$i = 1, 2, 3$

$a_i = Tr(M_i)$　$i = 1, 2, 3$

$a_4 = Tr(M_3 M_2 M_1)$

Thm (Iwasaki)

$\mathcal{F}_a$ が non sing. $\Longleftrightarrow D(a) := w(a)^2 \prod_{i=1}^{4} (a_i^2 - 4) \neq 0$

fiberが smooth

ただし $w(a) := \prod_{\substack{\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = 1 \\ \varepsilon_l \in \{\pm 1\}}} (\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3 + a_4) - \prod_{i=1}^{3} (a_i a_4 - a_j a_k)$　$(i, j, k) = (1, 2, 3)$ 巡回置換

問題 (Iwasaki)　$w(a)$ は何か？

discriminant の意味

⇝ P_{VI} の幾何を用いて考える.

◎ Facts

Thm (Inaba, Iwasaki, Saito)　$n = 4$ No of reg. sing

$t_0 \in T'$ fix

$$\begin{tikzcd} \mathcal{E} \arrow[r, "RH"] \arrow[d] & \mathcal{F} \arrow[d] \\ M \arrow[r, "\nu"] & A \end{tikzcd}$$

P_{VI} の Okamoto 初期値空間族

P_{VI} の parameter space

RH は $\mathcal{F} \to A$ の特異点同時解消を与える //

対称性と例外曲線 (Okamoto, Watanabe, Saito)

1. Painleve VI 方程式は Bäcklund 変換群として $\widetilde{W}(D_4)$ の対称性をもつ

2. $\kappa \in M$ が $\widetilde{W}(D_4)$ の鏡映面上にある

$\Longleftrightarrow$ $\mathcal{E}_\kappa$ (ファイバー) が (-2)-curve を含む

(**Rem** (-2) curves の配置と sing. のタイプが一致することは手計算でも示せる)

写像 ν : $M \xrightarrow{\Lambda_4} A$

$$\kappa = (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \longmapsto a = (a_1, a_2, a_3, a_4)$$
$$= (2\cos\pi\kappa_0,\ 2\cos\pi\kappa_1,\ 2\cos\pi\kappa_t,\ -2\cos\pi\kappa_\infty)$$

$$\begin{tikzcd} \mathfrak{h}_R \arrow[r, "\exp"] \arrow[rr, bend right, "\widetilde{\omega}_R"'] & H_R \arrow[r] & H_R/W(R) \end{tikzcd}$$

$H_R = \mathrm{Spec}\,\mathbb{C}[P_R]$

P_R: weight lattice

シュバレーの Th. $\mathbb{C}[P]^{W(R)} = \mathbb{C}[\chi_1, \cdots \chi_\ell]$ $\ell = \dim \mathfrak{h}_R$

χ_i: fundamental character

i.e. $H_R/W(R) = \mathrm{Spec}\,\mathbb{C}[\chi_1, \cdots, \chi_\ell]$

◎ Observation

1. $\nu^*(\{D(a)=0\}) = \bigcup \widetilde{W}(D_4)$ に関する鏡映面

2. $A \simeq H_{A_1^{\oplus 4}}/W(A_1^{\oplus 4}) \simeq \mathfrak{h}_{A_1^{\oplus 4}}/\widetilde{W}(A_1^{\oplus 4})$

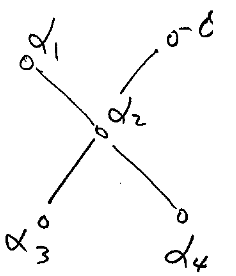
$$\mathfrak{h}_{A_1^{\oplus 4}} \cong \mathfrak{h}_{D_4} \cong M \curvearrowleft \widetilde{W}(D_4) > W(A_1^{\oplus 4})$$

出てくる sing. のタイプ

D_4, (A_1, A_1, A_1, A_1) A_3, A_2, A_1

(A_1, A_1, A_1) (A_1, A_1)

◎ $H_{D_4}/W(D_4)$ と $H_{A_1^{\oplus 4}}/W(A_1^{\oplus 4})$ の比較



$R(A_1^{\oplus 4}) \subset R(D_4)$

Simple root $\alpha_1, \theta, \alpha_3, \alpha_4$ $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ … simple root

fundamental weight

$\frac{\alpha_1}{2}, \frac{\theta}{2}, \frac{\alpha_3}{2}, \frac{\alpha_4}{2}$ ω_1 ω_2 ω_3 ω_4

fundamental character

φ_1 φ_2 φ_3 φ_4 $\chi_1, \chi_2, \chi_3, \chi_4$

$\varphi_1 = e^{\frac{\alpha_1}{2}} + e^{-\frac{\alpha_1}{2}}$

fundamental character の関係を見る（シュバレーのTh.）

Lem

$$(※※)\begin{cases} x_1 = \varphi_1\varphi_2 + \varphi_3\varphi_4 \\ x_2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \varphi_1\varphi_2\varphi_3\varphi_4 - 4 \\ x_3 = \varphi_2\varphi_3 + \varphi_1\varphi_4 \\ x_4 = \varphi_2\varphi_4 + \varphi_1\varphi_3 \end{cases}$$

$$\begin{tikzcd}
\mathfrak{h}_{A_1^{\oplus 4}} \arrow[r, phantom, "\simeq"] \arrow[d, "\widetilde{\omega}_{A_1^{\oplus 4}}"'] & \mathfrak{h}_{D_4} \arrow[r, phantom, "\simeq"] \arrow[d, "\widetilde{\omega}_{D_4}"] & M \\
H_{A_1^{\oplus 4}}/W(A_1^{\oplus 4}) \arrow[r] & H_{D_4}/W(D_4) &
\end{tikzcd}$$

ϕ

$$\begin{tikzcd}
M \arrow[r] \arrow[d, phantom, "\exists\ \wr|"] & A \arrow[d, phantom, "\exists\ \wr|"] & (※) \arrow[d, phantom, "\|"] \\
\mathfrak{h}_{A_1^{\oplus 4}} \arrow[r] & H_{A_1^{\oplus 4}}/W(A_1^{\oplus 4}) & (※※)
\end{tikzcd}$$

θ_2, θ_4 は入れかえ

Prop $\mathcal{F} = \phi^*\mathcal{S}$ $\quad \mathcal{S} := \{(x, X) \in \mathbb{C}^7 \mid f(x, X) = 0\}$

$$\begin{tikzcd}
\mathcal{F} \arrow[d] & \mathcal{S} \arrow[d] \\
H_{A_1^{\oplus 4}}/W(A_1^{\oplus 4}) \arrow[r, "\phi"] & H_{D_4}/W(D_4)
\end{tikzcd}$$

Rem $\quad U\widetilde{W}(D_4)$ の鏡映面 $= \{\widetilde{\omega}_{D_4}$ の critical points$\}$

$\{x = (x_1, \cdots, x_4) \in H_{D_4}/W(D_4) \mid J^2(x) = 0\}$

$= \{\widetilde{\omega}_{D_4}$ の critical values$\}$

J : Weyl 分母 $\qquad J^2$: キヤク

$\quad\ \cap \qquad\qquad\qquad\qquad \cap$

$\mathbb{C}[P_{D_4}] \qquad\qquad \mathbb{C}[P_{D_4}]^{W(D_4)}$

Thm $\quad D = \phi^* J^2$

Rem $\quad \{w(a) = 0\} = \nu(\{\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty = 1\})$

§2 Naruki · Sekiguchi's modified Cayley family との関係

$$\begin{tikzcd}
\mathcal{F} \arrow[d] \arrow[r, dashed, no head, "?"] & \widetilde{\mathcal{S}} \arrow[d] \\
A & T
\end{tikzcd}$$

N. S. family

$W(E_6) \curvearrowright \widetilde{\mathcal{S}} = \{([X:Y:Z:W], (\lambda, \mu, \nu, \rho)) \in \mathbb{P}^3 \times T$

$\cup$

$W(D_4) \qquad\qquad\quad \mid g(X, Y, Z, W; \lambda, \mu, \nu, \rho) = 0\}$

$W(D_4) \curvearrowright T = \{(\lambda, \mu, \nu, \rho) \mid \lambda\mu\nu\rho \neq 0\}$

$$g(X, Y, Z, W, \lambda, \mu, \nu, \rho)$$

$$= \rho W [\lambda X^2 + \mu Y^2 + \nu Z^2 + (\rho-1)^2(\lambda\mu\nu\rho-1)^2 W^2$$
$$+ (\mu\nu+1)YZ + (\lambda\nu+1)XZ + (\lambda\mu+1)XY$$
$$-(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X + (\mu+1)Y + (\nu+1)Z\}]$$
$$+ XYZ$$

$$\tilde{\mathcal{S}}^* := \tilde{\mathcal{S}} \times_{\operatorname{Spec}\mathbb{C}(Q)} \operatorname{Spec}\mathbb{C}[P]$$

$$\begin{tikzcd} W(E_6) \arrow[d, phantom, "\cup"] & & \\ W(D_4) \arrow[r, bend left, "\curvearrowright"] & \tilde{\mathcal{S}}^* \arrow[d] & \tilde{\mathcal{S}} \arrow[d] \\ W(D_4) \curvearrowright & \operatorname{Spec}\mathbb{C}[P] \arrow[r] & \operatorname{Spec}\mathbb{C}(Q) \end{tikzcd}$$

P : weight lattice
$\cup$
Q root lattice

Thm

$$\begin{tikzcd} \tilde{\mathcal{S}}^*/W(D_4) \arrow[r, phantom, "\simeq"] \arrow[d] & \bar{\mathcal{S}} \arrow[d] & \mathcal{S} \arrow[l, hook'] \arrow[dl] \\ H_{D_4}/W(D_4) \arrow[r, phantom, "\simeq"] & H_{D_4}/W(D_4) & \end{tikzcd}$$

$$\mathbb{C}(Q) \hookrightarrow \mathbb{C}(P)$$

$$\operatorname{Spec}\mathbb{C}(Q) \longleftarrow \operatorname{Spec}\mathbb{C}(P)$$

Date H15・7・10

吉岡 康太 氏 「安定層のモジュライ空間のコホモロジーへのリー環の作用について」

X : K3

$(H^*(X,\mathbb{Z}), \langle\ ,\ \rangle)$: Mukai lattice

$\langle x,y\rangle := -\int_X x^\vee \wedge y$, $x=(x_0,x_1,x_2)\in H^0\oplus H^2\oplus A^4$
$x^\vee=(x_0,-x_1,x_2)$

$= \int_X x_1\wedge y_1 - x_2\wedge y_0 - x_0\wedge y_2$

$\cong H^2(X,\mathbb{Z})\perp(H^0\oplus H^4)$

$\cong (-E_8)^{\oplus 2}\oplus \begin{pmatrix}0&1\\1&0\end{pmatrix}^{\oplus 3}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}$

E : coh. sheaf on X

$v(E) = ch(E)\sqrt{td_X}$: Mukai vector

$= (rk E, c_1(E), \chi(E) - rk E)$

Def H : X上の ample div

$M_H(v) := \{E \mid E : H\text{-stable}, v(E)=v\}$

(Mukai) $M_H(v)$: hol. symplectic form をもつ mfd
次元 $=\langle v,v\rangle+2$

Date ・ ・

Brill-Noether locus

E_0 : excep. v.b. $\Longleftrightarrow$ $Ext^1(E_0,E_0)=0$
H-stable

$\Longrightarrow \langle v(E_0)^2\rangle = -2$
つまり (-2)-vector.

◎ twisted degree : G : fix. v.b.

$deg_G(E) = deg(E\otimes G^\vee)$

$= (c_1(E\otimes G^\vee), H)$

$v\in H^*(X,\mathbb{Z})$ が minimal condition をみたす

$v(E)=v$ なる E に対し、

$deg_G(E) = \min\{deg_G E' > 0 \mid E' \in \overset{\text{Guthendick gp}}{\overset{\downarrow}{K(X)}}\}$

㊖ $G=\mathcal{O}_X$ $\mathrm{Pic}X=\mathbb{Z}\cdot H$
$\Longrightarrow v$: minimal
$\Longleftrightarrow v=(r,H,a)$

Brill-Noether locus

$M_H(v)_{E_0,n} := \{E\in M_H(v) \mid \dim Hom(E_0,E)=n\}$

BN locus を考えた理由：

(-2)-reflection (Mukai)

$$\begin{tikzcd} D(X) \arrow[r] & D(X) & & X\times X \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ E \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & R\pi_{2*}(\pi_1^*(E)\otimes\mathcal{E}) & X & & X \end{tikzcd}$$

$\mathcal{E} := \mathrm{Ker}(E_0^\vee \boxtimes E_0 \xrightarrow{ev} \mathcal{O}_\Delta)$

理想的には.

$$\begin{tikzcd} E \arrow[rrrr, bend right, mapsto] & 0 \arrow[r] & \mathrm{Hom}(E_0, E)\otimes E \arrow[r, "ev"] & E \arrow[r] & \mathrm{Coker}(ev) \arrow[r] & 0 \end{tikzcd}$$

① $M_H(v)_{E_0,n}$: Grassman-bdle $/ M_H(v-nv_0)_{E_0,0}$

② $N_{M_H(v)_{E_0,n}/M_H(v)} = \Omega_{rel}\,\mathrm{Grass}$

→ リー環の表現があるか？ (*)

類似の Grass. str. は Nakajima's quiver var. にも現われる.

⇓

homology gp H_* 上に Kac-Moody Lie alg の表現を作った。(キセク)

(*) 部分的に) Yes　・Nakajima　sl_2 の表現の構成

今日 これを一般化する.

$E_1, \cdots, E_n$: excep. に対し表現を作る.

$(\deg_G E_0 = 0.)$

Examples

X : K3, H : ample

G : semistable v.b, $\langle v(G)^2 \rangle = 0$

→ moduli = 2次元 (K3)

Assume $G = \bigoplus_{i=0}^{n} E_i^{\oplus a_i}$

where $\frac{\deg E_i}{\mathrm{rk}\, E_i} = \frac{\deg G}{\mathrm{rk}\, G} \Longleftrightarrow \deg_G(E_i) = 0.$

$$\frac{\chi_G(E_i)}{\mathrm{rk}\, E_i} = \frac{\chi_G(G)}{\mathrm{rk}\, G} = 0 \qquad (\chi_G(-) := \chi(-\otimes G^\vee))$$

$\langle \overset{v_0}{\overset{\shortparallel}{v(E_0)}}, \overset{v_1}{\overset{\shortparallel}{v(E_1)}}, \cdots, \overset{v_n}{\overset{\shortparallel}{v(E_n)}} \rangle$

: affine 型の Lie alg を生成

$\shortparallel$

$\mathfrak{g}$

Assume $a_0 = 1$

$F \subset E$

$$\frac{\chi_G(F(nH))}{\mathrm{rk}\, F} < (\leq) \frac{\chi_G(E(nH))}{\mathrm{rk}\, E}$$

$n \gg 0$

$l := \min\{\deg_G E > 0 \mid E \in K(X)\}$

$\bar{\mathfrak{g}}$: $v(E_1)$ – $v(E_n)$ で生成される Lie alg (finite)

$\bar{\mathfrak{h}}$: Cantan of $\bar{\mathfrak{g}}$

$\bar{\mathfrak{g}}_\alpha$: root sp of α

$\theta = \sum a_i v_i$: highest root of $\bar{\mathfrak{g}}$

$\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{g}} \oplus \mathbb{C} \cdot c \oplus \mathbb{C} \cdot d$

$\curvearrowright \langle v_i, \underline{v} \rangle id$

$e_{v_i} = 1 \otimes \bar{e}_{v_i}$, $f_{v_i} = 1 \otimes \bar{e}_{-v_i}$, $\boxed{h_{v_i}} = 1 \otimes \bar{h}_{v_i}$ $1 \le i \le n$

$e_{v_0} = t \otimes \bar{e}_{-\theta}$, $f_{v_0} = t^{-1} \otimes \bar{e}_{\theta}$, $h_{v_0} = -\sum a_i h_{v_i} + c$

$\bigoplus_{\deg_G(v) = l} H_*(M_H(v))$ に $\mathfrak{g}$ の作用が作れる。

ただし $\langle v, v(G) \rangle \neq 0$

$$c|_{H_*(M_H(v))} = \sum_i a_i \langle v_i, v \rangle id = \langle v(G), v \rangle id_{H_*(M_H(v))}$$

例 G : v.b. $\mathrm{rk}\, G = (H^2)$, $c_1(G) = H$

$v = (1 + (D, H), -D, a)$

$\deg_G(v) = (H.H)$
$= \min\{\deg_G(E') > 0 \mid E' \in K(X)\}$

$C_1, C_2, \dots, C_n$: irr. (-2) curve.

$v_i = ((C_i, H), -C_i, 0) \Rightarrow \deg_G v_i = 0$

Lem $^\exists$ stable v.b. E_i with $v(E_i) = v_i$

・ $H = \underset{\uparrow}{n} H' \Rightarrow E_i$: ~~μ~~-stable

n: 十分でかい

$\mathfrak{g}$: Lie alg gen by $C_1, \dots, C_n$

$$\mathfrak{g} \curvearrowright \bigoplus_{v = (1 + D, H), -D, a)} H_*(M_H(v))$$

特に $\pi : X \to \mathbb{P}^1$: elliptic K3 with section $\mathfrak{g}$'に適用可能

例 Purely 1-dim sheaf

$$v(E) := (c_1(E), \chi(E)) \in NS(X) \times \mathbb{Z}$$

$$\langle v(E)^2 \rangle := \langle c_1(E)^2 \rangle$$

E_i : Purely 1-dim sheaf

$$E_i \otimes K_X \cong E_i, \quad \langle v(E_i)^2 \rangle = -2$$

同様に

$$\mathfrak{g} \curvearrowright \bigoplus_{x_i \in \mathbb{Z}} H_*(M_H^G(v + \sum x_i v_i))$$

ここで v は
- $\chi_G(v) = 1$
- $M_H(v) =$ smooth.

・**特に** $X \to \mathbb{P}^2$ 9 pt blow-up

$$|-K_X| \ni Y = \sum a_i C_i \quad C_i : (-2)$$

$$(c_1(E), K_X) < 0 \Rightarrow \text{smooth}$$

$$\mathfrak{g} \curvearrowright \bigoplus H_*(M_H(v))$$

・$\pi : X \to C$: elliptic surface. $^\exists$section σ

$$\mathfrak{g} \curvearrowright \bigoplus_D H_*(M_H(\sigma + D, 1)) \quad (D.f) = 0$$

Lem $G \in K(X)$, st. $G > 0$, E_1 : μ-stable v.b. $\deg_G E_1 = 0$

E : μ-stable sheaf

$$\deg_G E = \min\{\deg_G E' > 0 \mid E' \in K(X)\}$$

$\Rightarrow$

(1) $0 \to E_1 \to \hat{E} \to E \to 0$: non-trivial

$\Rightarrow \hat{E}$: μ-stable

(2) $V \subset \mathrm{Hom}(E_1, E)$ について ① or ② が成立する.

$$\phi : V \otimes E_1 \longrightarrow E$$

① ϕ : surj. in codim 1, $\ker\phi$: μ-stable

② ϕ : inj, coker ϕ : μ-stable. $\Rightarrow$ $\mathcal{E}xt^1(V \otimes E_1 \to E$,

$\parallel$

μ-stable

$D(X)$: bdd. derived cat. of $\mathrm{ch}(X)$

Def $\mathbb{E} \in D(X)$, $\deg_G \mathbb{E}$: minimal

$\mathbb{E}$: stable $\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $H^i(\mathbb{E} \overset{L}{\otimes} \mathbb{C}_p) = 0$, $i \neq -1, 0$ $^\forall p \in X$

① or ② 成立

① $H^i(\mathbb{E}) = 0$ $i \neq 0$, $H^0(\mathbb{E})$: stable

② $H^i(\mathbb{E}) = 0$ $i \neq -1, 0$. $H^{-1}(\mathbb{E})^\vee$: stab

$H^0(\mathbb{E})$: 0-di

Coherent system

$S = \{E_1, E_2, \cdots, E_n\}$: finite set of μ-stable v.b

$\deg_G E_i = 0$

- $E_i \otimes k_x \overset{\wedge}{=} E_i$,, $E_i \in S$, E_i : rigid

$$\mathcal{R}^{(n)}_{E_i}(v) := \{ (\mathbb{E}, U) \mid \mathbb{E} \in M_H(v) , U \subset \mathrm{Hom}(E_i, \mathbb{E}), \underset{\dim U = n}{} \}$$

Zariski tangent sp :

$$\mathrm{Ext}^1(U \otimes E_i \to \mathbb{E}, \mathbb{E}) / \mathrm{End}(U)$$

obst. $\ker(\mathrm{Ext}(U \otimes E_i \to \mathbb{E}, \mathbb{E}) \to \mathrm{Ext}^2(\mathbb{E}, \mathbb{E}) \xrightarrow{tr} H^2(\mathcal{O}_X))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\|}{\mathbb{C}}$

$= 0$

$\Longleftrightarrow \mathrm{Ext}^2(\mathbb{E}', \mathbb{E}) = \mathbb{C}$

$\mathbb{E}' = [U \otimes E_i \to \mathbb{E}]$

$\Rightarrow \mathcal{R}^{(n)}_{E_i}(v)$: smooth

$$\dim \mathcal{R}^{(n)}_{E_i}(v) = \frac{1}{2} (\dim M_H(v) + \dim M_H(v - nv_i))$$

$$\begin{tikzcd}
 & & \mathbb{E}' \\
(\mathbb{E}, U) \arrow[rru, bend left] \arrow[d] & \mathcal{R}^{(n)}_{E_i}(v) \arrow[r, "\tilde{w}"] \arrow[d] \arrow[l, phantom, ""] & M_H(v - nv_i) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \\
\mathbb{E} \arrow[r, phantom, "\in"] \arrow[ru, phantom, "\pi"] & M_H(v) &
\end{tikzcd}$$

$$\pi \times \tilde{w} : \mathcal{R}^{(n)}_{E_i}(v) \longrightarrow M_H(v) \times M_H(v - nv_i)$$

closed immersion

Contraction of operators

$$\begin{tikzcd}
f^{(n)}_{v_i} : H_*(M_H(v - nv_i)) \arrow[r] & H_*(M_H(v)) \\
x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & p_{2*}(p_1^*(x) \cap P^{(n)}_{E_i}(v)) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \\
(-1)^{nr(v)} p_{1*}(p_2^*(y) \cap P^{(n)}_{E_i}(v)) & y \arrow[l, mapsto]
\end{tikzcd}$$

:

$$r(v) = \frac{1}{2}(\dim M_H(v - nv_i) - \dim M_H(v)) = -\langle v_i, v \rangle - 1$$

$$hw|_{H_*(M_H(v))} = \langle v_i, v \rangle \, id_{H_*(M_H(v))}$$

Thm S が (i) $\langle v_i, v_j\rangle \leq 1$ ${}^{\forall} i, j$

or

(ii) S : affine , $\langle \delta, v\rangle \neq 0$.

$$\Rightarrow \quad [h_{v_i}, e_{v_j}] = -\langle v_i, v_j\rangle e_{v_j}$$

$$[h_{v_i}, f_{v_j}] = \langle v_i, v_j\rangle e_{v_j}$$

$$[e_{v_i}, f_{v_j}] = \delta_{ij} h_{v_i}$$

$\Rightarrow$ integrable rep of $\mathfrak{g}$

Proof $i=j$: Nakajima

$i \neq j$

$e_{v_i} f_{v_j}$:

$$\begin{tikzcd}
M_H(v-n_iv_i) \times M_H(v-n_iv_i-n_jv_j) \times M_H(v-n_jv_j) \arrow[d, "\mathbb{P}^{n_j}_{E_j} \times M(v-n_jv_j) \cap M(v-n_iv_i) \times \mathbb{P}^{(n_i)}_{E_i}"] \\
M_H(v-n_iv_i) \times M_H(v-n_jv_j)
\end{tikzcd}$$

$$\begin{tikzcd}
M(v-n_iv_i-n_jv_j) \arrow[d, phantom] & M(v-n_iv_i) \arrow[l, "e_{v_j}"'] \arrow[d, "f_{v_i}"] & E_1 \\
M(v-n_jv_j) \arrow[from=u, "f_{v_i}"'] & M(v) \arrow[l, "e_{v_j}"] & E \\
E_2 & &
\end{tikzcd}$$

W : 成分のvのくり返し

$$M(v-n_iv_i) \times M(v) \times M(v-n_jv_j)$$

$$W(\mathbb{P}^{n_i}_{E_i}) \times M(v-n_jv_j) \cap M(v-n_iv_i) \times W(\mathbb{P}^{n_j}_{E_j})$$

$$\begin{tikzcd}
 & v_i\text{の}E_j \arrow[r, phantom, "="] \arrow[d] & v_j\text{の}E_j' \arrow[d] & \\
v_i\text{の}E_i \arrow[r] \arrow[d, phantom, "\wr"] & E \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & v_i\text{の}E_i[1] \arrow[d, phantom, "\|"] \\
v_i\text{の}E_i \arrow[r] & E_2 \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & v_i\text{の}E_i[1] \\
 & v_i\text{の}E_j[1] \arrow[r, phantom, "="] & v_j\text{の}E_j[1] &
\end{tikzcd}$$

S の条件の下, 交わりが横断的である事を示す.

univ. ext. & univ. div

$S = \{E_1, E_2, \cdots E_n\}$, $\deg_G E_i = 0$ $1 \le i \le n$

$E_i \otimes K_X \cong E_i$

$\mathcal{S} \subset Coh(X)$: F with $gr(F) = \bigoplus_i E_i^{\oplus n_i}$

Def An exact triple

$$F \to \mathbb{E} \to \hat{\mathbb{E}} \to F[1]$$

が $\mathbb{E}$ の univ. div. w.r.t S とは

① $F \in \mathcal{S}$

② $\hat{\mathbb{E}}$: stable complex. $Hom(E_i, \hat{\mathbb{E}}) = 0$ $1 \le i \le n$.

univ. div は 存在すれば一意.

∵ $F' \to \mathbb{E} \to \hat{\mathbb{E}}' \to F'[1]$

$$\underset{\overset{\shortparallel}{0}}{Hom(F'[1], \hat{\mathbb{E}})} \to Hom(\mathbb{E}', \hat{\mathbb{E}}) \to Hom(\mathbb{E}, \hat{\mathbb{E}}) \to \overset{\overset{0}{\shortparallel}}{Hom(F', \hat{\mathbb{E}})}$$

$S = \{v_i, v_j\}$

$v \longrightarrow v - \sum_i n_i v_i$

$\langle v_i, v_j \rangle \le 1$

$P_{E_i} \cap P_{E_j}$

$$\begin{tikzcd} & v_i \otimes E_i \arrow[d] \\ v_j \otimes E_j \arrow[r] & \mathbb{E} \end{tikzcd}$$

$$Ext^2(v_i \otimes E_i \oplus v_j \otimes E_j - \mathbb{E}, \mathbb{E}) \cong \mathbb{C}$$

No.

Date

No.

Date 03・7・11

有理曲面とパンルヴェ方程式の幾何 II

坂井 秀隆

2. 曲面の同型類のパラメータ付け

① $\mathbb{P}^2$ の 9点の配置

ナイーブには

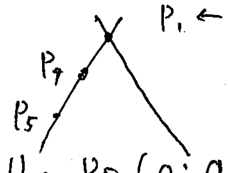
$$PGL(3)\backslash\begin{pmatrix} x_1 & x_2 & & x_9 \\ y_1 & y_2 & \cdots & y_9 \\ z_1 & z_2 & & z_9 \end{pmatrix}/(\mathbb{C}^{\times})^9$$

但し, 1) infinitely near な点をとるなど
$D=\sum m_i D_i$ の型によって配置に制限がある.

2) ここでのパラメーター付けは blowing-down structure による.

<u>例</u> $E_7^{(1)}$-曲面 （P_{II} の初期値空間のコンパクト化）

$$P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow P_6 \leftarrow P_7 \leftarrow P_8 \leftarrow P_9$$



$P_4: (0:0:1)$, $P_5\ (0:a_1:1)$

$P_1: (0:1:0) \leftarrow P_2: \left(\frac{x}{y}, \frac{z}{x}\right)=(0,0)$

$$\leftarrow P_3: \left(\frac{x}{y}, \frac{yz}{x^2}\right)=(0,0)$$

$$\leftarrow P_6: \left(\frac{x}{y}, \frac{y^2z}{x^3}\right)=(0,1)$$

$$\leftarrow P_7: \left(\frac{x}{y}, \frac{y(y^2z-x^3)}{x^4}\right)=(0,0)$$

$$\leftarrow P_8: \left(\frac{x}{y}, \frac{y^2(y^2z-x^3)}{x^5}\right)=(0,-t)$$

$$\leftarrow P_9: \left(\frac{x}{y}, \frac{y(y^2(y^2z-x^3)+tx^5)}{x^6}\right)=(0,\underline{-a_0})$$

$a_1+a_0\neq 0 \longrightarrow a_1+a_0=1$ と規格化できる

t, a_1 : 2次元のパラメーター

$a_1+a_0=0 \longrightarrow \dim|-K_X|=1$

② 周期写像を使うパラメーター付け

X上の $\overset{X}{\text{正則2形式}}$ と $H_2(X,\mathbb{Z})$ の coupling を考えたい。
(必ずしも存在しないので) → 有理2形式

$X-D$ での正則2形式と $H_2(X-D,\mathbb{Z})$ とのカップリング

$H_2(X-D\ \mathbb{Z})$

$$0 \to H_1(D_{red},\mathbb{Z}) \to H_2(X-D,\mathbb{Z}) \to Q(R^\perp) \to 0$$

$D=\sum m_i D_i \Rightarrow D_{red}=\sum D_i$

$Q(R^\perp) := Q(R)^\perp \subset Pic(X)$

$Q(R)=\sum \mathbb{Z} D_i$

ルート系 $\underset{\text{vertical leaves}}{R}$, $\underset{\text{パラメーター対称性}}{R^\perp}$ の2つがでてくる

$\hat{\chi}(E)=\int_E \omega \qquad E\in H_2(X-D,\mathbb{Z})$

ω: 有理2形式で D でのみ極をもつ。

これは

$\chi:\ Q(R^\perp) \to \mathbb{C}/\chi(H_1(D_{red},\mathbb{Z}))$

を導く。

$$\operatorname{rank} H_1(D_{red},\mathbb{Z}) = \begin{cases} 2 & \text{elliptic type} \\ 1 & \text{multiplicative type} \\ 0 & \text{additive type} \end{cases}$$

によって曲面を分類する。

χ の値はそれぞれ
$\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$
$\mathbb{C}/\mathbb{Z} \quad \sim \mathbb{C}^\times$
$\mathbb{C}$

χ の値は $\mathbb{P}^2$ の9点ブローアップから作った具体的な曲面に関しては簡単に計算できる。

$E_7^{(1)}$ - surface

$R=E_7^{(1)}$

$R^\perp = A_1^{(1)} \qquad \alpha_1=\varepsilon_4-\varepsilon_5,\ \alpha_0=3\varepsilon_0-\varepsilon_1-\varepsilon_2-\varepsilon_3-2\varepsilon_4-\varepsilon_6-\varepsilon_7-\varepsilon_8-\varepsilon_9$

$\chi(\alpha_1)=a_1,\ \chi(\alpha_0)=a_0$

χ は ω の定数倍だけ不定性がある。→ $a_1+a_0=1$

注意したいのは
ここでは "t" がでてこない。

① X の同型類　　② $X-D$ の同型類

(①と②でズレがあるのは) $X-D$ の同型写像で X の同型にのびないものがある
→ Painlevé 微分方程式

ズレがでてくるのは
$D_l^{(1)}$, $E_l^{(1)}$ 型のみ

これを Kodaira-Spencer map として具体的に X-E上の微分方程式として実現すると Painlevé 方程式がでてくる。(寺島, 斎藤)

3 affine Weyl群 対称性

2でやったパラメーター付けは blowing-down structure によって違うものを考えることができた。

① 例外曲線の選び方

$Pic(X)=\mathbb{Z}\varepsilon_0+\mathbb{Z}\varepsilon_1+\cdots+\mathbb{Z}\varepsilon_9$

$Pic(X)$ の基底のとり方だけ不定性がある。

定義 $\sigma\in Aut(Pic(X))$ が Cremona isometry であるとは次を満たすことである。

1) $\sigma(\mathcal{F})\cdot\sigma(\mathcal{G})=\mathcal{F}\cdot\mathcal{G}$ for $\forall\mathcal{F},\forall\mathcal{G}\in Pic(X)$
2) $\sigma(K_X)=K_X$
3) effective divisors のなす半群 $Pic^+(X)$ に対し $\sigma(Pic^+(X))=Pic^+(X)$

$D_i \in \mathrm{Pic}^+(X)$ であるので Cremona isometry は $Q(R)$を保存している。
ここには Dynkin automorphism くらいしか作用できない。
$Q(R^\perp)$には $W(R^\perp)$ の作用がある。
だいたい $\mathrm{Aut}(R) \ltimes W(R^\perp) \underset{\text{だいたい}}{\cong} \widetilde{W}(R^\perp) := \mathrm{Aut}(R^\perp) \ltimes W(R^\perp)$
extended Weyl群
$Cr(X)$: Cremona isometries のなす群

<u>定理</u> $R \neq A_6^{(1)}, A_7^{(1)}, A_7^{(1)\prime}, A_8^{(1)}, D_7^{(1)}, D_8^{(1)}$
$$Cr(X(R)) = \widetilde{W}(R^\perp)_{\Delta^{\mathrm{nod}}}$$

$G_S := \{ \sigma \in G \mid \sigma(S) = S \}$
$\Delta^{\mathrm{nod}} := \{ D_i$ 以外の (-2) curve $\}$
Rについて除いた部分も同様だが、簡単な有限群による拡大くらい
違うだけ (決定される)

4 離散 Painlevé 方程式

$\mathbb{Z} \subset Cr(X)$ なので
離散力学系を構成することができる。

$E_6^{(1)}$-曲面 (P_{IV} の初期値空間のコンパクト化)
$Cr(X) = \widetilde{W}(A_2^{(1)})_{\Delta^{\mathrm{nod}}}$ が blowing-down structure の入れ換えとして作用
一般には rank 2 の lattice を含んでいるので退化が考えられる。
$E_6^{(1)}$-曲面 → $E_7^{(1)}$-曲面
$$\text{d-}P_{II} \longrightarrow \begin{cases} \text{alt. d-}P_{I} \\ P_{II} \ (\text{微分方程式}) \end{cases}$$

うまい退化をとると離散 Painlevé 方程式の極限として
Painlevé 微分方程式が得られる。

具体的にはどのように構成するか。

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\text{パラメトリゼーション } \alpha"'] & X \arrow[d, "\sigma(\alpha)"] \\ \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^2 \end{tikzcd}$$

$E_6^{(1)}$-曲面
$f = -\frac{y}{x}, \quad g = t - \frac{x}{z} + \frac{y}{x}$

$w_1: (a_1, a_2, a_0; t; f, g) \mapsto \left(-a_1, a_2+a_1, a_0+a_1; t; f + \frac{a_1}{t-f-g}, g - \frac{a_1}{t-f-g}\right)$

$w_2: (a_1, a_2, a_0; t; f, g) \mapsto \left(a_1+a_2, -a_2, a_0+a_2; t; f, g + \frac{a_2}{f}\right)$

$w_0: (a_1, a_2, a_0; t; f, g) \mapsto \left(a_1+a_0, a_2+a_0, -a_0, t : f - \frac{a_0}{g}, g\right)$

$\sigma_1: (a_1, a_2, a_0; t, f, g) \mapsto (-a_2, -a_1, -a_0; t; t-f-g, g)$

$\sigma_2: (a_1, a_2, a_0; t, f, g) \mapsto (-a_1-a_0, -a_2; t; g, f)$

$T = w_2 \circ w_1 \circ \sigma_1 \circ \sigma_2$
$(a_i; t; f, g) \mapsto \left(a_1, a_2-1, a_0+1; t; t-f-g+\frac{a_0}{g}, t-g-\bar{f}-\frac{a_2-1}{\bar{f}}\right)$

$(f, g) \mapsto (\bar{f}, \bar{g})$
$$\begin{cases} \bar{f} = t - f - g + \frac{a_0}{g} \\ \bar{g} = t - g - \bar{f} + \frac{a_2-1}{} \end{cases}$$
<u>a_0, a_2 … 差分の時間と思える</u>

パラメータの離散的時間発展
additive type → 普通の差分
multiplicative type → q-差分
elliptic type → 楕円差分

No.

Date

A_3'' −曲面 multiplicative

$q-P_{VI}$ … ・Riccati 解 として q-hypergeometric function で書けるような解をもつ.

・線形 q-差分方程式のコネクション保存変形

$q-P_{VI}$:

$$\frac{f\bar{f}}{b_7 b_8} = \frac{\bar{g} - \enclose{circle}{qb_1}}{\bar{g} - b_3}\;\frac{\bar{g} - \enclose{circle}{qb_2}}{\bar{g} - b_4}$$

$$\frac{g\bar{g}}{b_3 b_4} = \frac{f - \enclose{circle}{b_5}}{f - b_7}\;\frac{f - \enclose{circle}{b_6}}{f - b_8}$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & ; f, g \\ b_5 & b_6 & b_7 & b_8 & \end{pmatrix} \longmapsto \begin{pmatrix} qb_1 & qb_2 & b_3 & b_4 & ; \bar{f}, \bar{g} \\ qb_5 & qb_6 & b_7 & b_8 & \end{pmatrix}$$

A_0'' −曲面 elliptic 差分 Painlé equation

・梶原, 増田, 野海, 太田, 山田

Ricati 解が hypergeometric function の楕円 version で書ける.

Painlevé系は Cremona 変換を起源にもつ.