The classification of toric weakened Fano 3-folds

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X : smooth projective variety $/\mathbb{C}$

$\dim X = d$

$\rho = \rho(X)$: Picard number

Def. X : Fano $\overset{\text{def}}{\Longleftrightarrow}$ $-K_X$: ample

X : weak Fano $\overset{\text{def}}{\Longleftrightarrow}$ $-K_X$: nef and big (wF)

Def. (Minagawa) X : wF

X : weakened Fano ($w^{nd}F$)

$\overset{\text{def}}{\Longleftrightarrow}$ i) X : not Fano

ii) $\exists \varphi : \mathcal{X} \longrightarrow \Delta$ small deformation

s.t. $\varphi^{-1}(0) \cong X$, $\varphi^{-1}(t)$: Fano $\forall t \in \Delta \setminus \{0\}$.

Ex. $\mathbb{F}_2 \overset{\text{diform}}{\rightsquigarrow} \mathbb{P}^1 \times \mathbb{P}^1$.

Rem. $(d=2)$ wF, not Fano $\Longleftrightarrow$ $w^{nd}F$.

$(d=3)$ ・$\rho = 2$ (Minagawa) : 5 type ・$\rho \geqq 3$: 3

"toric w^{nd} Fano 3-folds" ― Manuscripta. Math. 109 (2002)

§1. toric wF.

toric variety $X \overset{1:1}{\longleftrightarrow}$ fan Σ in $\mathbb{Z}^d (= N) \subset \mathbb{R}^d$ $(X = X_\Sigma)$

Notation $G(\Sigma) := \{$ the primitive generators of 1-dim. cones of $\Sigma \}$

$X = X_\Sigma$: smooth projective toric variety

Def. $P = \{x_1, \dots, x_m\} \subset G(\Sigma)$ "primitive collection" (p.c.) ← ここで油性
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いたことが分か
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$\overset{\text{def}}{\Longleftrightarrow}$ i) P does not generate a cone in Σ.

ii) $P \setminus \{x\}$ generates a cone in Σ $(\forall x \in P)$.

$$\bigcup_{\sigma\in\Sigma} \sigma = \mathbb{R}^d \qquad (\because X : \text{projective})$$

$\exists!\, \sigma(P) \in \Sigma$ s.t. $x_1+\cdots+x_m \in \mathrm{Rel.int.}(\sigma(P))$

$$x_1+\cdots+x_m = a_1y_1+\cdots+a_ny_n$$

$$\{y_1,\dots,y_n\} = \sigma(P)\cap G(\Sigma).$$

$a_1,\dots,a_n > 0$ primitive relation (p.r.)

$\deg P := m - (a_1+\cdots+a_n)$

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{Z}^d)^\vee \arrow[r] & \mathbb{Z}^{G(\Sigma)} \arrow[r] & \mathrm{Pic}\, X \arrow[r] & 0 \\ & a \arrow[r, mapsto] \arrow[u, phantom, "\cup" rotate=90] & (\langle a, x\rangle)_{x\in G(\Sigma)} \arrow[u, phantom, "\cup" rotate=90] & & \end{tikzcd}$$

rat. function

toric div

$$A_1(X) \cong \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{Z})$$
$$\cong \mathrm{Hom}(\mathbb{Z}^{G(\Sigma)}/(\mathbb{Z}^d)^\vee, \mathbb{Z})$$
$$\cong ((\mathbb{Z}^d)^\vee)^\perp \subset \mathrm{Hom}(\mathbb{Z}^{G(\Sigma)}, \mathbb{Z})$$

$$A_1(X) \cong \{(a_x)_{x\in G(\Sigma)} \in \mathrm{Hom}(\mathbb{Z}^{G(\Sigma)}, \mathbb{Z})$$

Thus, P : p.c. $\longmapsto r(P) \in A_1(X)$

Thm. (Batyrev, Reid)

$$\overline{NE}(X) = \sum_{P:\text{p.c.}} \mathbb{R}_{\geq 0}\, r(P)$$

P : extremal $\overset{\text{def}}{\iff} r(P) \subset$ e.r. $\overset{\text{def}}{\iff}$ 1-dim. face.

Rem. $\deg P = (-K_X . r(P))$

Prop. X : Fano $\iff \deg P > 0$ for $\forall P$: p.c.

wF $\iff \deg P \geq 0$ for $\forall P$: p.c.

◎ classification of toric wF.

(d = 2) $\{\text{smooth toric wF}\} \underset{\text{toric crepant resolution}}{\overset{\text{anti canonical model}}{\rightleftarrows}} \{\text{Gor. toric Fano}\}$

16 type 16 type (Batyrev)

(d=3) {smooth toric wF}/flops $\underset{1:1}{\rightleftarrows}$ {Gor. toric Fano}

$\sum_{i=1}^{4319} a_i$ type $(a_i > 0)$ 4319 type (Kreuzer-Skarke)

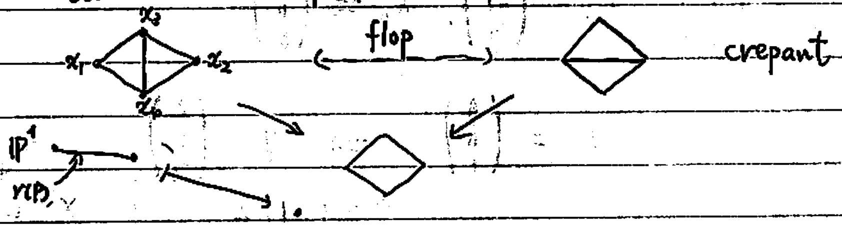
w^{nd} F ⇒ 15 type

§2. Classification

◎ types of e.r. of toric wF 3-folds

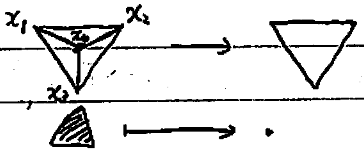
(I) $x_1 + x_2 = x_3 + x_4$ e.p.r.

curve ⟼ pt.



(II) $x_1 + x_2 + x_3 = a x_4$ $(a = 0, 1, 2, 3)$ e.p.r.

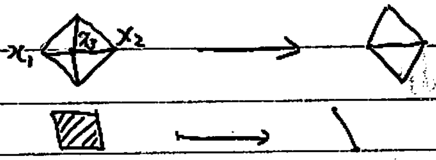
(II) div. ⟼ pt.



a=0 ⇒ D ($\mathbb{P}^2$-bdl)
a=1 ⇒ E2
a=2 ⇒ E5
a=3 ⇒ crepant

(III) $x_1 + x_2 = a x_3$ $(a = 0, 1, 2)$ e.p.r.

div. ⟼ curve



a = 0 ⇒ C ($\mathbb{P}^1$-bdl)
a = 1 ⇒ E1
a = 2 ⇒ crepant.

Thm. (Minagawa) X : wF

X : w^{nd} F.

⟺ every primitive crepant contraction is of (0,2)-type.

$\varphi : X \longrightarrow \overline{X}^{\text{prim.}}$ crepant contraction. (0,2)-type.

$\overset{\text{def}}{\Longleftrightarrow}$ i)

$$\begin{tikzcd} X \arrow[r] & \overline{X} \\ E \arrow[u, phantom, "\cup"] \arrow[r] & \overline{C} \arrow[u, phantom, "\cup"] \\ \text{div.} & \text{curve.} \end{tikzcd}$$

ii) $\overline{C} \cong \mathbb{P}^1$ $(g(C) = \underset{\sim}{0})$

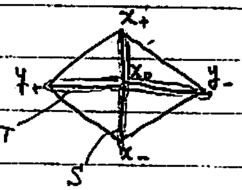
iii) $\varphi|_E : E \longrightarrow \overline{C}$ $\mathbb{P}^1$-bdl.

iv) $(-K_{\overline{X}} . \overline{C}) = \underset{\sim}{2}$ (0,2)-type.

◎ toric case

$\Rightarrow$ $x_+ + x_- = 2x_0$ e.p.r. ($\because$ crepant, i))

Put $x_+ := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $x_- = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $x_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$



$y_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ $y_- = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$a = 0, 1, 2$ ($\because X$: wF).

Rem. $E = \overline{\text{orb}\, x_0} \cong \mathbb{F}_a$.

Fix this (0,2)-type contraction ($\because$ not Fano)

$S := \mathbb{R}x_+ + \mathbb{R}x_- \subset \mathbb{R}^3$

$T := (\mathbb{R}_{\geq} y_+ + \mathbb{R}x_0) \cup (\mathbb{R}_{\geq 0} y_- + \mathbb{R}x_0)$

$I := G(\Sigma) \setminus \{x_0, x_+, x_-, y_+, y_-\}$

Prop. One of the following holds:

(a) $I \subset S$.

(b) $I \subset T$ and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, $a = 0$.

$\because$) $\nexists$ flopping contraction

$\{z_1, z_2\}$: p.c. s.t. $z_1 + z_2 \neq 0$.

$\Longrightarrow$ $\exists z_3 \in G(\Sigma)$ s.t. $z_3 \in \text{Rel.int}(\mathbb{R}_{\geq 0} z_1 + \mathbb{R}_{\geq 0} z_2)$

case-by-case classification $(a = 0, 1, 2)$

(1) $a = 0$ $E \cong \mathbb{P}^1 \times \mathbb{P}^1$

(a) $I \subset S \Rightarrow X \cong \mathbb{P}^1 \times W$ where W : w^{nd} del Pezzo

$16 - 5 = 11$ type.

(b) $I \subset T$ X is a toric surface bdl / $\mathbb{P}^1$

fiber : toric dP fan in T

2 type.

(2) $a = 1$ $E \cong \mathbb{F}_1$

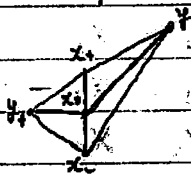
$I \subset S$ X is a toric surface bdl / $\mathbb{P}^1$.

fiber : toric w^{nd} dP fan in S

2 type.

(3) $a = 2$ X : surface bdl

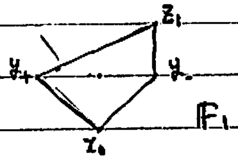
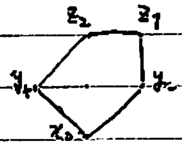
fiber : w^{nd} fan in S.



$\Rightarrow y_+ + y_- = 2x_+$ e.p.r.

$\Rightarrow (0,2)$-type $\Rightarrow$ impossible.

◎ Result

$(a=0)$	$X_{3 \leftarrow \text{Picard number}}^{0 \leftarrow a=0}$	X_4^0
fiber	 $\mathbb{F}_1$	 S_7
e.p.r.	$\begin{cases} x_+ + x_- = 2x_0 \\ y_+ + y_- = 0 \\ x_0 + z_1 = y_- \end{cases}$	$\begin{cases} x_+ + x_- = 2x_0 \\ x_0 + z_1 = y_- \\ y_- + z_2 = z_1 \\ y_+ + z_1 = z_2 \end{cases}$

$$\begin{tikzcd}
X_4^0 \arrow[r, "\text{deformation}"] \arrow[d, "\text{blow-up}"'] & Y_4^0 \arrow[d, "\text{blow-up}"] \\
X_3^0 \arrow[r, squiggly, "\text{deformation}"] \arrow[d] & Y_3^0 \arrow[d] \\
\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)) \arrow[r, squiggly, "\text{deformation}"] & \mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))
\end{tikzcd}$$

Y_4^0 : No. 8 (M-M) not toric

Y_3^0 ; No. 31 toric

No.

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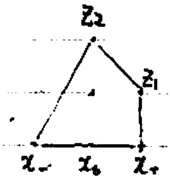
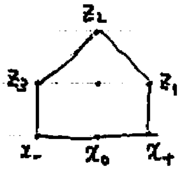
$(a=1)$	X_4^1	X_5^1
fiber		
c.r.	4 本	6 本.

$$\begin{tikzcd} X_5^1 \arrow[r, squiggly] \arrow[d] & Y_5^1 \arrow[d] & \text{No. 2} & \text{not toric} \\ X_4^1 \arrow[r, squiggly] & Y_6^1 & \text{No. 12} & \text{toric} \end{tikzcd}$$

☆

Y_3^0
$$\left\{\begin{array}{l} x_+ + x_- = y_- \\ y_+ + y_- = 0 \\ x_0 + z_1 = y_- \end{array}\right. \longrightarrow \overline{Y_3^0} \quad \text{wF}$$

Y_4^1
$$\left\{\begin{array}{l} x_1 + x_3 = x_2 \\ x_2 + x_5 = x_1 \\ x_1 + x_4 = x_5 \\ x_6 + x_7 = x_1 \end{array}\right. \quad \begin{array}{l} \longrightarrow \overline{Y_4^1} \\ \\ \longrightarrow \overline{Y_4^1} \quad \text{wF.} \end{array}$$

No.

Notes on toric varities from Mori theoretic viewpoint

Date 2002・9・16

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§ Intro

Theorem (S. Mori) (Cone Theorem)

X : smooth proj var $/ k = \bar{k}$

$$\Rightarrow \overline{NE}(X) = \overline{NE}(X)_{K_X \geqq 0} + \sum \mathbb{R}_{\geqq 0}[l]$$

where $l \simeq \mathbb{P}^1$

$0 < -K_X \cdot l \leqq \dim X + 1$

Theorem (…. Y. Kawamata)

X : normal proj var $/ \mathbb{C}$

D : effective $\mathbb{Q}$-div s.t. (X, D) : klt

$$\Rightarrow \overline{NE}(X) = \overline{NE}(X)_{K_X + D \geqq 0} + \sum \mathbb{R}_{\geqq 0}[l]$$

where $l \simeq \mathbb{P}^1$

$0 < -(K_X + D) \cdot l \leqq 2 \dim X$) ⟵ Bend & Break (Miyaoka-Mori)

Question (K. Matsuki ?) $0 < -(K_X + D) \cdot l \leqq \dim X + 1$?

§ Toric varities

Theorem (Cone Theorem)

$X = X(\underset{\substack{\uparrow \\ \text{fan}}}{\Delta})$: n-dim proj toric var $/ k = \bar{k}$

$N_1(X) := Z_1(X)_{\mathbb{R}} / \equiv$ ⟵ 1-cycles with $\mathbb{R}$-coefficients modulo numerical equiv

$NE(X) := \sum_{1\text{-cycles}} \mathbb{R}_{\geqq 0}[C] \subset N_1(X)$

$$\begin{tikzcd} Z_1(X) \arrow[r] & N_1(X) \\ C \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & {[C]} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$\Rightarrow NE(X)$: spanned as a convex cone by a finite number of extremal rays (Reid …)

$D = \sum_j d_j D_j$: $\mathbb{Q}$-divisor s.t. $0 \leqq d_j \leqq 1$ for $\forall j$

D_j : irr torus invariant div

Assume that $K_X + D$: $\mathbb{Q}$-Cartier

$\Rightarrow$ For each extremal ray $\mathbb{R}_{\geq 0}[C]$ $\exists$ $(n-1)$-dim cone $\tau \in \Delta$ s.t. $[V(\tau)] \in \mathbb{R}_{>0}[C]$ $-(K_X + D) \cdot V(\tau) \leqq n+1$

$\left(\underset{\substack{\shortparallel \\ \text{orbit}}}{V(\tau)} \simeq \mathbb{P}^1 \right)$

Moreover, we can choose τ s.t.

$-(K_X + D) \cdot V(\tau) \leqq n = \dim X$

unless $X \simeq \mathbb{P}^n$ & $\sum_j d_j < 1$

Question O.K. for toric var

Rem Characterization of $\mathbb{P}^n$ for toric varities

(Cho - Miyaoka - S - B)

Rem X : toric var

$D = \sum d_j D_j$ as above

$K_X + D$: $\mathbb{Q}$-Cartier $\Rightarrow$ (X, D) : lc $\supset$ klt

Cor (Strong Version of Fujita's Conj)

$\underset{\substack{\nearrow \\ \text{Proj. toric}}}{X} = X(\Delta)$ $D = \sum d_j D_j$ $K_X + D$: $\mathbb{Q}$-Cartier

L : line bundle on X

(1) Assume that $(L . C) \geqq n$

for $\forall C$: torus invariant curve

$\Rightarrow$ $K_X + D + L$: nef

unless $X \simeq \mathbb{P}^n$ $\sum d_j < 1$ $L \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.

(2) Assume that $(L . C) \geqq n+1$

$\Rightarrow$ $K_X + D + L$: ample

unless $X \simeq \mathbb{P}^n$ $D = 0$ $L \simeq \mathcal{O}_{\mathbb{P}^n}(n+1)$

Remark $\quad$ Mustă proved this Cor. on the assumption that X : smooth toric proj. D : reduced ($d_j = 0$ or 1) by his vanishing theorem. (homogeneous coordinate ring)

Tohoku

aterreer obtained similar results when $D = 0$

($\mathbb{Q}$- very ampleness ?)

Tohoku

Rem $\quad X$: toric

H : nef Cartier div

$\Leftrightarrow |H|$: free

X : smooth toric $\quad H$: ample $\Leftrightarrow H$: v. ample.

§. Toric Fano with $\rho = 1$

Def ($\mathbb{Q}$-fac. toric. Fano with $\rho = 1$) $\ni$ (weighted proj space)

$N \simeq \mathbb{Z}^n$: fix

$\{v_1 \cdots v_{n+1}\}$: set of primitive vectors s.t.

$N_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0} v_i$

$\sigma_i := \langle v_1 \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n+1} \rangle \qquad 1 \leq i \leq n+1$

Δ : fan generated by σ_i and its faces for $\forall i$

$X = X(\Delta)$: $\mathbb{Q}$-fac toric Fano with $\rho = 1$

(Rem $\quad \sum \mathbb{Z} v_i = N \Leftrightarrow X = X(\Delta)$: weighted proj space

$\mu_{ij} := \sigma_i \cap \sigma_j$ for $i \neq j \quad (n-1)$-dim cone

We can write

$$\sum a_i v_i = 0 \quad \text{where } a_i \in \mathbb{Z}_{>0}$$
$$\text{g.c.d}(a_1 \cdots a_{n+1}) = 1 \qquad a_1 \leqq \cdots \leqq a_{n+1}$$

⊙ $\Big($ $\mathrm{mult}(\sigma) := [\underset{\sigma\cap N + (-\sigma)\cap N}{\underset{\shortparallel}{N_\sigma}} : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k]$ index

σ : k-dim simplicial cone primitive.

$v_1 \cdots v_k$, first lattice pts along the edges of σ

$$\Rightarrow \quad -K_X \cdot V(\mu_{n,n+1}) = \underbrace{\frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right)}_{\leqq n+1} \frac{\operatorname{mult}(\mu_{n,n+1})}{\underset{\geqq 1}{\operatorname{mult}(\sigma_n)}} \leq n+1$$

key
Prop If $X \not\cong \mathbb{P}^n \Rightarrow \exists\, (l, m)$ s.t.
$-K_X \cdot V(\mu_{l,m}) \leqq n$

Sketch of Proof

Assume $-K_X \cdot V(\mu_{k,n+1}) > n$ for $\forall k$

$\Rightarrow \operatorname{mult}(\sigma_k) = \operatorname{mult}(\mu_{k,n+1}) \quad \forall k$

$\Rightarrow a_k \mid a_{n+1} \quad \forall k \Rightarrow a_1 = \cdots = a_{n+1} = 1$

$\Rightarrow -K_X \cdot V(\mu_{i,j}) = n+1 \quad \forall (i,j)$

$\Rightarrow \operatorname{mult}(\sigma_j) = \operatorname{mult}(\mu_{i,j}) \quad \forall i \neq j$

$\Rightarrow \operatorname{mult}(\sigma_i) = 1$ for $\forall i$

$\Rightarrow X \simeq \mathbb{P}^n$

$\lightning$

- Theorem is o.k. for $\mathbb{Q}$-fac toric Fano with $\rho = 1$

§ Proof of Theorem ($\mathbb{Q}$-fac Case)

$X = X(\Delta)$: $\mathbb{Q}$-fac. $\rho \geqq 2$

$R = \mathbb{R}_{\geq 0}[C]$ $(K_X + D)$-neg extremal ray

$\exists$ extremal cont

$\exists \varphi_R : X \underset{\text{contraction}}{\longrightarrow} Y$, by Reid

$\exists\, \sigma \in \Delta$ s.t.

$P = V(\sigma) \subset X$ $\mathbb{Q}$-fac toric Fano

s.t. $\varphi_R(P) = pt$

By adjunction

$$\dim P + 1 \geqq -K_P \cdot \underset{\simeq \mathbb{P}^1}{\underline{V(\tilde{\tau})}} \geqq -(K_X + D) \cdot \underset{\simeq \mathbb{P}^1}{\underline{V(\tilde{\tau})}}$$

◎ P : stratum of $X(\Delta)$

⇒ We can compare K_P with $K_X|P$ without difficulties

§ Proof of Theorem (general case)

Lem A $X = X(\Delta)$, not $\mathbb{Q}$-fac

$K_X + D$: $\mathbb{Q}$-Cartier

$\Rightarrow \exists f : \tilde{X} \to X$ small $\tilde{X}$: $\mathbb{Q}$-fac

$K_{\tilde{X}} + \tilde{D} = f^*(K_X + D)$

Lem A easy consequence of log MMP (c.f. MCPR ?)

? ? Batyrev

§ Reference

◎ Notes to appear in Tohoku

トーリックの世界 ー 森理論入門 ー

H 14 公開講座用テキスト

(Homepage にある)

No.

Date '02 · 9 · 17

Geometry of Calibrations, Calabi-Yau, hyperkähler, G_2, Spin(7) structures I

後藤竜司 (大阪大学)

§ Introduction

(X, g) Riemannian manifold
(irreducible non-symmetric)

$\mathrm{Hol}(X)$ Holonomy group … Lie group
"Xの曲がり方を量る"

Berger's Holonomy group の分類リスト

(X, g) $\mathrm{Ricci}_g \equiv 0$ とする

$\Rightarrow$ 4つの群

	$SU(n)$	$Sp(m)$,	G_2,	$Spin(7)$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
	Calabi-Yau manifolds	Hyperkähler manifold	G_2 manifold	Spin(7) manifold.
$\dim_{\mathbb{R}}$	$2n$	$4m$	7	8
• 平行な spinor がある	$\sigma_{\mathbb{C}}$	$\sigma_{\mathbb{C}}$	$\sigma_{\mathbb{R}}$	$\sigma_{\mathbb{R}}$
• 微分形式	Ω hol n-form ω Kähler form	$\omega_I, \omega_J, \omega_K$ 3つのKähler form $I^2 = J^2 = K^2 = -1$ $= IJK$	φ 3-form ψ 4-form associative co-associative	Φ 4-form Cayley form.
• 変形理論 変形空間	Smooth (Tian, Ran-Kawamata)		smooth	smooth
• Calibrated submanifold	Special Lagrangean	hol. Lagrangean	Co associative 4-manifold	Cayley submanifold.
	$\mathbb{C}$	$\mathbb{H}$	$\underbrace{\qquad\qquad}_{\mathbb{O}}$	

今日　G_2 の幾何
明日　変形理論　(Calibration から見た)

§1-1　$\mathbb{O}$: the octanions から G_2 へ

$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ as $\mathbb{R}$ vector space

$\mathbb{H}$ the quaternion

積構造　$(a,b)(c,d) = (ac - \bar{d}b,\ da + b\bar{c})$

Cayley-Dickson process.

$$\mathbb{R} \rightsquigarrow \mathbb{C} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{O}$$

非可換　non-associative

$x \in \mathbb{O}$　$x = (a,b)$

conjugate　$\bar{x} = (\bar{a}, -b)$

norm　$\|x\|^2 = \|a\|^2 + \|b\|^2$

$$\bar{\bar{x}} = x, \quad \mathrm{Re}\,(x\bar{y}) = \langle x, y \rangle$$

$$\langle x, y \rangle = 0 \Rightarrow xy = -\bar{y}\bar{x}$$

$$x(yw) = -\bar{y}(\bar{x}w)$$

$$(wy)x = -(w\bar{x})\bar{y}$$

$$G_2 = \mathrm{Aut}(\mathbb{O})$$
$$:= \{ g \in GL(\mathbb{O}) \mid g(xy) = g(x)g(y) \}$$

14-dim Lie group.　単連結

$g(1 \cdot 1) = g(1)^2 \Rightarrow g(1) = 1.$

$G_2 \subset SO(\mathrm{Im}\,\mathbb{O})$　　O : orthogonal group

$$\cong SO(7)$$

<u>Def</u> (associative 3-form). $\varphi(x,y,z) = \langle x, yz \rangle$

$(x, y, z \in \mathrm{Im}\,\mathbb{O})$

skew-symmetric

$\mathbb{O} = \mathbb{R}\{x_0, x_1, \cdots, x_7\}$ Orthonormal basis

$\varphi = \sum_{1 \leq i,j,k \leq 7} C_{ijk}\, x_i \wedge x_j \wedge x_k$ C_{ijk} : $\mathbb{O}$の構造定数

$V = \mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$

$\varphi \in \Lambda^3 V^*$

$*$: Hodge star 作用素

$\psi = *\varphi$ $(** = (-1))$

Co-associative 4-form

$\varphi \wedge \psi = \frac{1}{7}\, \underline{\mathrm{vol}}$

volume form. $x_1 \wedge \cdots \wedge x_7$

<u>Theorem</u> (Bryant)

$G_2 = \{ g \in GL(7) \mid g^*\varphi = \varphi \}$

isotropy subgroup of φ.

moduli of φ

$GL(7)/G_2$ 35-dim

$x_0 = (1, 0)$ $x_4 = (0, 1)$

$x_1 = (i, 0)$ $x_5 = (0, -i)$

$x_2 = (j, 0)$ $x_6 = (0, -j)$

$x_3 = (k, 0)$ $x_7 = (0, -k)$

basis of $\mathbb{O}$

$$\varphi = x_{123} - x_{145} - x_{167} - x_{246} - x_{275} - x_{347} - x_{356}$$

$x_{\alpha\beta\gamma} = dx_\alpha \wedge dx_\beta \wedge dx_\gamma$

$\mathrm{Im}\,\mathbb{O} = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$

x_1, x_2, x_3 $x_4 \cdots x_7$

$\varphi = x_{123} - dx_1 \wedge k_I - dx_2 \wedge k_J - dx_3 \wedge k_K$

$$\begin{cases} k_I = x_{45} + x_{67} \\ k_J = x_{46} + x_{75} \\ k_K = x_{47} + x_{56} \end{cases}$$

ℍ上の hyperkähler structure

k_I : $\mathbb{C}^2$上の Kähler form.

$k_J + \sqrt{-1}\, k_K$ $\mathbb{C}^2$上 hol. 2-form.

$$g \in \{ g \in GL(\mathbb{H}) \mid g^* k_I = k_I,\ g^* k_J = k_J,\ g^* k_K = k_K \}$$
$$\cong$$
$$Sp(1) \cong SU(2)$$

$\therefore\ Sp(1) \hookrightarrow G_2$

$$\mathrm{Im}\,\mathbb{O} = \mathbb{R}\langle x_1 \rangle \oplus \overset{W}{\overset{\shortparallel}{\mathbb{R}\langle x_2, \cdots, x_7 \rangle}}$$

$$\Omega = (dx_2 + \sqrt{-1}\, dx_3) \wedge (dx_4 - \sqrt{-1}\, dx_5) \wedge (dx_6 - \sqrt{-1}\, dx_7)$$

$$\omega = dx_2 \wedge dx_3 - dx_4 \wedge dx_5 - dx_6 \wedge dx_7$$

$$\omega \in \Lambda^2 W^*, \quad \Omega \in \Lambda^3 W^* \otimes \mathbb{C}$$

$$\Rightarrow \quad \varphi = dx_1 \wedge \omega - \Omega^{Re}$$
$$\psi = \frac{1}{2} \omega \wedge \omega + \Omega^{Im} \wedge x_1$$
$$(\Omega = \Omega^{Re} + \sqrt{-1}\, \Omega^{Im})$$

$$SU(3) \cong \{ g \in GL(W) \mid g^* \Omega = \Omega,\ g^* \omega = \omega \}$$

$W \cong \mathbb{C}^3$ と見て Ω hol 3-form

ω Kähler form.

$\therefore\ SU(3) \hookrightarrow G_2$

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§1-2 Spinor から G_2 へ

V : vector space , $\langle\ ,\ \rangle$ metric $\longrightarrow \mathbb{R}$

$\bigotimes V = \sum_i \otimes^i V$

$\mathcal{J} = (\{ v\otimes v + \|v\|^2\cdot 1 \mid v\in V\}$ で生成されるイデアル$)$

$Cl(V) = \bigotimes V/\mathcal{J}$ Clifford algebra

積構造 $v_1\cdot v_2 + v_2\cdot v_1 = -2\langle v_1, v_2\rangle\cdot 1.$

<u>Prop</u>. $\mathcal{A}$: associative alg with unit 1 $/\mathbb{R}$

$f : V \longrightarrow \mathcal{A}$ linear map s.t

$f(v)\cdot f(v) + \|v\|^2\cdot 1 = 0$

$\Rightarrow$ f extends to an algebra hom

$\tilde{f} : Cl(V) \longrightarrow \mathcal{A}$ $(\tilde{f}|_V = f)$

$f(v)f(v) + f(\|v\|^2) = 0$ $\longrightarrow$
$\overset{\|}{\|v^2\|\cdot 1}$

- 8 dim Clifford algebra

$Cl(8) \cong M(16) \cong End(\mathbb{O}\oplus\mathbb{O})$ $\mathbb{O}\cong\mathbb{R}^8$

$f : \overset{\mathbb{O}}{\overset{\|}{V}} \longrightarrow M(16)$

$u \xmapsto{f} \begin{pmatrix} 0 & R_u \\ -R_{\bar{u}} & 0 \end{pmatrix}$ $R_u(x) = x\cdot u$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\cap$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad M(8)$

$\Rightarrow$ [illegible]

$f(u)\cdot f(u) + \|u\|^2\cdot 1 = 0$

$\because)$ $(wx)y = -(w\bar{y})\bar{x}$

<u>$Cl^{even}(8)$</u> $=$ ([illegible] で生成されているもの)
$e_{i_1}\cdots e_{i_k}$ ($e_0 \cdots e_7$: $\mathbb{O}$の基底)

$Cl^{even}(8) \cong M(8)\oplus M(8)$

$\begin{pmatrix} 0 & R_1 \\ -R_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & R_{e_i} \\ -R_{e_i} & 0 \end{pmatrix} = \begin{pmatrix} -R_{e_i} & 0 \\ 0 & -R_{e_i} \end{pmatrix}$

$\mathbb{O} \supset \underset{\mathbb{R}^7}{\underset{\|\wr}{Im\,\mathbb{O}}} \ni e_1, \cdots, e_7$

$Cl^{even}(8) \simeq Cl(7)$ $\simeq M(8)\oplus M(8)$
$e_0 e_i \longleftarrow\!\!\!\shortmid\ e_i$

$S^{\pm} := \mathbb{O}$ 8 dim vector space; $M(8) = \mathrm{End}(S^{\pm})$

$Cl(7) \simeq \mathrm{End}(S^+) \oplus \mathrm{End}(S^-)$

$S^{\pm}$: positive (negative) Spinor representation.

$Cl(7) = \Lambda^* V^*$

$S^+ \otimes S^+ \cong S^+ \otimes (S^+)^* \simeq \mathrm{End}(S^+) \hookrightarrow Cl(7) \simeq \Lambda^* V^*$

Spinor を2つ tensor すると 微分形式 と思える.

<u>Prop.</u> $\sigma \in S^+$ $\|\sigma\| = 1$.

$\Rightarrow$ $\sigma \otimes \sigma = 1 + \varphi + \psi + \mathrm{vol}$.

φ : associative 3-form

ψ : co-associative 4-form.

X : G_2-manifold

<u>Example</u> 1. Kummer type (Joyce)

T^7 $\mathbb{R}$-dim 7 torus

$\Gamma \subset G_2$: finite subgroup

T^7/Γ : G_2-orbifold $\underset{\text{Smoothing}}{\rightsquigarrow}$ X^7: smooth G_2-mfd

2. Fano 3-fold から G_2 へ (Donaldson-Kovalev)

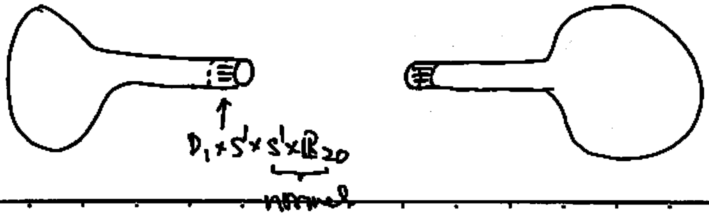
$\overline{W}$: Fano 3-fold

D : anti canonical divisor $\simeq K3$.

$W = \overline{W} \setminus D$ non-compact Calabi Yau

$W \times S^1$: non-compact G_2-manifold

$M_1 = W_1 \times S^1$. $W_2 \times S^1 = M_2$



$D_1 \times \underset{\overset{\uparrow}{\text{out}}}{S^1} \times \underset{\overset{\uparrow}{\text{int}}}{S^1} \times \mathbb{R}_{\geq 0}$ $\qquad$ $D_2 \times \underset{\overset{\uparrow}{\text{out}}}{S^1} \times \underset{\overset{\uparrow}{\text{int}}}{S^1} \times \mathbb{R}_{\geq 0}$

$S^1_{out} \longleftrightarrow S^1_{int}$
$S^1_{int} \longleftrightarrow S^1_{out}$ で $M^7 = (M_1^7 \amalg M_2^7)/\sim$

M^7 compact G_2-mfd

<u>Def</u> of G_2-manifold

X : 7-manifold

$T_xX \simeq V (\simeq \mathrm{Im}\,\mathbb{O})$ associative 3-form φ

$\mathcal{A}(T_xX) \simeq GL(7)/G_2$ associative 3-form 全体

$\mathcal{A}(X) = \bigcup_{x \in X} \mathcal{A}(T_xX) \xrightarrow[GL(7)/G_2\text{-bundle}]{} X$

$\mathcal{E}_{G_2}(X) = \Gamma(X, \mathcal{A}(X))$ Global sections.

X 上の G_2 structure $\Leftrightarrow$ $\varphi \in \mathcal{E}_{G_2}(X)$

- $(X, \varphi \in \mathcal{E}_{G_2}(X))$ s.t. $d\varphi = 0$, $d(*\varphi) = 0$

(φが決まる metric g_φ)

を G_2-manifold という

単に φ が closed なものは比較的簡単.

$d*\varphi = 0$?

X^7 上 $\exists \varphi \in \mathcal{E}_{G_2}(X) \Leftrightarrow w_2(X) = 0$ i.e X: spin mfd
(Stiefel Whitney class)

σ : spinor $\quad \sigma \otimes \sigma \rightsquigarrow \varphi$

(1) Y : K3,

$X = Y \times T^3 \Rightarrow X$: G_2-manifold.
3-torus

$\because$) $Sp(1) \hookrightarrow G_2$

$$\varphi = dt_1\wedge dt_2\wedge dt_3 + dt_1\wedge k_I + dt_2\wedge k_J + dt_3\wedge k_K$$

$$\psi = \mathrm{vol}_Y + dt_2\wedge dt_3\wedge k_I + dt_3\wedge dt_1\wedge k_J + dt_1\wedge dt_K$$

closed

(2) Y : Calabi-Yau 3-fold

$X = Y \times S^1$ G_2-manifold $SU(3) \hookrightarrow G_2$

(Y, ω, Ω) $\varphi = dt\wedge\omega + \Omega^{Re}$

$\psi = \frac{1}{2}\omega\wedge\omega - \Omega^{Im}\wedge dt$

$$d\varphi=0,\ d\psi=0$$
$$\Updownarrow\ (\Downarrow !)$$
$$\nabla\varphi=0,\ \nabla\psi=0$$
$$\mathrm{Iso} = \{g\in GL \mid g^*\varphi=\varphi\}$$
$$\cup$$
$$\mathrm{Hol}$$

Thm X^7 : compact G_2-manifold. $b_1(X)=0$

$\Longrightarrow \mathrm{Hol}(X) = G_2$ i.e X is irreducible G_2-manifold

$$\begin{tikzcd} \sigma\in S^+ \arrow[d] \\ X \end{tikzcd}$$

spinor bundle $\|\sigma\|=1$

$$\sigma\otimes\sigma = 1+\varphi+\psi+\mathrm{vol}.$$

$$S^+\otimes S^+ \cong \Lambda^*$$

$$\|\overset{\overset{d+d^*}{\shortparallel}}{D^{cl}}(\sigma\otimes\sigma)\| = \|D\sigma\|^2 + \|\nabla\sigma\|^2$$

$$\sigma\otimes\sigma = 1+\varphi+\psi+\mathrm{vol}.$$

$d\varphi=0$ & $d^*\varphi=0 \Leftrightarrow \nabla\varphi=0$

Spin Geometry (Lawson-Michaelson)

$G_2 \curvearrowright \Lambda^*$ representation

$\Lambda^0,$

Λ^1 irreducible

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$$

$\underset{\text{self dual}}{\Lambda^2_7 = \{i_v\varphi\in\Lambda^2 \mid v\in TX\}}$ $\Lambda^2_{14} \cong \underset{\mathrm{Lie}(G_2)}{\mathfrak{g}_2}$ anti-self dual

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$$\Lambda^3 = \underset{\underset{\mathbb{R}\langle\varphi\rangle}{\shortparallel}}{\Lambda^3_1} \oplus \Lambda^3_7 \oplus \underbrace{\Lambda^3_{27}} \quad \big\} *$$

$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$$

$$\Lambda^5 \qquad \vdots$$

$\mathfrak{gl}(7) \curvearrowright \Lambda^3$

$\cup\!|$

$\mathfrak{so}(7) \oplus \mathrm{Sym}(7)$

$\cup\!|$

$\mathfrak{so}(7) \oplus \underset{\substack{\uparrow \\ \text{trace free}}}{\mathrm{Sym}_0(7)} \oplus \mathbb{R}1$

$$\Lambda^3 = \mathfrak{gl}(7)/\mathfrak{g}_2$$

$$\cong \underset{\substack{\uparrow \\ 7\,\mathrm{dim}}}{\mathfrak{so}(7)/\mathfrak{g}_2} \oplus \underset{\substack{\uparrow \\ 27\,\mathrm{dim}}}{\mathrm{Sym}_0(7)} \oplus \underset{\substack{\uparrow \\ 1\text{-dim}}}{\mathbb{R}1}$$

$\Lambda^1_7 \simeq \Lambda^2_7 \simeq \Lambda^3_7$
as rep. space

Thm X : compact G_2-manifold.

$$H^2(X) = H^2_7 \oplus H^2_{14}$$

$$H^3(X) = H^3_1 \oplus H^3_7 \oplus H^3_{27}$$

$$H^4(X) = H^4_1 \oplus H^4_7 \oplus H^4_{27}$$

Fano の 3 G_2 Λ

$$\begin{tikzcd} \overline{W} \arrow[d, "\pi"'] \\ \mathbb{P}^1 \ni p \end{tikzcd}$$

$\overline{W}$ cpx 3-fold $\pi_1(\overline{W}) = e$

K3 fibration

$$\underset{K3}{D = \pi^{-1}(p)} \in |-K_{\overline{W}}|$$

$$W = \overline{W} \setminus D \qquad K_W \simeq \mathcal{O}_W$$

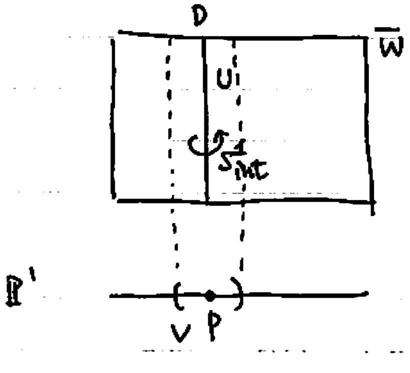
Thm (Tian-Yau) $\exists$ Kähler metric ω, $\exists$ hol 3-from Ω
complete

(Monge-Ampère eq.) $\longrightarrow$ $\Omega \wedge \overline{\Omega} = c\,\omega^3$ $\quad c$: const

$(\Rightarrow \mathrm{Ricci}_\omega = 0)$

$\nabla\omega = 0,\ \nabla\Omega = 0$

$\mathrm{Hol}(W) \hookrightarrow SU(3)$



$U \simeq D \times V \qquad V \hookrightarrow \mathbb{C}(z)$

$z = e^w \quad w = t + i\theta$

$$U \setminus D = D \times (V \setminus \{p\})$$

$$= D \times S^1_{\mathrm{int}} \times \mathbb{R}_{>0}$$

$$\omega_0 = dt \wedge d\theta + k_I .$$
$$\Omega_0 = (dt + \sqrt{-1}\, d\theta) \wedge (k_J + \sqrt{-1}\, k_K) \qquad \text{on } U$$

(k_I, k_J, k_K) HK metric on K3 D,

Tian-Yau metricの近似.

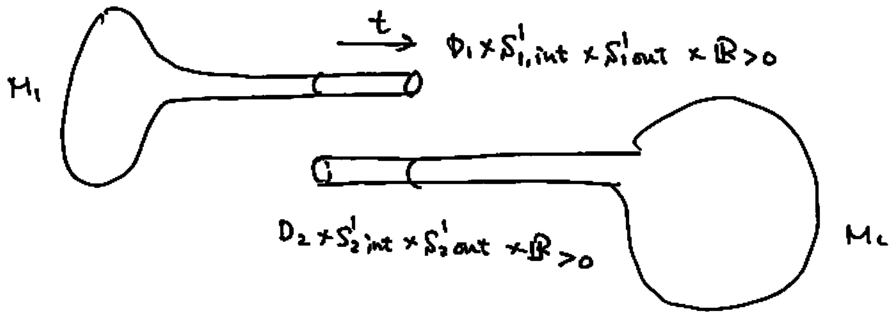
$$\left.\begin{aligned} \omega - \omega_0 &= i\partial\bar\partial u \\ \Omega - \Omega_0 &= d\psi \end{aligned}\right\} \text{ on } U$$

$(\overline{W_1}, D_1)$, $(\overline{W_2}, D_2)$ $2 > t > 2 < \delta$

$M_i = W_i \times S^1_{out}$ G_2-manifold (not irreducible)

$U_i \setminus D_i = D_i \times S^1_{i,int} \times \mathbb{R}_{>0}$

$(U_i \setminus D_i) \times S^1_{out} = D_i \times S^1_{i\,int} \times S^1_{i\,out} \times \mathbb{R}_{>0}$



$$F : D_1 \times S^1_{1,int} \times S^1_{1,out} \times \underset{[T,T+1]}{\cancel{\mathbb{R}_{>0}}} \longrightarrow D_2 \times S^1_{2\,int} \times S^1_{2\,out} \times \underset{[T,T+1]}{\cancel{\mathbb{R}_{>0}}}$$

$$(x, \theta_{int}, \theta_{out}, t) \longmapsto (f(x), \theta_{out}, \theta_{in}, -t+2T+1)$$

$M = M_1 \amalg M_2 / \sim_F$ cpt 7-mfd $f : D_1 \longrightarrow D_2$ diffeo a between K3

<u>matching condition</u> $(D_i, k^i_I, k^i_J, k^i_K)$

$h : H^2(D_2, \mathbb{Z}) \longrightarrow H^2(D_1 \mathbb{Z})$ preserve cup product

$$h \otimes \mathbb{C} : \begin{aligned} [k^2_I] &\to [k^1_J] \\ [k^2_J] &\to [k^1_I] \\ [k^2_K] &\mapsto [-k^1_K] \end{aligned}$$

$\xRightarrow[\text{Torelli}]{}$ $^{\exists} f : D_1 \longrightarrow D_2$ diffeo ~~with these condition~~

$h = f^*$

Thm D_1, D_2 matching condition を満たす

$\rightsquigarrow M = M_1^{\circ} \amalg M_2 /\underset{F}{\sim}$ は G_2-manifold.

- $\pi_1(M) = \pi_1(W_1) \times \pi_1(W_2)$

Counting BPS states via holomorphic anomaly equations

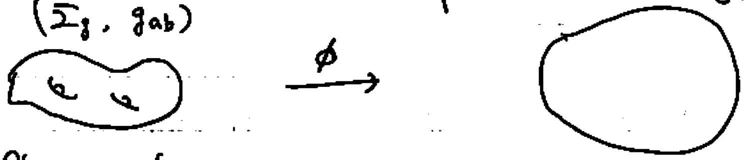
Date '02 · 9 · 17

細野 忍 (東大数理)

§0. Introduction

(1) Gromov-Witten invariant

$$F(G,B) = \int dg_{ab} \int_{\substack{\phi:\Sigma_g \to X \\ C^\infty \text{ map}}} e^{-S(G,B;g_{ab})}$$



Riemann surface

g_{ab}. Riemannian metric

X : CY manifold , topological limit

$$F(G,B) \rightsquigarrow \int_{(\text{moduli of } \Sigma_g)} \int_{\substack{\phi:\Sigma_g\to X \\ \text{hol map}}} e^{-\int_{\Sigma_g}\phi^*(\overset{\text{Kähler form}}{k})}$$

$$= \sum_{\beta \in H_2(X,\mathbb{Z})} N_g(\beta)\, e^{2\pi i \int_\beta k}$$

$$k_t = J_1 t_1 + \cdots + J_r t_r \quad (t_1,\cdots,t_r \in \mathbb{C})$$

$J_1,\cdots,J_r$: a positive base of $H^{1,1}\cap H^2(X,\mathbb{Z})$

'90 Candelas et al.

$F_0(t)$ genus $=0$ case

$\mathcal{M}_g(\beta) =$ moduli of stable maps $(C_g ; \phi: C_g \to X)$ $[\phi(C_g)] = \beta \in H_2(X,\mathbb{Z})$

$$\text{"}\dim \mathcal{M}_g(\beta)\text{"} = 3g-3 + \underbrace{(h^0(C_g,\phi^*T_X) - h^1(C_g,\phi^*T_X))}_{\dim_{\mathbb{C}} X\cdot(1-g) + \int_{C_g} c_1(\phi^*T_X))}$$

$\dim_{\mathbb{C}} X = 3 \Rightarrow$ expected dim $= 0$

Kontsevich, Behrend-Manin , Givental, Lian-Liu-Yau

$\overline{M}_g(\beta)$, virtual fundamental class

$F_0(t)$ の計算

"Mirror Thm"

$$N_g(\beta) = \int_{\overline{M}_g(\beta)} \cdots\cdots$$

[toric hypersurface $\rho(X)=1$

$H_2(X,\mathbb{Z}) = \mathbb{Z}\langle l \rangle$ l : line

~~[illegible]~~

Example $X = (5) \subset \mathbb{P}^4$

$N_g(\beta) = N_g(d \cdot l)$

$$F_0(t) = \frac{5}{6}t^3 - \frac{50}{12}t - \frac{200}{(2\pi i)^3}\zeta(3) + 2875q + \frac{4876875}{4}q^2$$

BCOV: $F_1(t) = \cdots$

$F_2(t) = \cdots$

$\vdots$

Thm (Pandharipande)

$$F_g(t) = \chi(X)\int_{M_g}[c_{g-1}^{g}(\text{Hodge bundle})]^3 + O(q)$$

(2) # of BPS states

$N_g(\beta) \in \mathbb{Q} \rightsquigarrow \exists\, \tilde{N}_g(\beta) \in \mathbb{Z}$

$$\sum_{g=0}^{\infty}\sum_{\beta\in H_2} N_g(\beta)q^\beta\lambda^{2g-2}$$

$$= \sum_{h=0}^{\infty}\sum_{k\geq 1}\sum_{\beta}\tilde{N}_h(\beta)\frac{1}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2h-2} q^{k\beta}$$

eg $(5) \subseteq \mathbb{P}^4$

$\tilde{N}_0(d)$: 2875, 609250, 317206375, ...

予想.
数え幾何的意味は
不明.

Problem

1) Geometric meaning of $\tilde{N}_g(\beta)$?

2). Calculation of $N_g(\beta)$ $(g \geq 1)$

BCOV hol. anomaly eq.

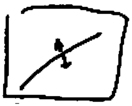
+ local mirror symmetry $(\frac{1}{2}K3)$

3) Large N duality

e.g. $\mathbb{P}^1 \subset X$

$\mathcal{O}(-1)\oplus\mathcal{O}(-1)$

$$\tilde{N}_g([\mathbb{P}^1]) = \begin{cases} 1 & g=0 \\ 0 & g>0 \end{cases}$$



$\wr$ contraction & smoothing

Vanishing S^3

$$\sum_{k\geq 1} \frac{1}{k}\left(2\sin\frac{k\lambda}{2}\right)^2 e^{-kt} = \log Z(S^3)$$

U(n)-chern simons gauge theory

$$Z(S^3) = \int DA\, e^{\int(AdA + \frac{2}{3}g_s A^3)}$$

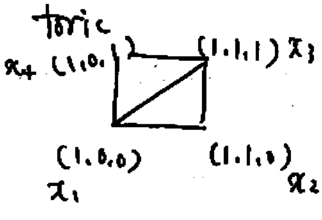
$(N\to\infty)$
fixing $g_s N$

$\lambda \longleftrightarrow g_s$

$t \longleftrightarrow g_s N$

- Large N duality

toric

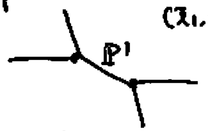


dual graph $\longleftrightarrow$

$\mathbb{P}_\Sigma = (\mathbb{C}^4\setminus Z)/\mathbb{C}^* \;/\!\!/\; \mu_\Sigma^{-1}(a)/\!\!/U(1)$

moment map

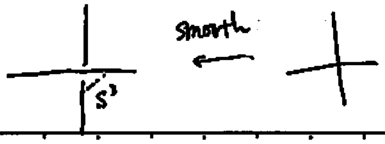
$\mu_\Sigma : \mathbb{C}^4 \longrightarrow \mathfrak{g}^* = u(1)^*$
$\| $
$A^1 \oplus \mathbb{R}$



$(x_1,\cdots,x_4)$
$\to |x_1|^2 + |x_4|^2 - |x_1|^2 - |x_3|^2$

* nbd of (-1,-1) curve

$\swarrow$ contraction $\mathfrak{B}_2 =$



(Gopakumar-Vafa, '98)

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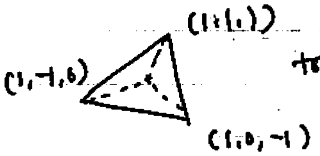
$\mathcal{O}(-3) \longrightarrow \mathbb{P}^2$



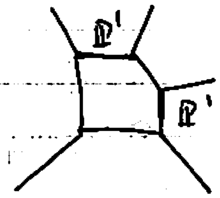
total space of $\mathcal{O}(-3)$

2pt

↑ blow-up

total space of K_X

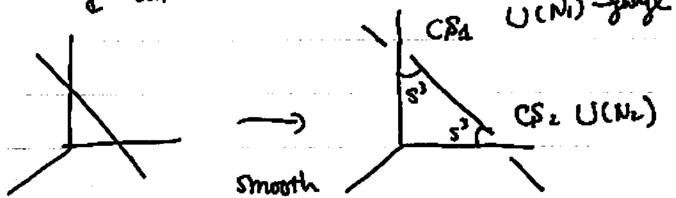
$$\begin{tikzcd} & \mathbb{P}^2 \\ K_X \arrow[r] & ⋇\ \times \arrow[u] \end{tikzcd}$$

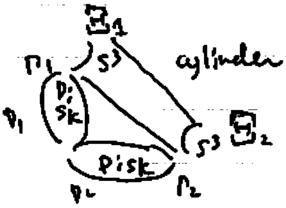


$h . E_1, E_2$

$\beta = d h + m_1 E + m_2 E_2$

↓ contraction

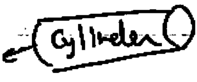




$H_2(X, L_1 \cup L_2)$
⇓
β

~~(CS$_1$ + CS$_2$ + CS interaction)~~

$$S^{(1)}_{int} = -\sum \frac{-e^{-dt}}{d} \mathrm{Tr}\, U_1^d \cdot \mathrm{Tr}\, U_2^d$$



$$S^{(2)}_{int} = \sum_{d \geq 1} \frac{i e^{-dt}}{2d \sin \frac{d g_s}{2}} \mathrm{Tr}\, V_1^d + \sum_{d \geq 1} \frac{i e^{-dt}}{2d \sin \frac{d g_s}{2}} \mathrm{Tr}\, V_2^d$$

disk D_2

D_1 disk

$$+ 2 \sum_{d \geq 1} \frac{e^{-2t}}{d} \mathrm{Tr}\, V_1^d \cdot \mathrm{Tr}\, V_2^d$$

$$U_i = P e^{\int_{\Xi_i} A_i}$$

$$V_i = P e^{\int_{\Gamma_i} A_i}$$

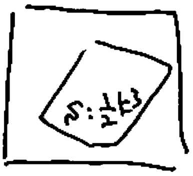
$D_1 \cup D_2$

$$\{\text{GW theory } N_g(\beta)\} \longleftarrow \{CS_1 + CS_2 + CS\text{-interaction}\}$$

$\lambda, \underbrace{S_1, S_2}_{\text{Kähler class of parameter}}, t$ $\qquad$ $g_s, \ , t$

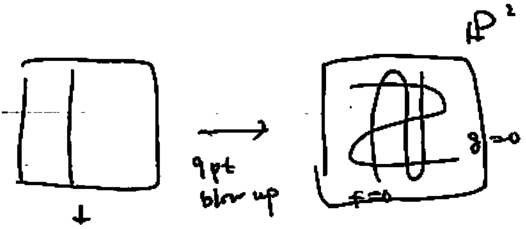
g_sN_1, g_sN_2

§2. $\frac{1}{2}K3$



S: rat. elliptic

X: CY 3-fold



elliptic fibration

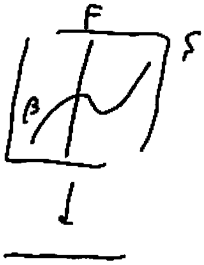
f, g cubic

$$\mu f + \lambda g = 0 \subset \mathbb{P}^2\times\mathbb{P}^1$$
$$\downarrow$$
$$\mathbb{P}^2$$

$$H^2(S, \mathbb{Z}) = \mathbb{Z}H + \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_9$$

Def. $$F_g(q,p) = \sum_{\beta\in H_2} N_g(\beta)\, q^{(\beta, e_9+F)}\, p^{(\beta\cdot F)}$$

"$\beta \subset S$"



$F = 3H - e_1 - e_2 - \cdots - e_9$

e_9 section

$$= \sum_{n\geq 1} Z_{g;n}(q)\cdot p^n$$

← counting fcn of n-sections

Claim (1) $$Z_{g;n}(q) = \frac{q^{\frac{n}{2}}}{\eta(q)^{12n}} P_{g;n}(E_2, E_4, E_6)$$

$P_{g;n} \in \mathbb{Q}[E_2, E_4, E_6]$ Quasi-modular form

22)

(2) $$\frac{\partial Z_{g,n}}{\partial E_2} = \frac{1}{24} \sum_{g'+g''=g} \sum_{s=1}^{n-1} s(n-s) Z_{g',s} Z_{g'',n-s} + \frac{n(n+1)}{24} Z_{g-1,n}$$

(3) $$Z_{0,1}(q) = \frac{q^{\frac{1}{2}} E_4}{\eta^{12}(q)} \leftsquigarrow \Theta_{E_8}$$

(Saito - Takahasi - Hosono)
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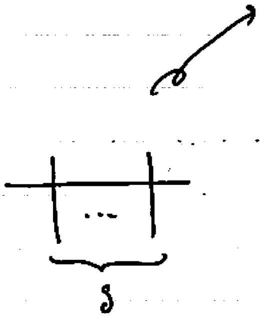
$\underline{n=1}$ $$Z_{1,1} = \frac{q^{\frac{1}{2}}}{\eta^{12}} E_2 . E_4$$

$$Z_{2,1} = -\frac{1}{1440} \frac{q^{\frac{1}{2}}}{\eta^{12}} E_4 (5E_2^2 + E_4),$$

$\vdots$



$Z_{g,1}$

$$\sum_{g=0}^{\infty} Z_{g,1} \lambda^{2g-2} = \left(\frac{\frac{\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^2 \frac{}{\underbrace{\prod (1-q^n)^8 (1-e^{-i\lambda} q^n)^2 (1-e^{i\lambda} q^n)^2}_{\text{"}}} E_4$$

$\underline{\text{cf}}$ Göttsche's formula for $\chi(S^{[n]})$)

No.

Date 2002 · 9 · 17

(General elephants of) three-fold divisorial contractions

川北 真之 (東京大学)

Higher-dim'l alg. geom.

← the theory of minimal models /$\mathbb{C}$

X variety ·············> $\tilde{X}$ good variety

mild sing's { flips

divisorial contr'ns

dim = 3 Mori

⇝ Explicit study of 3-folds

e.g. Sarkisov program

Mori fibre space

flips ← Mori, Kollár

divisorial contr'ns

- terminal sing.

X terminal $\overset{\text{def}}{\Leftrightarrow}$ X normal, $\mathbb{Q}$-Gor ← K_X $\mathbb{Q}$-Cartier

$\exists f : Y \to X$ resol'n

$K_Y = f^* K_X + \sum a_i E_i$ $(a_i \in \mathbb{Q})$

s.t. $a_i > 0$

- $\dim X = 2$: X terminal ↔ X smooth

$\dim X = 3$: X Gor. : term. ⇔ isol. cDV (Reid 1983)

$\overset{\Downarrow}{P}$

non-Gor. (by Mori 1985)

cDV = compound Du Val

$\overset{\text{def}}{\Leftrightarrow}$ $P \in H$ gen. hypersurface

H : Du Val

A_n, D_n, E_n

↕

cA_n, cD_n, cE_n

analytic category の中で考える

Def $f : \underset{\text{term}}{Y} \to \underset{\text{term}}{X}$ divisorial contr'n

$\overset{\text{def}}{\Leftrightarrow} \begin{cases} E := \operatorname{Exc} f & \text{prime divisor} \\ -K_Y & f\text{-ample} \end{cases}$

Main Object

dim = 3

$$\begin{tikzcd} f : Y \arrow[r] & X \\ E \arrow[u, phantom, "\cup"] \arrow[r] & f(E) = P \ \text{pt} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Gor ⇔ Isol cDV

Rmk $f(E) = C$ curve

E (as val'n) : X bl-up along C

- Y sm (Mori 1982)
  Gor. (Cutkosky 1988)

- $P \in X$ $\left\{ \begin{array}{l} \text{quotient} \\ \text{ODP} \end{array} \right.$ (Kawamata 1996) (Corti 2000)
  → f unique

- Focus on general elephants of Y
  $\overset{\text{def}}{\Updownarrow}$ (Reid)
  general ~~ele~~member of $|-K|$

Reid : <u>general elephant conjecture</u> (GE)

in dim 3 , in a suitable situation where $-K$ ample, general elephant $\in |-K|$ is Du Val.

この予想の背景

(1) Reid : $P \in X$ 3-dim Gor. term.
$\Leftrightarrow$ isol. cDV
$\Leftrightarrow$ total space of 1-para. smoothing of a Du Val sing.
$\Leftrightarrow$ $P \in H$ gen. h-s : Du Val (GE) holds

$P \in X$ term. (GE) holds

(2) Mori (88) : the existence of 3-fold flips

$$\begin{tikzcd} f : Y \arrow[r] & X \\ C \arrow[u, phantom, "\cup"] \arrow[r] & P \arrow[u, phantom, "\in"] \end{tikzcd}$$

$|-K_Y| \ni S$ Du Val along C
$\Rightarrow$ reduce to flopping case

(GE)

(3) Fano 3-fold (with mild sing's) (GE) holds : Shokurov, Reid, Takagi (1979) (1983) (2000)

Main thm

$$\begin{tikzcd} f : Y \arrow[r] & X & \text{3-dim div. contr.} \\ E \arrow[r] \arrow[u, phantom, sloped, "\subset"] & P \arrow[u, phantom, sloped, "\in"] & \text{Gor pt.} \end{tikzcd}$$

$\Rightarrow |-K_Y| \ni S$ general elephant has Du Val sing's along E $\square$

$$\begin{tikzcd} (\mathrm{GE}) \arrow[d, Rightarrow] & (\text{Numerically class'n of } f) \arrow[l, Rightarrow] \arrow[dl, Rightarrow] \\ (\text{Classification of } f) & \end{tikzcd}$$

- $-K_Y$ f-ample $\Rightarrow$ $-E$ f-ample
  $K_Y = f^*K_X + \underset{>0}{\underline{a}}E$

$Y = \mathrm{Proj}_X \bigoplus_{i \geq 0} f_*\mathcal{O}_Y(-iE) \xrightarrow{f} X$

$i \in \mathbb{Z}$. $0 \to \mathcal{O}_Y((i-1)E) \to \mathcal{O}_Y(iE) \to Q_i \to 0$

$0 \to f_*\mathcal{O}_Y((i-1)E) \to f_*\mathcal{O}_Y(iE) \to H^0(Q_i) \to 0$ $i < a$

by Kawamata - Viehweg vanishing (82)

$\dim f_*\mathcal{O}(iE)/f_*\mathcal{O}_Y((i-1)E)$

$= h^0(Q_i)$

$= \chi(Q_i)$

$= \chi(\mathcal{O}_Y(iE)) - \chi(\mathcal{O}_Y((i-1)E))$

calculated by sing. R-R (by Reid)^{88}

- Y term. sing.

term. sing. $\xrightarrow[\sim\sim\sim]{\text{deformation}}$ $\{$quotient singularities$\}$

Y term. sing $\rightsquigarrow$ basket : $\{Q : \frac{1}{r_Q}(1, -1, \overset{\text{residue}}{\overline{a}} v_Q)\}$

$J := \{(r_Q, v_Q)\}$ $\qquad$ $K_Y = f^* K_X + aE$

Thm $\overset{\text{(NC)}}{=}$ numerical classification $\rightsquigarrow$ restr'n of sing's of Y
One of the followings holds.

type	$\dim \mathcal{O}_X / f_* \mathcal{O}_Y(-2E)$	J	a
0	$\geqslant 2$	$\# J \leq 3$	1
I	1	$\{(7,3)\}$ $\{(3,1),(5,2)\}$	2
II$_a$	2	$\{(r,2)\}$	2 or 4
II$_b$	2	$\{(r_1,1)\}$ $\{(r_2,1)\}$	$(r_1+r_2)/r_1r_2 E^3 \geqslant 2$
III	3	$\{(r,1)\}$	$(1+r)/rE^3 \geqslant 2$
IV	4	ϕ	2

Rmk (1) $\# J \leq 3 \Rightarrow \# \{$non-Gor sing's of $Y\} \leq 3$

ex $x_1^2 + x_2^3 + x_3^3 + x_4^6 = 0$ $\quad$ cD_4
wtd bl-up $(3, 2, 2, 1)$ $\qquad$ $J = \{(2,1)\ (2,1)\ (2,1)\}$
$\qquad$ Y $\quad 3 \times \frac{1}{2}(1,1,1)$

Rmk (2) type I : the only case where $m_p = f_* \mathcal{O}(-2E)$
$\Rightarrow P \in X$ $\quad cE_7$, cE_8

ex cE_7 $\quad x_1^2 + x_2^3 + x_2 x_3^3 + x_4^7$ $\qquad$ wtd bl-up
cE_8 $\quad x_1^2 + x_2^3 + x_3^5 + x_4^7$ $\qquad$ $(7, 5, 3, 2)$

No.

Date . .

- (NC) + (GE) ⇒ class'n

$P \in X$ sing	:	good ⟷ bad	
		sm cA.	cD, cE.
description	:	Simple ←	complicated
$\# f$	:	large →	Small

$K_Y = f^*K_X + aE$

class'n (sm, cA_n)

restr'n of f ⇐ restr'n of a (cD_n, cE_n)

Thm $a \le 4$

Rmk ∃ examples of $a = 1, 2, 3$

? $a = 4$ → cD_4, cD_5, cD_6

Thm Assume $P \in X$ sm, cA_n

⇒ f : weighted blow-up

e.g. If $P \in X$ sm

then x_1, x_2, x_3 wtd bl-up $(1, r_1, r_2)$

r_1, r_2 : coprime

unless

$$\begin{tikzcd} Y \arrow[d] & \supset E \arrow[d] & \ni Q \\ X & \ni P & \end{tikzcd}$$

Q: $y_1^2 + y_2^2 + y_3^2 + y_4^3 = 0 / \frac{1}{4}(1, 3, 3, 2)$

P: cA_2, $x_1x_2 + x_3^3 + g_{\geq 4}(x_3, x_4) = 0$

$a = 3$

unless の 部分の.

ex $x_1^2 + x_2^2 + x_3^3 + x_1 x_4^2 = 0$

wtd bl-up $(4.3,2.1)$

---

idea of the proof of (GE) type III

$$\begin{tikzcd} Y \supset E \ni Q \arrow[d, "f"'] & \\ X \ni P & \end{tikzcd} \quad \frac{1}{r}(1,-1,a) \quad \begin{tikzcd} Q \in H \underset{f}{\sim} -E \arrow[d] \\ H_X \ni P \text{ gen. h-s} \end{tikzcd} \quad f^*H_X = H + E$$

$H \cap E \quad H^1(H \cap E) = 0$

$H \cap E = \bigcup \mathbb{P}^1 \supset C = \mathbb{P}^1 \ni Q$

- $Y \supset C \ni Q$ descript

$$\begin{tikzcd} Y^\# \arrow[d, "\text{index-1 cover}"'] & \supset & C^\# \arrow[d] & \supset & \overset{\text{irr}}{C^+} & x_1, x_2, x_3 \\ Y & \supset & \underset{t}{C} & \ni & Q & \end{tikzcd}$$

$$(x_1|_{C^+}, x_2|_{C^+}, x_3|_{C^+}) = (t^{\frac{a_1}{r}}, t^{\frac{a_2}{r}}, t^{\frac{a_3}{r}})$$

$$\frac{1}{r}(a_1, a_2, a_3) \in \mathbb{Z}\frac{1}{r}(1,-1,a) + \mathbb{Z}^3$$

$\Big\downarrow$ delicate arguments

$a_3 = 1$ ☆

$$(Y \supset C \ni Q) \simeq \left(\mathbb{C}^3_{x_1x_2x_3} \supset [x_3\text{-axis}] \ni 0\right) \Big/ \tfrac{1}{r}(1,-1,a)$$

$$☆ \quad \Leftrightarrow \quad H \cap E = C$$
$$\mathcal{O}_Y(-K) \otimes \mathcal{O}_C / \text{tors} = \mathcal{O}_{\mathbb{P}^1}(1)$$

$H \cap E = C \ni Q$

- $aH \sim -aE \sim -K \Rightarrow |-K_Y|$ free outside C
- $\underset{\substack{\rotatebox{90}{$\in$} \\ S \text{ general}}}{f_*\mathcal{O}_Y(-K)} \twoheadrightarrow H^0(\mathcal{O}(1)) \Rightarrow \begin{cases} |-K_Y| \text{ free outside } Q \\ S \cap C = \{{}^\exists P, Q\} \end{cases}$

(GE) $\Leftarrow$ S Du Val at Q

$$(S \cdot C) = (\underset{\substack{\wr \\ -aE}}{S} \cdot \underset{\substack{\wr \\ -E}}{H} \cdot E) = aE^3 = 1 + \frac{1}{r}$$

$$\Rightarrow \begin{array}{l} (S \cdot C)_P = 1 \\ (S \cdot C)_Q = \frac{1}{r} \end{array}$$

$S = (x_3 + \dots = 0)$
$\Rightarrow$ Du Val A_{r-1}
$(Q \in Y \quad cA_{r-1})$

Rmk (1) type O, I $\Rightarrow$

$$\begin{tikzcd} Y \arrow[d] & \supset & H \arrow[d] & : \text{Du Val} \\ X & \supset & H_X & \ni P \end{tikzcd}$$

(2) type II, III, IV $\Rightarrow$ $Q \in \underset{\text{gen. elephant}}{S}$ type Z

$\Updownarrow$

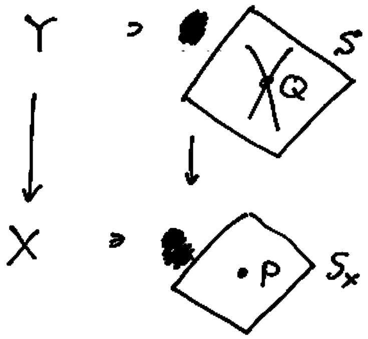
$Q \in Y$, $Q \in S' \subset Y$ type Z
gen. elephant
of a germ $Q \in Y$

$$\begin{tikzcd} Y \arrow[d] & \supset S \arrow[d, "\text{partial resol'n}"] & \text{gen. el.} & \supset E|_S = \text{irred} \\ X & \supset S_X & \text{Du Val} & \ni P \end{tikzcd}$$

almost

ex $x_1^2 + x_2^2 x_3 + x_3^{2r} + x_4^r$ $r \geq 3$ odd

wtd bl-up $(r, r, 1, 2)$



$Y \ni Q \rightsquigarrow \frac{1}{r}(1, -1, 2) \times 2$

(IIb)

- Application to classn cA_n $(n \geq 2)$

$P \in X \simeq x_1 x_2 + g(x_3, x_4) = 0$

$m_i := \mathrm{mult}_E \operatorname{div}(x_i)$

- $x_4 \mapsto x_4 + x_i \Rightarrow m_4 = 1$
- $x_3 \mapsto x_3 + h(x_4) \Rightarrow x_3 + \check{h}(x_4) \notin \mathcal{O}_Y(-(m_j+1)E)$
- $x_1 x_2 + g_{\leq m_1+m_2}(x_3, x_4) + g_{> m_1+m_2} = 0$

  m_1 m_2
- wtd-ord $=: d_0 \leqq m_1 + m_2$

  $g_{d_0} =$ product of $x_{34}, x_3 + * x_4^{m_3}, \ldots$

  $\rightsquigarrow d_0 = m_1 + m_2$

wtd-bl-up $(m_1, m_2 = m_3 \mid \exists m_4)$
$$m_4 = 1$$

$$\begin{tikzcd} Z \arrow[d, "g"'] & \supset & F \arrow[d] \\ X & \ni & p \end{tikzcd}$$

$K_Z = g^*K_X + m_3 F$

- If $(m_3 \geq a) \Longleftrightarrow m_3 = a$ & $g = f$

$\Uparrow$ (GE)

(GE) : $\exists$ Du Val $\in |-aE|$

No.

Geometry of Calibrations. Calabi-Yau, hyperkähler, G_2, Spin(7) structures II

Date '02 · 9 · 18

後藤竜司 （大阪大学）

1. 変形理論
2. 昨日の構成の続き （G_2-manifold）

(X, g) Riemannian manifold

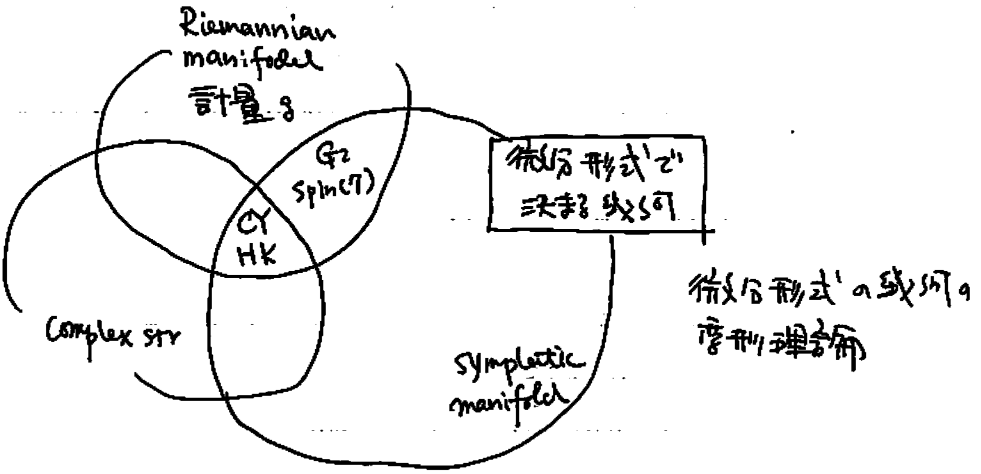
$\mathrm{Ricci}_X \equiv 0$, irreducible, non-symmetric

⇒ Holonomy

	$SU(n)$	$Sp(m)$	G_2	Spin(7)
変形	unobstructed	unobstructed	smooth	smooth

何故これらの4つの幾何構造の変形では obstruction が消えるのか？

統一的な方法で変形理論を作れないか？



昨日の G_2 Geometry

φ: associative 3-form

ψ: co-associative 4-form が重要だった

$\mathrm{Im}\,\mathbb{O}$ 上に 3-form φ が自然に決まる

$$\updownarrow \text{ Hodge } *$$
$$\psi = *\varphi$$

計量は φ, から決まる

$$g(u, v) = \frac{1}{6}(i_u\varphi)\wedge(i_v\varphi)\wedge\varphi$$

i_u: 内部積

$\varphi = dx_{123} + dx_1\wedge k_I + dx_2\wedge k_J + dx_3\wedge k_K$ $u = \frac{\partial}{\partial x_1}$ ⇒ $g(u,u) = \underbrace{k_I\wedge k_J\wedge dx_{123}}_{\text{volume form}}$

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G_2 の変形は、微分形式 (φ, ψ) の変形を考えればよい

$V = \mathrm{Im}\,\mathbb{O}$ として、V 上の G_2 str.

$$\mathcal{A}_{G_2}(V) = \{ (\rho_g \varphi, \rho_g \psi) \in \Lambda^3 \oplus \Lambda^4 \mid g \in GL(V) \}$$

(φ, ψ) : $\mathrm{Im}\,\mathbb{O}$ に入る canonical G_2-str.

X : cpt 7-manifold

$$\mathcal{A}_{G_2}(X) = \bigcup_{x \in X} \mathcal{A}_{G_2}(T_x X) \longrightarrow X$$

$$\mathcal{E}_{G_2}(X) = \Gamma(X, \mathcal{A}_{G_2}(X)) \qquad C^\infty\text{-sections}$$

$$\overset{\cup\!\!\!\!\;\mathrm{l}}{(\varphi, \psi)} \qquad \text{almost } G_2\text{-str.}$$

$d\varphi = 0$, $d\psi = 0$ のとき $(X, (\varphi, \psi))$: G_2-manifold

$$\mathfrak{M}_{G_2}(X) = \frac{\{ (\varphi, \psi) \in \mathcal{E}_{G_2}(X) \mid d\varphi = 0, d\psi = 0 \}}{\mathrm{Diff}_0(X)} : G_2\text{-str. の moduli}$$

$\mathrm{Diff}_0(X) =$ (indentity component of $\mathrm{Diff}(X)$)

Calabi-Yau の場合

$V \cong \mathbb{R}^{2n}$

Ω : cpx n-form $\in \Lambda^n V \otimes \mathbb{C}$

Def $SL_n(\mathbb{C})$ structure Ω

$\Leftrightarrow$ $\mathrm{Ker}\,\Omega = \{ v \in V_{\mathbb{C}} \mid i_v \Omega \equiv 0 \}$

(1) $\mathrm{Ker}\,\Omega \cap \overline{\mathrm{Ker}\,\Omega} = \{0\}$

(2) $\dim_{\mathbb{C}} \mathrm{Ker}\,\Omega = n$

$\Omega : SL_n(\mathbb{C})$-str $\Longrightarrow$ I_Ω : almost cpx str.

$v \in \mathrm{Ker}\,\Omega \Rightarrow I(v) = -\sqrt{-1}\, v$

$v \in \overline{\mathrm{Ker}\,\Omega} \Rightarrow I(v) = \sqrt{-1}\, v$

<u>例</u> $V \otimes \mathbb{C}$ $\Omega = z_1 \wedge \cdots \wedge z_n$ $z_i = x_i + \sqrt{-1}\, y_i$

$\ker \Omega = \mathbb{C}\langle \bar{z}_1, \ldots, \bar{z}_n \rangle = T^{0,1}$

$\overline{\ker \Omega} = T^{1,0}$

$$\mathcal{A}_{SL}(V) = \{ \Omega \in \Lambda^n V^* \otimes \mathbb{C} \mid SL_n(\mathbb{C})\text{-str} \} \overset{\text{transitive}}{\circlearrowleft} \underset{\substack{\text{IS} \\ GL(2n,\mathbb{R})}}{GL(V)}$$

$$\text{isotropy} = \{ g \in GL(V) \mid g^*\Omega = \Omega \}$$
$$= SL_n(\mathbb{C})$$

X : $2n$-dim cpt manifold.

$$\mathcal{A}_{SL}(X) = \bigcup_{x \in X} \mathcal{A}_{SL}(T_x X) \longrightarrow X$$

$GL(2n,\mathbb{R}) / SL_n(\mathbb{C})$ - bundle

$$\mathcal{E}_{SL}(X) = \Gamma(X ; \mathcal{A}_{SL}(X)) \quad C^\infty\text{- sections}$$

$\Omega \in \mathcal{E}_{SL}(X) \rightsquigarrow I_\Omega$: almost cpx str.

$I_\Omega \in \mathrm{End}(TX)$ $I_\Omega^2 = -1$

<u>Lem</u> $d\Omega = 0 \Rightarrow I_\Omega$ integrable

(proof)

N-N a Thm $\{\theta^1, \ldots, \theta^n\}$ $T^{*1,0}$ a local frame

I_Ω integrable $\Leftrightarrow d\theta^i \in \Lambda^{2,0} \oplus \Lambda^{1,1}$

$\big((X, I)$ cpx mfd $\Rightarrow \{dz_1, \ldots, dz_n\} \big)$

$\theta^i \wedge \Omega = 0$ $\big(\because \Omega$: $(n,0)$ form w.r.t $I_\Omega \big)$

$\downarrow d$

$d\theta^i \wedge \Omega = \theta^i \wedge d\Omega = 0$

$\Rightarrow d\theta^i \in \Lambda^{2,0} \oplus \Lambda^{1,1}$ ▨

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$SL_n(\mathbb{C})$ - structure の moduli

$$\mathcal{M}_{SL}(X) = \frac{\{\Omega \in \mathcal{E}_{SL}(X) \mid d\Omega = 0\}}{\mathrm{Diff}_0(X)}$$

$$\Big\downarrow \ \mathbb{C}^*\text{- bundle} \qquad \leftarrow \text{multiplication} \quad \Omega \mapsto \lambda\Omega \quad \lambda \in \mathbb{C}^*$$

$$\mathcal{N} = \{\text{cpx manifold}, K \cong \mathcal{O}\}$$

Def. (Calabi-Yau structure)

$V \cong \mathbb{R}^{2n}$

$\Omega \in \mathcal{A}_\Omega(V)$

ω : real 2-form s.t

(1) $\Omega \wedge \omega = 0$

(2) $\Omega \wedge \bar{\Omega} = c\,\omega^n$ c: 定数, nのみによる

(3) $g(u,v) = \omega(I_\Omega u, v) \Rightarrow g$: positive definite

$\Rightarrow (\Omega, \omega)$ CY str

$$\mathcal{A}_{SU}(V) = \{\text{CY str on } V\} \curvearrowleft GL(2n, \mathbb{R})$$

$\quad \wr\!\wr$

$GL(2n,\mathbb{R})/SU(n)$

X: cpt 2n-manifold

$$\mathcal{A}_{SU}(X) = \bigcup_{x \in X} \mathcal{A}_{SU}(T_xX) \longrightarrow X$$

$$\mathcal{E}_{SU}(X) = \Gamma(X, \mathcal{A}_{SU}(X))$$

$$\mathcal{M}_{SU}(X) = \{(\Omega,\omega) \in \mathcal{E}_{SU}(X) \mid d\Omega = 0, d\omega = 0\} / \mathrm{Diff}_0(X)$$

$\circledast$ $d\Omega = 0, d\omega = 0 \Leftrightarrow \nabla\Omega = 0, \nabla\omega = 0$

$\mathrm{Hol} \hookrightarrow SU(n)$

一般化にする

V: $\mathbb{R}$-dim $= n$ vector space

$\Phi_V^0 = (\phi_V^1, \cdots, \phi_V^l) \in \bigoplus_{i=1}^{l} \Lambda^{p_i} V^*$ fix.

$\mathcal{A}(V) = \{ \rho_g \Phi_V^0 \in \bigoplus_i \Lambda^{p_i} V^* \mid g \in GL(V) \}$

$H = \{ g \in GL(V) \mid \rho_g \Phi_V^0 = \Phi_V^0 \}$ $\rho: GL(V) \to \bigoplus_i \mathrm{End}(\Lambda^{p_i} V^*)$

X: compact n-manifold.

$\mathcal{A}(X) = \bigcup_{x \in X} \mathcal{A}(T_x X) \longrightarrow X$

G/H-bundle

$\mathcal{E}(X) = \Gamma(X, \mathcal{A}(X))$ C^∞ sections.

$$\mathcal{M}(X) = \frac{\{ \Phi \in \mathcal{E}(X) \mid d\Phi = 0 \}}{\mathrm{Diff}_0(X)}$$

◎ $\mathcal{M}(X)$ は いつ smooth な多様体になるか?

$\widetilde{\mathcal{M}}(X) = \{ \Phi \in \mathcal{E}(X) \mid d\Phi = 0 \}$

$\Phi \in \widetilde{\mathcal{M}}(X)$ fix

$E^0 = \{ i_v \Phi = (i_v \phi^1, \cdots, i_v \phi^l) \in \bigoplus_i \Lambda^{p_i - 1} \mid v \in TX \}$

$E = (E^0$ から生成される Λ^*-module$)$ ← $C^\infty(X)$ で

$E = \bigoplus_{k \geq 0} E^k$

$E^k = \{ \alpha \wedge i_v \Phi \mid \alpha \in \Lambda^k, v \in TX \}$

$E^1 = \{ \hat{\rho}_\xi \Phi \in \bigoplus \Lambda^{p_i} \mid \xi \in gl(TX) = \mathrm{End}(TX) \}$

$\hat{\rho}$: は $GL \to \bigoplus \mathrm{End}(\Lambda^{p_i})$ の微分表現

$\xi = \sum \xi^j \theta^i \otimes i_{v_j}$

$\hat{\rho}_\xi \Phi = \sum \xi_i^j \theta^i \wedge i_{v_j} \Phi.$

Prop. graded Λ^*-module $E = \bigoplus_{k \geq 0} E^k$ は differential module w.r.t. d

∵) $d : E^0 \longrightarrow E^1$

$d\, i_v \Phi \in E^1$ を見る

$\quad \shortparallel$

$L_v \Phi$ Lie微分

$\mathcal{E}(X)$ は diffeo で不変 $\Rightarrow L_v \Phi = T_\Phi \mathcal{E}(X)$ tangent space 差

$T_\Phi \mathcal{E}(X) = E^1 = \{ \hat{\rho}_\xi \Phi \mid \xi \in gl(TX) \}$ //

$$\begin{tikzcd}
0 \arrow[r] & E^0 \arrow[r, "d"] \arrow[d] & E^1 \arrow[r, "d"] \arrow[d] & E^2 \arrow[r] \arrow[d] & \cdots & (\#_\Phi) \\
0 \arrow[r] & \bigoplus_i \Gamma(\Lambda^{p_i-1}) \arrow[r] & \bigoplus_i \Gamma(\Lambda^{p_i}) \arrow[r] & \bigoplus_i \Gamma(\Lambda^{p_i+1}) \arrow[r] & \cdots & (dR)
\end{tikzcd}$$

de Rham complex の直和

$\mathrm{per}_k : H^k(\#_\Phi) \longrightarrow \bigoplus_i H^{p_i+k-1}(X)$

Def $\mathcal{A}(V)$ が elliptic orbit

$\underset{\text{def}}{\Longleftrightarrow}$ $(\#_\Phi)$ が elliptic complex ← (V上の[illegible] 作用素の symbol.)

$\Leftrightarrow 0 \to E^0(V) \xrightarrow{u \wedge \cdot} E^1(V) \xrightarrow{u \wedge \cdot} E^2(V) \to \cdot$ symbol complex

$\forall \underset{\neq 0}{u} \in V^*$ が exact

$\mathcal{A}(V)$ が metrical $\Leftrightarrow$ isotropy $H \subset O(V)$
($\exists$ metric on V)

Def. $\Phi \in \mathcal{E}(X)$ $d\Phi = 0$ is topological calibration

$\Leftrightarrow$ $\mathrm{per}^1 : H^1(\#) \longrightarrow \bigoplus_i H^{p_i}(X)$
$\mathrm{per}^2 : H^2(\#) \longrightarrow \bigoplus_i H^{p_i+1}(X)$ } injective

$\mathcal{A}(V)$ topological orbit

$\Leftrightarrow \forall \Phi \in \mathcal{A}(X)$, $\forall X$: compact n-manifold

Φ が topological calibration

Theorem $\mathcal{A}(V)$: elliptic, metrical, Topological

$\Rightarrow$ $m(X)$: smooth finite dimensional manifold.

In particular $m(X)$ Hausdorff.

Thm. $\Phi \in \widetilde{m}(X)$ $\quad Per^2 : H^2(\#) \longrightarrow \bigoplus_i H^{p_i+1}(X)$

$\Rightarrow$ Φのまわりでの変形は "unobstructed".

Thm $\mathcal{A}(V)$ elliptic metrical topological

$$\begin{aligned} Per : m(X) &\longrightarrow \bigoplus_i H^{p_i}(X) \\ \Phi &\longmapsto [\Phi]_{dR} \end{aligned}$$

$\Rightarrow$ Per is locally injective

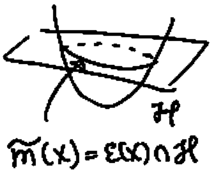
Thm $\mathcal{A}_{SU}$, $\mathcal{A}_{Sp}$. $\mathcal{A}_{G_2}$, $\mathcal{A}_{Spin(7)}$ は全て
metrical, elliptic, topological

Def $\Phi \in \widetilde{m}(X) = \{ \Phi \in \mathcal{E}(X) \mid d\Phi = 0 \}$

Φ が unobstructed

$\Leftrightarrow$ $\forall a \in \mathbb{H}^1(\#_\Phi)$ $\exists \Phi_t \in \widetilde{m}(X)$ $\frac{d}{dt}\Phi_t|_{t=0} = a$

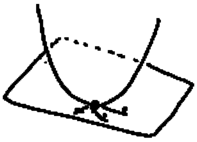
$\mathcal{E}(X)$ 等質空間 "=" $GL(TX)/H(TX)$ Hilbert manifold



$\bigcap$

$\bigoplus_i \Lambda^{p_i} \supset \mathcal{H} = \{ \text{closed form} \}$

infinitesimal tangent sp $T_\Phi \mathcal{E}(X) \cap \mathcal{H}$.



← obstructed

$\Phi^0 \in \widetilde{m}(X)$ fix

$g \in GL(TX)$

$$\begin{tikzcd} TX \arrow[rr, "g"] \arrow[dr] & & TX \arrow[dl] \\ & X & \end{tikzcd}$$

$g|_{T_xX} \in GL(T_xX)$

$\Phi = \rho_g \Phi^0$

$a \in End(TX)$

$a(t) = a_1 t + \frac{1}{2!} a_2 t^2 + \cdots$

$g(t) = \exp a(t) \in GL(TX)[[t]]$

$\rho_{g(t)} \Phi^0$, $d\hat{\rho}_{a_1} \Phi^0 = 0$ とする

a_1 に対して $d\rho_{g(t)} \Phi^0 = 0$: 解くべき方程式 (★)

$d\rho_{g(t)} \Phi^0 = \sum_{k=1}^{\infty} \frac{1}{k!} dR_k t^k$ とおく

(★) $\Leftrightarrow$ $dR_k = 0$ $(k = 1, 2, \cdots)$

$R_1 = \hat{\rho}_{a_1} \Phi^0$, $dR_1 = 0$.

$dR_2 = \frac{1}{2!} (d\hat{\rho}_{a_2} \Phi^0 + d\hat{\rho}_{a_1} \hat{\rho}_{a_1} \Phi^0)$

$dR_2 = 0 \Leftrightarrow d\hat{\rho}_{a_2} \Phi^0 = - d\hat{\rho}_{a_1} \hat{\rho}_{a_1} \Phi^0$ なる a_2 を取る

$d\hat{\rho}_a \hat{\rho}_a \Phi^1 = (-i_{N(a_1,a_1)} + \underbrace{L_{a_1^2}}_{0}) \Phi^0$

$L_{a_1^2} = [\rho_{a_1^2}, d]$

$N(a,a)$: Nijenhuis tensor

a $a \in End(TX) \Rightarrow N(a,a) \in \Lambda^2 \otimes TX$

$i_{N(a,a)} \Phi \in E^2$

$L_{a_1^2} \Phi = d\rho_{a_1} \Phi = 0$.

$\hat{\rho}_{a_2}\Phi \in E^1$

方程式 $dR_2=0 \iff d\hat{\rho}_{a_2}\Phi^0 = -d\hat{\rho}_{a_1}\hat{\rho}_{a_1}\Phi^0$

$\rotatebox{90}{$\in$}$ E^2

$$\begin{tikzcd}
E^1 \arrow[r] & E^2 \arrow[r] \arrow[d] & (\#) \\
 & \oplus\Lambda^{p_i} \to \oplus\Lambda^{p_i+1} &
\end{tikzcd}$$

Per^2_2 injective $\Rightarrow$ $d\hat{\rho}_a\hat{\rho}_a\Phi^0 \equiv 0 \in H^2(\#)$

$\Rightarrow \exists a_2$ $\quad \hat{\rho}_{a_2}\Phi^0 = -d^*d\underset{\substack{\uparrow\\ \text{Green operator.}}}{G}(d\hat{\rho}_{a_1}\hat{\rho}_{a_1}\Phi)$

Inductiveに $\quad a_1,\cdots,a_k$

$dR_{k-1}=0 \qquad d\hat{\rho}_{a_{k+1}}\Phi^0 = -\mathrm{Ob}(a_1,\cdots,a_k)$

$\rotatebox{90}{$\in$}$ E^2 : d-exact

$\Downarrow$

$\exists a_{k+1}$

$\Rightarrow$ $\exists g(t)$ べき級数 $\quad d\rho_{g(t)}\Phi^0=0, \quad \frac{d}{dt}\rho_{g(t)}\Phi^0 = \hat{\rho}_{a_1}\Phi^0$

$g(t)$の収束を示す.

・・・・

$SL_n(\mathbb{C})$ str

$\Omega \in \widetilde{m}_{SL}(X)$

$\{ i_v\Omega \mid v \in TX \} \in \Lambda^{n-1,0}$

$\{ \hat{\rho}_\xi\Omega \mid \xi \in gl(TX) \} = \Lambda^{n,0} \oplus \Lambda^{n-1,1}$

$$\begin{tikzcd}
E^0 \arrow[r] & E^1 \arrow[r] & E^2 \arrow[r] & {} \\
\Lambda^{n-1,0} \arrow[u, phantom, "\rotatebox{90}{$\cong$}"] \arrow[r, "d"] & \Lambda^{n,0}\oplus\Lambda^{n-1,1} \arrow[u, phantom, "\rotatebox{90}{$\cong$}"] \arrow[r] & \Lambda^{n,1}\oplus\Lambda^{n-1,2} \arrow[u, phantom, "\rotatebox{90}{$\cong$}"] \arrow[r] & {} \\
\Lambda^{n-1} \arrow[u, hook] \arrow[r] & \Lambda^{n} \arrow[u, hook] \arrow[r] & \Lambda^{n+1} \arrow[u, hook] \arrow[r] & (dR)
\end{tikzcd}$$

No.

Date

Kählerとすると

$$H^0(\#) = H^{n-1,0}$$
$$H^1(\#) = H^{n,0} \oplus H^{n-1,1} \hookrightarrow H^n$$
$$H^2(\#) = H^{n,1} \oplus H^{n-1,2} \hookrightarrow H^{n+1}$$

$SL_n(\mathbb{C})$ structure は unobstructed

No.

Date '02 · 9 · 18

Derived categories for complex symplectic varieties.

並河 良典 (阪大)

§1 Introduction

<u>Def</u> X : complex in symplectic mfd. of 2ℓ dim

$\overset{\text{def}}{\Longleftrightarrow}$ X: Compact Kähler mfd, $\pi_1(X) = \{1\}$

$\exists\,\Omega$: holo non-degenerate 2-form

$\left(\Leftrightarrow \bigwedge^{\ell}\Omega\right.$: nowhere vanishing $\left.\right)$

s.t. $H^0(X, \Omega_X^2) = \mathbb{C}[\Omega]$

$H^2(X, \mathbb{Z})$ admits a symmetric bilinear form q_X (Beauville - Bogomolov form) <u>K3のとき</u>. $q_X = \langle\ ,\ \rangle_X$ cup積.

$$q_X(\alpha) := {}^{\exists}(\text{const})\left[\frac{\ell}{2}\int_X (\Omega\bar{\Omega})^{\ell-1}\alpha^2 + (1-\ell)\int_X \Omega^{\ell}\bar{\Omega}^{\ell-1}\alpha \cdot \times \int_X \Omega^{\ell-1}\Omega^{\ell}\alpha \right]$$

<u>Examples</u>

1. $\mathrm{Hilb}^n(S)$ S: K3-surface
2. $K^n(A)$ A: Abelian surface.
3. K3 surface 上の stable torsion free sheaves の moduli

No.

Date

Def X, Y : complex irr. symp. mfds

$$X \sim Y \iff (H^2(X,\mathbb{Z}), q_X) \cong (H^2(Y,\mathbb{Z}), q_Y)$$

(P) equivalent — Hodge isometry

period

$$\begin{tikzcd} \arrow[r] & K^{i-1} \arrow[r] \arrow[d] & K^i \arrow[r] \arrow[d] & K^{i+1} \arrow[r] \arrow[d, "\varphi"] & {} \\ \arrow[r] & L^{i-1} \arrow[r] & L^i \arrow[r] & L^{i+1} \arrow[r] & {} \end{tikzcd}$$

quasi-iso

$$X \sim Y \iff D(X) \xrightarrow{\sim} D(Y)$$

(D) equivalent — ∃ equivalence

derived category

φ: equivalence $\iff \exists \psi : D(Y) \to D(X)$ s.t.

$\varphi \circ \psi \simeq \mathrm{id}_{D(Y)}$

$\psi \circ \varphi \simeq \mathrm{id}_{D(X)}$

quasi-inverse of φ

$X \sim Y$

birat. equ

$X \sim Y$

iso. equ

$K^2(A), K^2(\hat{A})$

$$\begin{tikzcd} (\text{iso. equ}) \arrow[r, Rightarrow] & (\text{birat. equ}) \arrow[l, bend right, Leftarrow, "\times"'] \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\text{conj.}"] & (\text{P-equivalence}) \arrow[l, bend right, Leftarrow, "\times"'] \arrow[d, shift left, Rightarrow, "?"] \\ & & (\text{D-equivalence}) \arrow[u, shift left, Rightarrow, "\times"] \arrow[ul, Rightarrow, bend left] \end{tikzcd}$$

Aim of the talk

① Partial answer to the conj.

② D-equivalence for cpx. symp. orbifolds and orbifold cohomology.

§2. Mukai flops and D-equivalence

$$\begin{tikzcd} & & E \subset \widetilde{X} \arrow[dll] \arrow[dl, "p"'] \arrow[dr] & \\ (\mathbb{P}^n \simeq) Y \subset & X \arrow[rr, dashed, "\varphi"] \arrow[dr] & & X^+ \supset Y^+ (\simeq \mathbb{P}^n) \arrow[dl] \\ & & \bar{X} & \end{tikzcd}$$

dim n subvar.

smooth proj. var. of dim 2n

smooth proj. var. of dim 2n

$N_{Y^+/X^+} \simeq \Omega^1_{\mathbb{P}^n}$

$N_{Y/X} \simeq \Omega^1_{\mathbb{P}^n}$

$$\begin{tikzcd} & \hat{X} := X \times_{\bar{X}} X^+ \arrow[dl, "q"'] \arrow[dr, "q^+"] & \\ X \arrow[dr] & & X \arrow[dl] \\ & \bar{X} & \end{tikzcd}$$

$\hat{X} = \widetilde{X} \cup Y \times Y^+$
N.C. variety with 2-comp.

$$\phi : \mathbb{R}p^+_* \circ \mathbb{L}p^* : \ D(X) \to D(X^+)$$

$$\psi : \mathbb{R}q^+_* \circ \mathbb{L}q^* : \ D(X) \to D(X^+)$$

Thm

(1) ϕ is not fully faithful.

(2) ψ is an equivalence.

Rem 1 Y. Kawamata proved independently this theorem by a different method.

Rem.2 Bondal-Olrov.

flop

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "p"'] \arrow[dr, "p^+"] & \\ (\mathbb{P}^n \cong) Y^n \subset X^{2n+1} & & X^+ \supset Y^+ (\cong \mathbb{P}^n) \end{tikzcd}$$

$N_{Y/X} \cong \mathcal{O}(-1)^{\oplus n+1}$ $\qquad$ $N_{Y^+/X^+} \cong \mathcal{O}(-1)^{\oplus n+1}$

$$\phi : \mathbb{R}p^+_* \mathbb{L}p^* : \underset{\text{equivalence}}{D(X) \to D(X^+)}$$

Cor. X, Y : complex (irr) symp mfd :
- projective
- 4次元.

$$X \underset{\substack{\text{birat}\\ \text{equ}}}{\sim} Y \Rightarrow X \underset{\text{D-equ.}}{\sim} Y$$

∵)

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{birat}"] \arrow[d, dashed] & & Y \\ \underbrace{X_1 \arrow[r, dashed] & X_2} \arrow[r, dashed] & \cdots \arrow[ur, dashed] & \end{tikzcd}$$

Mukai flop

Outline of the proof

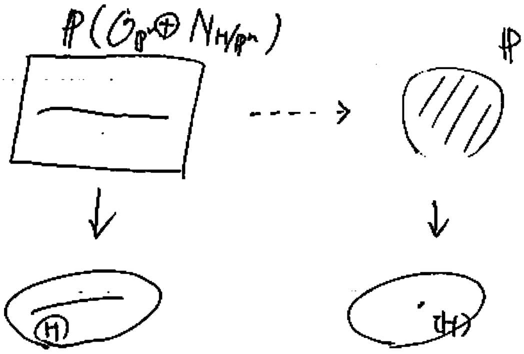
(Special case) $\qquad$ $(:= \mathrm{proj}\ \mathrm{Sym}(\cdots))$

$$\begin{tikzcd} X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \Theta_{\mathbb{P}^n}) \arrow[dr, "\pi"'] & & \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}) =: Y \arrow[ll, hook'] \arrow[dl] \\ & \mathbb{P}^n & \end{tikzcd}$$

$X^+ \supset Y^+$ Its copy.

$$\begin{tikzcd}
& \tilde{X} \arrow[dl, "p"'] \arrow[dr, "p^+"] & \\
Y \subset X \arrow[rr, dashed, "\varphi"'] \arrow[d, "\pi"'] & & X^+ \supset Y^+ \arrow[d, "\pi^+"] \\
\mathbb{P}^n \supset H & & (\mathbb{P}^n)^*
\end{tikzcd}$$

T(H)



(1). $\phi(\pi^*\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{X^+}\otimes\pi_+^*\mathcal{O}_{\mathbb{P}^n}(-1)\otimes I_{Y^+}$

$\mathrm{Ext}^2(\pi^*\mathcal{O}_{\mathbb{P}^n}(1), \pi^*\mathcal{O}_{\mathbb{P}^n}(1))$

$\mathrm{Ext}^2(\phi(\pi^*\mathcal{O}_{\mathbb{P}^n}(1)), \phi(\pi^*\mathcal{O}_{\mathbb{P}^n}(1))) \simeq \mathrm{Ext}^2(I_{Y^+}, I_{X^+}) \neq 0$

(2) $D(X)$ is generated by $\Omega := \{\mathcal{O}_X(j)\otimes\pi^*\mathcal{O}_{\mathbb{P}^n}(k)\}_{\substack{-n\leq j\leq 0\\ -n<k\leq 0}}$

$$\psi(\mathcal{O}_X(j)\otimes\pi^*\mathcal{O}_{\mathbb{P}^n}(k)) = \mathcal{O}_{X^+}(j+k)\otimes\pi_+^*\mathcal{O}_{\mathbb{P}^n}(k)$$

この generator を用いて計算する.

$\psi^{-1} := \mathbb{R}q_*(\mathbb{L}q_+^*(\cdot)\otimes\omega_{\tilde{X}/X})$ ψの quasi-inverse.

(general case) X: 一般

$\Omega' = \{\mathcal{O}_x\}_{x\in X}$ spanning class of $D(X)$

Bridgeland: enough to show

Fully-faithfulness $\mathrm{Hom}^i(\mathcal{O}_x, \mathcal{O}_{x'}) \xrightarrow{\sim} \mathrm{Hom}^i(\varphi(\mathcal{O}_x), \varphi(\mathcal{O}_{x'}))$

$x, x' \in Y \subset X$ の時が問題

$\Rightarrow$ Special case より o.k.

Equivalence $\forall c \in \Omega'$ に対して

$$\varphi \circ \underset{\uparrow}{S_X}(c) = \underset{\nearrow}{S_{X^+}} \circ \varphi(c)$$ を示せばよい。

Serre functor

$$\begin{tikzcd} S_X: D(X) \arrow[r] & D(Y) \\ \cdot \arrow[r, mapsto] & \omega_X \otimes (\cdot)[m] \end{tikzcd}$$ Serre functor

S_{X^+} ,

§3.

Orbifold cohomology

$$\begin{tikzcd} M & G \arrow[l, bend right] \end{tikzcd}$$
cpx mfd   finite group

$\dim M = k$

$\exists$ nowhere vanishing holo k-form ω

$g^*\omega = \omega \ (\forall \omega)$

$Z := [M/G]$.

$g \in G$, $M^g := \{x \in M \mid gx = x\}$ $C(g)$: gの centralizer

<u>Def</u>

(1) $x \in M^g$

$\langle g \rangle \curvearrowright TM_x$ $\{e^{2\pi i r_1}, \dots, e^{2\pi i r_k}\}$

gの eigen-value.

$0 \le r_j < 1$ ($\forall j$)

$a(g) := \sum_{i=1}^{k} r_i$ (age)

(2) $H^*_{orb}(Z) := \bigoplus_{\underset{\text{Conj. class.}}{[g] \subset G.}} H^*(M^g/C(g))$ 係数 $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

grading:

$H^i_{orb}(Z) := \bigoplus_{[g] \subset G} H^{i-2a(g)}(M^g)^{C(g)}$

(3) $H^{p,q}_{orb}(Z) := \bigoplus_{[g] \subset G} H^{p-a(g), q-a(g)}(M^g)^{C(g)}$

(4) $H^*_{orb}(Z) = H^{even}_{orb}(Z) \oplus H^{odd}_{orb}(Z)$

Example A: Abelian surface.

$$Y := K^2(A) \quad \text{4-dim gen. Kummer variety}$$

$$N := \{(x,y,z) \in A^3 \mid x+y+z=0\} \curvearrowleft \underset{\underset{G}{\cup\!\!\shortmid}}{\mathfrak{S}_3}$$

置換

$$\begin{tikzcd}
Y \arrow[dr, "\text{Crepant resolution}"'] & & N \arrow[dl] \arrow[loop right, "G"] & & \hat{N} \arrow[dr] \arrow[loop right, "G \text{ (dual action)}"] & \\
& N/G & & & & \hat{N}/G.
\end{tikzcd}$$

$$\hat{N} := \{(\hat{x}, \hat{y}, \hat{z}) \in \hat{A}^3 \mid \hat{x} + \hat{y} + \hat{z} = 0\}$$

$$X := [N/G], \quad \hat{X} := [\hat{N}/G]$$

(上のG)

Prop. 自然な $\mathbb{C}$-linear bijection

$$\psi : H^{\text{even}}_{\text{orb}}(X, \mathbb{C}) \xrightarrow{\sim} H^{\text{even}}_{\text{orb}}(\hat{X}, \mathbb{C})$$

が存在して. ψ は.

$$\psi^{p,q} : H^{p,q}_{\text{orb}}(X, \mathbb{C}) \xrightarrow{\sim} H^{4-q, 4-p}_{\text{orb}}(\hat{X}, \mathbb{C})$$

を induce する.

$$\underset{\underset{\langle \underset{3}{\tau}, \underset{2}{\sigma} \rangle}{\shortparallel}}{G} \curvearrowright N \simeq A^2$$

$$(x,y,z) \mapsto (x,y)$$

$$\tau : (x,y) \mapsto (y, -x-y)$$

$$\sigma : (x,y) \mapsto (y, x)$$

$$G \curvearrowright \hat{N} \simeq \hat{A}^2$$

$$(\hat{x}, \hat{y}, \hat{z}) \mapsto (\hat{x}, \hat{y})$$

$$\tau : (\hat{x}, \hat{y}) \mapsto (-\hat{x}+\hat{y}, -\hat{x})$$

$$\sigma : (\hat{x}, \hat{y}) \mapsto (\hat{y}, \hat{x})$$

ψの構成

K-group of coherent G sheaf

topological -equi. K-group

$$\begin{tikzcd}
D^G(N) \arrow[r] \arrow[d, "\wr"] & K^G(N) \arrow[r, "\text{forgetful map}"'] \arrow[d, "\wr"] & K_G(N)\otimes\mathbb{C} \arrow[r, "\sim"] \arrow[d, "\wr"] & H^{even}_{orb}(X,\mathbb{C}) \arrow[d, "\psi"] \\
D^G(\hat{N}) \arrow[r] & K^G(\hat{N}) \arrow[r, "\text{forgetful map}"'] & K_G(\hat{N})\otimes\mathbb{C} \arrow[r, "\sim"] & H^{even}_{orb}(\hat{X},\mathbb{C})
\end{tikzcd}$$

$$\begin{tikzcd}
\alpha : K_G(N)\otimes\mathbb{C} \arrow[r, "\phi", "\sim"'] & \bigoplus_{[g]\subset G}[K(N^g)\otimes\mathbb{C}]^{C(g)} \arrow[r, "ch", "\sim"'] & \bigoplus_{[g]\subset G}[H^{even}(N^g)\otimes\mathbb{C}] \arrow[d, equal] \\
 & & H^{even}_{orb}(X,\mathbb{C})
\end{tikzcd}$$

φの構成 (Baum-Connes, Atiyah-Segal)

ϕ :

$E \in K_G(N)$

$\langle g\rangle \curvearrowright E|_{N^g} = \underbrace{E^1 \oplus \cdots \oplus E^l}_{分解}$ E^iの eigenvalue λ_i

$$\phi(E)_{[g]} := \sum_{i=1}^{l} \lambda_i [E_i] \in [K(N^g)\otimes\mathbb{C}]^{C(g)}$$

No.

Date '02 · 9 · 18

「FM partners of a K3 surface of Picard number 1」 小木曽 啓示 (東大)

(joint $\underset{w}{\wedge}$ with S. Hosono. B. Lian, S. T. Yau)

proj. smooth / $\mathbb{C}$

$$\{\text{proj. K3}\} \underset{\text{dense}}{\supset} \{\text{proj K3 } \rho = 1\} = \coprod \left\{ \begin{array}{l} \text{proj K3} \\ \rho = 1, \ \ \underset{(H^2)}{\overset{}{\underset{\shortparallel}{\deg}}} = 2n \end{array} \right\}$$

$$\text{Pic}\, S = NS(S) = \mathbb{Z}H$$

K3 : canonical には偏極をえらべない.

$\rho = 1$: Canonical (自動的に) きまる.

<u>定義</u> $FM(X) = \{ Y \mid D(Y) \underset{\text{equiv}}{\simeq} D(X) \} / \text{isom}$

(Fourier Mukai partners of X)

triangulated categories

$\mathcal{O}_X$ - module

$$\text{ob} D(X) \ni \begin{array}{c} \mathcal{O}_x \\ \wr\wr \\ \mathbb{C} \end{array}, \ x \in X$$

recover できれば ⇒ X が recover できる

<u>定理</u> (Bondal - Orlov)

K_X : ample 又は − ample

⇒ ($\mathcal{O}_x$ ($x \in X$) が復元できて) $Y \cong X$

⇓

$FM(X) = \{X\}$

$X : K3$ もう $\mathcal{O}_x$ を一般には復元しない.

$K_X = 0$

<u>例</u> $X : K3$ $\quad \overset{\overline{M_H}(v)}{\overset{\cup}{\underset{\substack{\downarrow \\ \text{ample}}}{M_H}(v)}} = \left\{ \begin{array}{l} \text{moduli space of } \overset{\swarrow \text{ w.r.t } H}{\overbrace{\text{stable}}} \text{ sheaves } F \\ \text{on } X \quad \text{s.t} \quad v(F) = v \end{array} \right\}$

$$v(F) = ch(F) \cdot \underset{\substack{\| \\ 1+w \\ w \in H^4(X, \mathbb{Z}) \\ \text{foundamental} \\ \text{class}}}{\underbrace{\sqrt{td_X}}} \in \hat{H}(X, \mathbb{Z}) = H^0 \oplus \underset{\substack{\cup \\ NS(X)}}{H^2} \oplus H^4$$

$$\hat{H}(X,\mathbb{Z}) \supset H^0 \oplus NS(X) \oplus H^4$$

// Mukai vector

$$(\underset{\substack{\| \\ rk}}{r}, \underset{\substack{\| \\ c_1}}{l}, \underset{\substack{\| \\ \frac{c_1^2}{2} - c_2 + r}}{s})$$

<u>仮定</u> $M_H(v)$ fine, compact

$$\text{fine} \nwarrow (r, s) = 1 \quad \text{compact} \uparrow (r, s) = 1$$

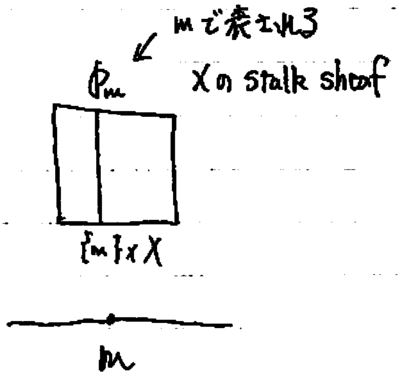
H : ample 2-dim

Mukai

$\Leftarrow \ (H^2) = 2rs \ , \ r \geq 1$

仮定 ⇒ $\mathcal{P}$: universal sheaf

$$\begin{tikzcd} & \mathcal{P} \arrow[d] & \\ & M\times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ M & & X \end{tikzcd}$$



$M = M_H(v)$ (⇒ K3)

$$\Phi^{\mathcal{P}}_{M\to X} : D(M) \longrightarrow D(X)$$

$$a \longmapsto \mathbb{R}q_*(\mathcal{P} \overset{\mathbb{L}}{\otimes} \mathbb{L}p^* x)$$

仮定の下で equivalence

$$\mathcal{O}_m \longmapsto \underbrace{\mathcal{P}_m}$$

点の str. sheaf とは程遠い.

<u>用語</u> X : K3

$(\hat{H}(X,\mathbb{Z}), \langle\ ,\ \rangle) \simeq U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ $\quad \langle (r, l, s), (r', l', s') \rangle$

Mukai lattice $\quad = (l, l') - rs' - r's = \langle v(F), v(G) \rangle$

$\cup$

$\widehat{NS}(X) \quad \underset{\text{R.R}}{=} \chi(F, G) = \sum (-1)^i \dim \mathrm{Ext}^i(F, G)$

$\underline{T(X)} = \widehat{NS}(X)^{\perp}$ in $\hat{H}(X, \mathbb{Z})$

$= NS(X)^{\perp}$ in $H^2(X, \mathbb{Z})$

trans lattice

ω_X : X 上の 2-form

$T(X)$: $T(X) \otimes \mathbb{C} \supset \mathbb{C}\omega_X$ となる $H^2(X, \mathbb{Z})$ の最小の prim. lattice

<u>基本定理</u> (Mukai Orlov - (Bridgeland - Maciocia))

Bridgeland

I X : k3 $Y \in FM(X) \Rightarrow Y$: k3

II X, Y : k3 次は同値

(1) $Y \in FM(X)$ ⋯ Categorical

(2) $(T(Y), \mathbb{C}\omega_Y) \underset{\exists H \cdot I}{\simeq} (T(X), \mathbb{C}\omega_X)$ ⋯ Arithmetical

(3) $(\hat{H}(Y, \mathbb{Z}), \mathbb{C}\omega_Y) \simeq (\hat{H}(X, \mathbb{Z}), \mathbb{C}\omega_X)$ ⋯ geometrical

(4) $Y \simeq {}^{\exists}$(2-dim, fine, compact moduli of stable sheaves on X w.r.t ${}^{\exists}$pol.)

(4)に関して

定理 (Mukai)

(1) $M_H(v)$: smooth

(2) $v = (r, \ell, s)$ $\langle v^2 \rangle = 0$ ($\Leftrightarrow$ $(H^2) = 2rs$)
$r \geq 1$
$(r, s) = 1$
H : ample
$\Bigg\}$ $\Rightarrow M_H(v)$ (4)の右辺の条件をみたす.

系 (2) 逆も成り立つ : ie $Y \in FM(X) \Rightarrow Y \cong M_{\underset{\downarrow}{H}}(v).\ {}^{\exists}\underset{H}{v}$ in (2)

primitive にとれる (後で)

Outline

I) derived category の equiv : Serre functor と可換

$$\begin{tikzcd}
D(X) \arrow[r, "\sim", "\Phi"'] \arrow[d, "S_X", "\otimes \overset{\mathcal{O}_X}{\overset{\parallel}{\overbrace{\mathcal{O}_X(k_X)}}}[2] = {[2]}"'] & D(Y) \arrow[d, "S_Y \ \otimes \mathcal{O}_Y(k_Y)[\dim Y]"] \\
D(X) \arrow[r, dash, "\sim", "\Phi"'] & D(Y) \arrow[l, bend left, "\Phi"]
\end{tikzcd}$$

$\Phi^{-1} = \Psi$: q.inv

$\otimes \mathcal{O}_Y(k_Y)[\dim Y]$
$= \Phi \circ [2] \circ \Psi = [2]$

$\Rightarrow \begin{cases} \dim Y = 2 \\ \mathcal{O}_Y(k_Y) \cong \mathcal{O}_Y \\ b_1(Y) = b_1(X) = 0 \end{cases}$

__Orlov__ $\Phi = \Phi^{\exists \mathcal{E} \in D(X\times Y)}_{X\to Y}$

$$\Downarrow$$

$$\begin{tikzcd}
D(X) \arrow[r, "\Phi_{X\to Y}"] \arrow[d, "ch"'] \arrow[dr, phantom, "\curvearrowright"] & D(Y) \arrow[d, "ch"] \\
\hat{H}(X, \mathbb{Q}) \arrow[r, "\sim"] & \hat{H}(X, \mathbb{Q})
\end{tikzcd}$$

$f^{ch(\mathcal{E})}$

odd、even 保つ.

II) $(2) \Rightarrow (3)$: Nikulin's prim. emb. Thm.

$\hat{H}$ と T の rank は 十分はなれている.

$\rightarrow T \hookrightarrow \hat{H}$ $\overset{\underline{\oplus U}}{\text{prim.}}$ unique up to $O(\hat{H})$

$(3) \Rightarrow (2)$ 自明

$(1) \Rightarrow \begin{matrix}(2)\\(3)\end{matrix} \Rightarrow (4) \Rightarrow (1)$

__$(1) \Rightarrow (3)$__ $\Phi : D(\overset{Y}{\cancel{X}}) \to D(\overset{X}{\cancel{Y}})$ equiv

Orlov $\Phi = \Phi^{\mathcal{E}}_{\underset{Y}{\cancel{X}} \to \underset{X}{\cancel{Y}}}$ $a \longmapsto \mathbb{R}\pi_{X*}(\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi_Y^* a)$

$^{\exists}\mathcal{E} \in D(X \times Y)$.

$$\begin{tikzcd}
D(Y) \arrow[r, "\Phi^{\mathcal{E}}_{Y\to X}"] \arrow[d, "\mathrm{ch}(\cdot)\sqrt{\mathrm{td}_X}"'] & D(X) \arrow[d, "\mathrm{ch}(\cdot)\sqrt{\mathrm{td}_X}"] \\
\hat{H}(Y, \underset{\sim}{\mathbb{Z}}) \arrow[r, "f^{\mathcal{E}}_{Y\to X}"'] \arrow[d, leftrightarrow] & \hat{H}(X, \mathbb{Z}) \\
\text{K3 : even} &
\end{tikzcd}$$

$$f^{\mathcal{E}}_{Y\to X}(y) = \pi_{X*}\big(\pi_Y^*(y)\underbrace{\pi_Y^*\sqrt{\mathrm{td}_Y}\,\mathrm{ch}(\mathcal{E})\,\pi_X^*\sqrt{\mathrm{td}_X}}_{\text{alg} \Rightarrow \text{Hodge}}\big)$$

(可換 (← G.R.R)
isometry ↙

逆 $f^{\mathcal{E}^\vee}_{X\to Y}$ でえられる.

(3) ⇒ (4)

$\varphi : \hat{H}(Y, \mathbb{Z}) \xrightarrow{\sim} \hat{H}(X, \mathbb{Z})$ H.I — ②

$(0, 0, 1) \longmapsto \boxed{(r, l, s)}$ $l \in NS(X)$

Mukai 先生の定理の条件をみたす.

$\langle \cdot^2 \rangle = 0 \longrightarrow (l^2) = 2rs$

· $\mathrm{ch}(\underline{mH})$: H.I も合成 $\begin{cases} r \geq 1 \\ l : \text{ample} \end{cases}$

$(r, l, s) \longleftarrow\!\!\shortmid (s, l, r)$ $\underline{(r, s) = 1}$

$(-1, 0, 0) \longmapsto (r', l', s')$

このl'を用いて ch(nl')を考えて

r, s, (l, l') 互いに素. (r, s) = 1 にできる.

$\hat{H}(M_\ell((r.\ell.s))) \xrightarrow{H.I} \hat{H}(\qquad)$ — ①

$(0.0.1) \longmapsto (r.\ell.s)$

②

$$(0.0.1)^\perp / \mathbb{Z}(0.0.1) \underset{H.I}{\simeq} (r.\ell.s)^\perp / \mathbb{Z}(r.\ell.s) \underset{H.I}{\simeq} (\qquad)$$

$\parallel$

$$H^2(Y, \mathbb{Z}) \xrightarrow[H.I]{\sim} H^2(M_H(v), \mathbb{Z})$$

$\Downarrow$

$Y \simeq M_H(v)$

<u>(4) ⇒ (1)</u>

$$\mathcal{P} \downarrow \underbrace{M_H(v)}_{\parallel \atop M} \times X$$

$$\Phi^{\mathcal{P}}_{M \to X} : D(M) \longrightarrow D(X)$$ equi ⟸ fully faithful $\left(\begin{array}{l} \omega_X \simeq \mathcal{O}_X \\ \omega_M \simeq \mathcal{O}_M \end{array}\right)$

$$\mathcal{O}_m \longmapsto \underbrace{\mathcal{P}_m}_{\text{stable}}$$

$\Downarrow$

Hom^i の間の同型ができる

$\rho=1$　n : 自然数 $\geqq 2$.　$\tau(n) = \#(\text{prime factor of } n)$
$(\tau(4)=1$、$\tau(6) = 2)$

Main Thm

X : K3　$\rho(X) = 1$,　$\deg X = 2n$　$\left(\Leftrightarrow NS(X) = \mathbb{Z}H,\ (H^2) = 2n\right)$

(1)　$|FM(X)| = 2^{\tau(n)-1}$

$(X \simeq M_H(n.H.2))$

(2)　$$FM(X) \underset{\text{bij}}{=} \left\{ \underbrace{M_H(r,H,s)}_{\text{完全代表系}} \,\middle|\, \underbrace{\boxed{(r,s)=1,\ r\geqq s,\ n=rs}}_{(*)} \right\}$$

注)　・Pado Stellari も (2) をえている. ((1)も用いて)

・(*)をみたす (r,s) の個数 $= 2^{\tau(n)-1}$

(1)がいえれば

(2) $\Leftarrow$ Prop
$$\left[\begin{array}{l} (*)\text{ をみたす } \overset{r\leq s\text{ でもよい}}{(r,s),\ (r',s')} \\ M_H(r,H,s) \simeq M_H(r',H',s') \\ \Rightarrow r=r' \text{ or } r=s' \end{array}\right.$$

(1) ⇔ (4) も使って便に.

(3) にもちこむ　(⇒ Torelli)

⇒ 算数の問題 (Lemma)

問題 (Lemma) r, s, l, m, k : 正の整数.

$rs = n$, $lm = nk^2$ $-mr - ls + 2nk = 1$

$r's' = n$, $l'm' = nk'^2$ $-m'r' - l's' + 2nk' = 1$

このとき.

(1) $r'm' - l's' \equiv rm - ls \pmod{2n} \Rightarrow r' = r$

(2) $r'm' - l's' \equiv -(rm - ls) \pmod{2n} \Rightarrow s' = r$

(1) について:

Thm (S. Hosono, B. Lian, S.T. Yau).

X : K3 ($\rho(X)$: 何でもよい)

$$|FM(X)| = \sum_{i=1}^{m} \left| O(S_i) \backslash O(A_{S_i}) / O(T(X), \mathbb{C}\omega_X) \right|$$

$$\underset{\text{genus}}{g(\underline{NS(X)})} = \underbrace{\{S_1, \cdots, S_m\}}_{\text{☆をみたす同型類.}} \quad S_i, NS(X) \text{ locally isom}$$

$\otimes \mathbb{Z}_p$
$\otimes \mathbb{R}$
で同型

$$\Leftrightarrow \begin{cases} \mathrm{sgn}(S_i) = \mathrm{sgn}\, NS(X) \\ ☆\ (A_{S_i}, q_{S_i}) \simeq (A_{NS(X)}, q_{NS(X)}) \end{cases}$$

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$A_S = S^{*}/\overset{\swarrow\,\text{even}}{\textcircled{s}}$: アーベル群.

$$\begin{tikzcd} q_S \arrow[d] & : & A_S \arrow[r] & \mathbb{Q}/2\mathbb{Z} \\ O(A_S) & & x \bmod S \arrow[r, mapsto] & x^2 \bmod 2\mathbb{Z} \end{tikzcd}$$

$$\begin{tikzcd} (A_{S_i}, q_{S_i}) \arrow[r, phantom, "\simeq"] & (A_{NS(X)}, q_{NS(X)}) \arrow[r, phantom, "\simeq", "\uparrow\,\text{自然}"'] & (A_{T(X)}, -q_{T(X)}) \\ O(\quad) & & \\ O(S_i) \arrow[u, "\text{自然}"] & & O_{\mathrm{Hodge}}(T(X), \mathbb{C}\omega_X) \arrow[uu] \arrow[uull, bend left] \end{tikzcd}$$

$$O(S_i)\backslash O(A_{S_i}) / O_{\mathrm{Hodge}}(T(X), \mathbb{C}\omega_X)$$

$T(X) \simeq T(Y)$ but $H^2(X,\mathbb{Z}) \underset{\text{H.I}}{\not\simeq} H^2(Y,\mathbb{Z})$

① $NS(X) = T(X)^{\perp} \not\simeq NS(Y)$ ⋯ genus

② $NS(X) \simeq NS(Y)$ だけど

$\underset{\substack{\cup\ \text{index} \\ NS(X)\oplus T(X)}}{H^2(X,\mathbb{Z})} \simeq H^2(Y,\mathbb{Z})$ にのびない

↑

各 orbit space

$\underline{\rho = 1}$ $\quad g(\underbrace{NS(X)}_{\underset{\mathbb{Z}H}{\parallel}}) = \{\mathbb{Z}H\}$

$$|FM(X)| = \Big| \underset{\underset{\{\pm id\}}{\parallel}}{O(NS(X))} \backslash O(A_{NS(X)}) / \underset{\underset{\{\pm id\}}{\parallel}}{O_{Hodge}} \Big|$$

Rem (Cor)

- $\rho(X) \geq 12 \Rightarrow FM(X) = \{X\}$
- $\rho(X) \geq 3$ & $\det NS(X)$: square free $\Rightarrow FM(X) = \{X\}$
- $\rho(X) = 2$ $\quad \det NS(X) = -\underset{\text{素数} (\equiv 1 \ (4))}{\underbrace{p}}$

$$\Rightarrow |FM(X)| = \frac{h(p)+1}{2}$$

$h(p)$ = class # of $\mathbb{Q}(\sqrt{p})$

$2^{\tau(n)-1}$ について:

$(\langle 2n\rangle^{\perp}$ in $\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}) \simeq \langle -2n\rangle \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$

$\langle 2n \rangle = \langle h \rangle$, $h \in \Lambda_{K3}$

$\Lambda_{2n} = \langle 2n\rangle^{\perp}$

$\mathcal{D}_n := \{ [\omega] \in \mathbb{P}(\Lambda_{2n} \otimes \mathbb{C}) \mid (\omega.\omega)=0,\ (\omega.\bar{\omega})>0 \}$ period dom of g of K3 pol of deg 2n

$\cup$ dense

$\mathcal{D}^1_n$ $\rho=1$

$\tilde{\Gamma}_n = (O(\Lambda_n) \to \mathrm{Aut}(\mathcal{D}_n))$

normal $\overrightarrow{\nabla}$ Im

$\Gamma_n = \{ g \in \tilde{\Gamma}_n \mid \exists f \in O(\Lambda_K)\ \ f(h)=h,\ \iota(f)=g \}$

arithmetic $\curvearrowright \mathcal{D}_n$ sym. ~~bdded~~ bdded dom of type IV

$$\begin{tikzcd}
\mathcal{D}_n/\Gamma_n \arrow[r] & \mathcal{D}_n/\tilde{\Gamma}_n & \tilde{\Gamma}_n/\Gamma_n \simeq \mathbb{Z}/2. \\
\mathcal{D}^2_n/\Gamma_n \arrow[u, phantom, "\text{dense}\ \cup"] \arrow[r] & \mathcal{D}^1_n/\tilde{\Gamma}_n & \\
X \arrow[u, phantom, "\in"] \arrow[r, mapsto] & FM(X) & \\
\text{moduli of K3},\ \rho=1,\ \deg=2n & \text{moduli of } FM(X) \text{ of } \rho=1,\ \deg=2n \arrow[uuur, bend right] &
\end{tikzcd}$$

Geometry of exceptional holonomy as a generalization of conifold singularity and geometric transition in M theory

Date '02 · 9 · 19

菅野 浩明 (名古屋大学)

PLAN
1. Supersymmetric (SUSY) compactification and special holonomy
2. Dynamics of string duality and conical singularity
   Ricci flat cone geometry
3. Conifold (ordinary double point) of CY_3
   Chern-Simons / top. closed string 対応
   Gromov · Witten
4. M-theory (G_2-lift) of conifold
   (Atiyah-Maldacena-Vafa, hep-th/0011256
   "M theory flop"
5. Spin(7) geometry, conifold like transition
   (work w/ Y. Yasui)
   Gibbons et al. Cvetic…

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1. SUSY and special holonomy.

$$X = \underset{\substack{\text{d-dim} \\ \text{Lorentz sp}}}{\underset{\nearrow}{\mathbb{R}^{d-1,1}}} \times \underset{\substack{\text{"gt" Riemannian} \\ \text{manifold.}}}{\underset{\uparrow}{M^n}}$$

$$\exists \text{SUSY on } \mathbb{R}^{d-1,1} \Leftarrow \text{Covariantly constant } \overset{\eta}{\text{spinor}} \text{ on } M^n$$

$$\Downarrow$$

$$(M^n : \text{Ricci flat})$$

Dim.	Holonomy	# of cov. const. spinors
4k+2	SU(2k+1)	(1,1)
4k	SU(2k)	(2,0)
4k	Sp(k)	(k+1,0)
7	G_2	1
8	Spin(7)	(1,0)

7, 8 → "N=1" SUSY

$\eta \Rightarrow$ calibration form φ $(d\varphi=0)$

$M \supset^{\forall} \Sigma$ cycle $\qquad \Updownarrow$ def

$$\text{"Volume}(\Sigma)\text{"} \geq \left|\text{"period}(\Sigma)\text{"}\right| \nwarrow \int_\Sigma \varphi$$

$$\underset{\text{mass}}{\parallel} \qquad \Updownarrow$$

$$\geq \ |\text{central charge}|$$

BPS bound.

等号成立 : Σ calibrated submanifold (BPS state)

(SUSY cycle) monopole, instanton

holomorphic cycle

special Lagrangian submanifold

2. Dynamics and singularity

Interesting dynamics $\longleftarrow$ $\exists$ massless BPS stats

$\Updownarrow$

collapsing SUSY cycle

$$\mathrm{vol}(\Sigma) = \int_\Sigma \varphi \longrightarrow 0$$

- S^1 cycle in T^2 $\longrightarrow$ massless monopole in Seiberg-Witten
- ADE type singularity in K3 $\longrightarrow$ gauge symmetry enhancement $U(1)^N \longrightarrow U(N)$
- Conifold in CY_3 $\longrightarrow$ massless Black Hole

"localization principle"

Ricci flat geometry of $\mathrm{vol}(\Sigma) \longrightarrow 0$

is well-approximated by the geometry of $N(\Sigma)$

$\uparrow$ normal bundle in M^n

$\mathrm{Vol}(\Sigma)=0 \Leftrightarrow$ a cone geometry

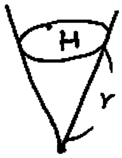
H for "horizon"

C(H) cone over H

conical singularity

C(H) Ricci flat $\Leftrightarrow$ H: $\overset{\text{cpt}}{\text{Einstein}}$.

Ricci-flat cone geometry



$$ds^2 = dr^2 + r^2\, d\Omega_H^2$$

$$R_{ab} = (m-1)\, g_{ab}.$$

dim	holonomy	"horizon" geometry H
n	trivial (flat)	S^{n-1}
$4k$	$Sp(k)$	3-Sasakian (→ Einstein)
$2k$	$SU(k)$	Einstein-Sasaki
7	G_2	nearly Kähler (weak $SU(3)$)
8	$Spin(7)$	weak G_2

weak $SU(3)$

on H $\left\{ \begin{aligned} d(\mathrm{Im}\,\Omega) &= -2\lambda\,(\omega\wedge\omega) \\ d\omega &= 3\lambda\,(\mathrm{Re}\,\Omega) \end{aligned} \right.$ $(\lambda \neq 0)$

on C(H) $\Phi = dr\wedge\omega + \mathrm{Re}\,\Omega$

$*\Phi = dr\wedge \mathrm{Im}\,\Omega - \frac{1}{2}\,\omega\wedge\omega$

$d\Phi = d(*\Phi) = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial r}(\mathrm{Re}\,\Omega) = 3\lambda\,(\mathrm{Re}\,\Omega)$

$\frac{\partial}{\partial r}(\omega\wedge\omega) = 4\lambda\,(\omega\wedge\omega)$

weak G_2

on H $d\Phi = \lambda *\Phi$ $(\lambda \neq 0)$

on C(H) $\Omega := dr\wedge\Phi + *\Phi.$

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§§ 3.4 a Review article Large N duality and Transition in Geometry

3. conifold geometry

local eq. $z_1^2+z_2^2+z_3^2+z_4^2=0$ (ordinary double pt)

$\Leftrightarrow$ $xy-uv=0$

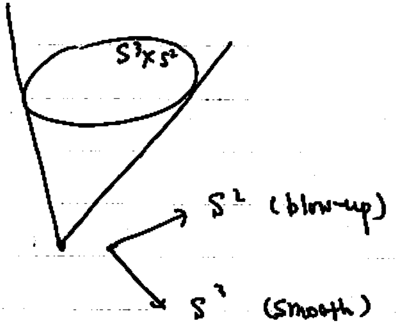
= a cone over $S^3\times S^2 = S^3\times S^3/U(1)$

$z_i = q_i + \sqrt{-1}p_i$

$\Sigma q_i^2 - \Sigma p_i^2 = 0$, $\Sigma p_i q_i = 0$

Intersection with S^7 : $\Sigma p_i^2 + \Sigma q_i^2 = 2$

$\Rightarrow$ top') $\square$ $\begin{cases} \Sigma p_i^2 = \Sigma q_i^2 = 1 \\ \Sigma p_i q_i = 0 \end{cases}$

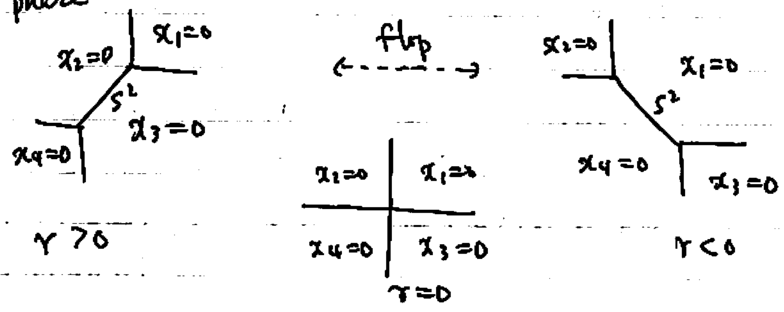


moment map $\mu_r = |x_1|^2 - |x_2|^2 - |x_3|^2 + |x_4|^2 - r$

conifold $= \{\mu_r = 0\}//U(1)$ $z = x_1x_2x_3x_4 = 0$

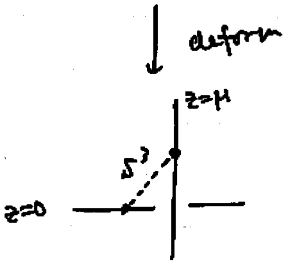
$x = x_1x_2$
$y = x_3x_4$
$u = x_1x_3$
$v = x_2x_4$
$z = x_1x_2x_3x_4$

Higgs phase



$\begin{cases} xy = z \\ uv = z-\mu \end{cases}$

Coulomb phase



$$\begin{tikzcd} & \text{conifold} \arrow[dl, "\text{resol}"'] \arrow[dr, "\text{deform}"] & \\ \mathcal{O}(-1)\oplus\mathcal{O}(-1) \arrow[d] & & T^*S^3 \\ S^2 & & \end{tikzcd}$$

large N dual

Large N-duality

N D6 brane wrapping $S^3 \subset T^*S^3$ $\xrightarrow{\text{top}}$ S^3 上の $U(N)$ Chern-Simons theory

N units of 2-form flux through S^2 of $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to S^2$

$\Big\downarrow$ top.

Gromov-Witten theory of local $\mathbb{P}^1$

4. Large N-duality as M-theory flop (Atiyah-Maldacena-Vafa)

M theory on X^7 (G_2-manifold)

$\Big\downarrow$ $U(1)$-action

IIA string theory on CY_3 (6 dim)

$U(1)$ acts on X as isometry

the following cases can be interpreted as IIA theory.

(1) $U(1)$ acts freely

$X/U(1)$ smooth $\quad$ X: $U(1)$ bundle over B

IIA theory on B with 2-form flux $=$ c_1 of $U(1)$ bundle

(2) $U(1)$ action has a fixed pt set W of codim. 4 in X

$X/U(1) \supset W$ codim. 3 singularity

$W \times \mathbb{R}^{3,1}$: D6 brane の world volume

Conifold geometry $=$ cone over $S^3 \times S^3/U(1)$

$\wr$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $\wr$

- G_2-geometry $=$ cone over $S^3 \times S^3$ (?)

$S^3 \times S^3$ has a <u>nearly</u> Kähler geometry

$\wr$

$C(S^3 \times S^3)$ is G_2 with a singularity

$S^3\times S^3$ $SU(2)\simeq S^3$ left invariant one form

"σ-model" $\longleftarrow$ $\{\sigma^i\}_{i=1}^3$ $\{\Sigma^i\}_{i=1}^3$

$\circ\,\sigma$

$$\omega = \sum_{i=1}^{3} \sigma^i \wedge \Sigma^i \qquad (d\omega \neq 0)$$

Non-compact G_2 metric on $S(S^3) \cong S^3\times\mathbb{R}^4$

$$ds^2 = \alpha^2 dr^2 + r^2(\sigma^i)^2 + \beta^2(\Sigma^i - \tfrac{1}{2}\sigma^i)^2$$

$$\alpha^{-2} = \left(1-\frac{a^3}{r^3}\right) \quad \beta^2 = \frac{r^2}{9}\left(1-\frac{a^3}{r^3}\right) \quad \gamma^2 = \frac{r^2}{12} \quad (r>a)$$

$r \longrightarrow a$ (near horizon geometry)

$$\rho = \frac{r}{3}\left(1-\frac{a^3}{r^3}\right)^{1/2} \longrightarrow 0$$

$$ds^2 \sim 4\underset{\substack{\shortparallel \\ \text{flat } \mathbb{R}^4}}{\left[d\rho^2 + \frac{\rho^2}{4}(\Sigma^i - \tfrac{1}{2}\sigma^i)^2\right]} + \frac{a^2}{12}(\sigma^i)^2$$

vol $(S^3) \sim a^3$

$r \longrightarrow \infty$ (asymptotic geometry)

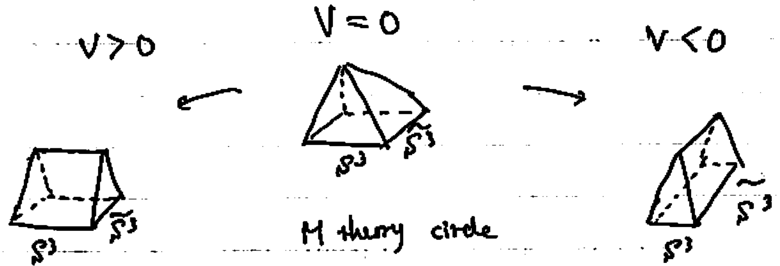
$$ds^2 \sim dr^2 + \frac{r^2}{9}\left((\sigma^i)^2 + (\Sigma^i)^2 - \sigma^i\cdot\Sigma^i\right)$$

a cone metric

a second Einstein metric on $S^3\times\tilde{S}^3$ $SU(2)^3$-Sym.

G_2-lift of conifold transition

$$\underset{S^3}{|z_1|^2 + |z_2|^2} - \underset{\tilde{S}^3}{(|z_3|^2 + |z_4|^2)} = \cancel{0}\,V$$



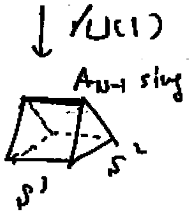
M theory circle

$\downarrow$

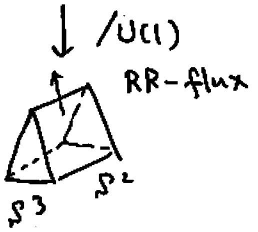
$\mathbb{Z}_N \subset U(1) \subset \tilde{S}^3$

$S^3\times C(\tilde{S}^1)$
$\simeq S^3\times\mathbb{R}^4$

$C(S^3)\times\tilde{S}^3$
$\simeq S\times\tilde{S}^3$

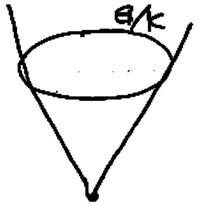


deformed conifold
with D6 branes wrapping S^3



resolved conifold
with N units of
RR-flux through S^2

5. Spin(7) geometry and geometric transition



"resolution" of $C(G/K)$

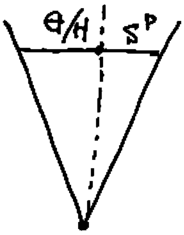
$$G \supset^{\exists} H \supset K$$

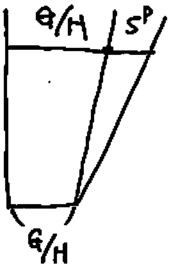
s.t $H/K \simeq S^p$ sphere

$$G/K \simeq G/H \times \underbrace{H/K}_{S^p}$$

$$C(G/K) \sim G/H \times C(S^p) = G/H \times \mathbb{R}^{p+1}$$

G/H 上の (p+1) dim vector bundle





Conifold case $G/K = SU(2) \times SU(2)/U(1)$

1. $H = U(2) \longrightarrow S^2 \times \mathbb{R}^4 \quad \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$
2. $H = SU(2) \longrightarrow S^3 \times \mathbb{R}^3 \quad T^*S^3$
3. $H = U(1) \times U(1) \longrightarrow (S^2 \times S^2) \times \mathbb{R}^2 \quad K_{\mathbb{F}_0}$

Spin(7) ← Cone over G/K weak G_2 structure

$$G/K = \begin{cases} Sp(2)/Sp(1) \\ SU(3)/U(1) \quad ◎ \end{cases}$$

Spin(7) case $G/K = SU(3)/U(1)$

1. $H = S(U(2)\times U(1)) \longrightarrow \mathbb{CP}^2 \times \mathbb{R}^4$ $\quad T^*\mathbb{CP}^2$
2. $H = SU(2) \longrightarrow S^5 \times \mathbb{R}^3$ $\quad (?)$
3. $H = U(1)\times U(1) \longrightarrow \underset{\substack{\shortparallel \\ SU(3)/T^2}}{F(2,1)} \times \mathbb{R}^2$ $\quad K_{F(2,1)}$

$$U(1) \sim \mathrm{diag}\,(e^{in\theta}, e^{im\theta}, e^{-i(n+m)\theta})$$

generic m, n ⇒ 1 のみ Spin(7) metric をもつ

$m=n$ ⇒ 1 と 3 が Spin(7) metric を持つ

$m=-n$ ⇒ 1 と $\underset{\uparrow}{2}^{(?)}$ が Spin(7) metric を持つ

formal power series solution
(Gibbons et al)

「Crepant resolution and the Philosophy of catspace」

中島 徹 (東京都立大学)

- non-commutative geometry

$$\begin{cases} \text{variety} \rightsquigarrow \text{coordinate ring} \\ \text{affine scheme} \leftsquigarrow \text{ring} \end{cases}$$

nonsing. proj. var $X \rightsquigarrow D(X)$: derived category

bounded

"non-commutative space" $\leftsquigarrow$ category

<u>$n \leq 3$</u> $X = \mathbb{C}^n/G$. $G \subset SL(n.\mathbb{C})$: finite

$\pi : \tilde{X} \to X$ crepant resolution

$$\boxed{D(\tilde{X}) \xrightarrow{\sim} D^G(\mathbb{C}^n)}$$

↑

FM

<u>$n \geq 4$</u> $\Rightarrow$ crepant resolution は一般に存在しない.

$\mathbb{C}^4/\mathbb{Z}_2$

Stringy Hodge number $h_{st}^{p.q}(X)$. X : Gorenstein ~~[illegible]~~ sing.

もし $\pi : \tilde{X} \to X$ crepant resolution

$$h_{st}^{p.q}(X) = h_{st}^{p.q}(\tilde{X}) \quad \text{(Batyrev)}$$

X : non-sing proj $/\mathbb{C}$ 、$\dim X = n$

ω_X : canonical bundle

E、F : coherent sheaf on X

$\mathrm{Ext}^i(E, F) \cong \mathrm{Ext}^{n-i}(F, E \otimes \omega_X)^\vee$ (Serre duality)

a、$b \in D(X)$

$\mathrm{Hom}(a, b) \cong \mathrm{Hom}(b, a \overset{\mathbb{L}}{\otimes} \omega_X[n])^\vee$

$\mathcal{A}$: $\mathbb{C}$ - linear triangulated category

$\mathrm{Hom}(a, b)$: $\mathbb{C}$ - vector space

$[1] : \mathcal{A} \to \mathcal{A}$ shift

$a \to b \to c \to a[1]$: distinguished.

$S : \mathcal{A} \to \mathcal{A}$ a Serre functor

$\Leftrightarrow$ ・S is an exact autoequivalence

・$\forall a$、$b \in \mathcal{A}$

$\mathrm{Hom}(a, b) \cong \mathrm{Hom}(b, S(a))^\vee$

例

$$\mathcal{A} = D(X) \Rightarrow S = (\cdot) \overset{\mathbb{L}}{\otimes} \omega_X[n]$$

Lemma $\mathcal{A}$ has a Serre functor ($\mathcal{A}$ is of finite type)

$\Leftrightarrow$ ${}^\forall a \in \mathcal{A}$

$$\begin{tikzcd} F_a : \mathcal{A} \arrow[r] & \mathrm{Vect}_{fd} \\ b \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & F_a(b) = \operatorname{Hom}(\overset{a}{\cancel{b}}, \overset{b}{\cancel{a}})^\vee \end{tikzcd}$$

$$\begin{tikzcd} G_a : \mathcal{A} \arrow[r] & \mathrm{Vect}_{fd} \\ b \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \cancel{F} G_a(b) = \operatorname{Hom}(b, a)^\vee \end{tikzcd}$$

が共に representable

(∵)

$S(a)$: representable objects of F_a $\Rightarrow$ S is Serre functor

$S^{-1}(a)$: 〃 G_a .

$\mathcal{A}$: of finite type

$\underset{\text{def}}{\Leftrightarrow}$ ${}^\forall a, b \in \mathcal{A}$ $\operatorname{Hom}^i(a, b)$ が有限次元の $\mathbb{C}$-v.s

$= 0$ for a.a.i

$h : \mathcal{A} \to \mathrm{Vect}_{fd}$ が cohomology functor

$\underset{\text{def}}{\Leftrightarrow}$ $a \to b \to c \to a[1]$ dist. triangle

$\Rightarrow$ $h(a) \to h(b) \to h(c)$: exact

h : of finite type

$\underset{\text{def}}{\Leftrightarrow}$ ${}^{\forall}a \in \mathcal{A}$, $h(a[-i]) = 0$ for a.a. i

$\mathcal{A}$: of finite type が catspace

$\Leftrightarrow$ ${}^{\forall}h : \mathcal{A} \to \mathrm{Vect}_{fd}$ cohomology functor of finite type
(contravariant
or covariant)

$\underline{h \text{ が representable}}$
↑
このとき $\mathcal{A}$ が saturated

<u>例</u>

- $D(\mathrm{Spec}\,\mathbb{C}) = D(\mathrm{Vect}_{fd})$ は catspace

- X : nonsing proj. variety $/\mathbb{C}$
  $D(X)$ は catspace

$\Big\}$ Bondal-Kapranov の Thm (1989)

- Λ : finite dim $\mathbb{C}$-algebra, finite homological dim

  $D(\Lambda\text{-mod})$(derived category of right Λ-module)

  は catspace

X : singular proj. var with rational sing.

$D(X)$: derived category

$f : Y \to X$: resolution

$f^* : D(X) \not\to D(Y)$

$D_{perf}(X) = \{ a \in D(X) \mid$ a is locally quasi-isomorphic to a finite complex of vecter bundle of finite rank $\}$

Lemma X : rat. sing

$f : Y \to X$: resolution

$f^* : D_{perf}(X) \longrightarrow D(Y)$ は fully faithful

No.

Date . .

$\mathcal{A}$: catspace が X の catspace resolution

($\mathcal{A} = \mathrm{Res}(X)$ と書く)

$\underset{\text{def}}{\Longleftrightarrow}$ $^{\exists} f : Y \to X$ resolution

$^{\exists}\mathcal{B} \subset D(Y)$: full subcategory

$\cup$

$D_{\text{perf}}(X)$

s.t $\mathcal{A} \underset{\text{equiv}}{\cong} \mathcal{B}$

Rem $D_{\text{perf}}(X)$ は of finite type だが

一般には saturated でない.

$\mathrm{Res}(X)$ は $D_{\text{perf}}(X)$ の "saturation" と見なせる.

$\omega_X \cong \mathcal{O}_X$ の場合に "crepant" catspace resolution

を作りたい

このとき $\mathrm{Res}(X)$ が crepant

$\underset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Res}(X)$ が Calabi-Yau catspace

$\mathcal{A}$: catspace が CY

$\underset{\text{def}}{\Longleftrightarrow}$ $\mathcal{A}$ has <u>trivial</u> Serre functor

ie $S \cong [m]$ $(m \in \mathbb{Z})$

X : Gorenstein canonical sing. $\omega_X \cong \mathcal{O}_X$

$\mathrm{Res}(X)$ が X の crepant $\underset{\text{catspace}}{\wedge}$ resolution

$\underset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Res}(X)$ is a CY catspace

$$\overset{\dim X = n+1}{}$$

$$\begin{array}{l} \pi : Y \to X \quad \text{resolution of discrepancy } a. \\ \ \cup \\ \ E : \text{unique exceptional divor } E \cong \mathbb{P}^n \end{array}$$

$\omega_Y = aE$

$N_{E/Y} = \mathcal{O}_{\mathbb{P}^n}(-1) \Rightarrow (n+1) = d(\underset{a}{\cancel{d}}+1)$

by canonical bundle formula

$$\mathcal{O}_{\mathbb{P}^n}(m) \in D(E)$$

$$\iota : \overset{\mathbb{P}^n}{\overset{\shortparallel}{E}} \hookrightarrow Y$$

$$\mathcal{B} = \langle \iota_* \mathcal{O}_{\mathbb{P}^n}(-n), \cdots, \iota_* \mathcal{O}_{\mathbb{P}^n}(-d) \rangle \subset D(Y)$$

↑ generator subcategory

$$^{\perp}\mathcal{B} = \{ a \in D(Y) \mid \mathrm{Hom}_{D(Y)}(a, b) = 0 \ \ {}^{\forall} b \in \mathcal{B} \}$$

↑ left orthogonal of $\mathcal{B}$

- $\mathrm{Res}(X) := {}^{\perp}\mathcal{B}$ は X の crepant catspace resolution
- $\mathrm{Res}(X)$ は catspace resolution.

$\mathcal{A}$: catspace

$E_0, \cdots, E_n \in \mathcal{A}$ が exceptional collection

$\underset{\text{def}}{\Longleftrightarrow}$ (1) ${}^{\forall}E_i$ が exceptional object

ie
$$\mathrm{Hom}^j(E_i, E_i) = \begin{cases} \mathbb{C} & j = 0 \\ 0 & j \neq 0 \end{cases}$$

2) $\mathrm{Hom}^h(E_i, E_j) = 0 \quad {}^{\forall} h \in \mathbb{Z}, \ {}^{\forall} i > j$

このとき

$\langle E_0, E_1, \cdots, E_n \rangle$ は catspace になる.

$\iota_* \mathcal{O}_{\mathbb{P}^n}(-n), \cdots, \iota_* \mathcal{O}_{\mathbb{P}^n}(-d)$ は.

$D(Y)$ の exceptional collection

$\therefore$ $\mathcal{B}$ は catspace

<u>Lemma</u> $\mathcal{A}$: catspace

$\mathcal{B} \subset \mathcal{A}$: subcatspace

$\Rightarrow$ ${}^{\perp}\mathcal{B}$ 〃

$\mathcal{B}^{\perp}$ 〃

$\therefore$ $\mathrm{Res}(X)$ は catspace

$H^i(\mathcal{O}_{\mathbb{P}^n}(-n)) = 0 \quad \forall i \in \mathbb{Z}$

$\vdots$

$H^i(\mathcal{O}_{\mathbb{P}^n}(-d)) = 0$ 〃 .

$a \in D_{\mathrm{perf}}(X)$

$$\Rightarrow \mathrm{Hom}(f^* a, \iota_* \mathcal{O}_{\mathbb{P}^n}(-m)) = \mathrm{Hom}(a, \underset{\overset{\shortparallel}{0}}{\underline{f_* \iota_* \mathcal{O}(-m)}}) = 0$$

$\therefore$ $D_{\mathrm{per}}(X) \subset {}^{\perp}\mathcal{B} = \mathrm{Res}(X)$

$i : \mathrm{Res}(X) \hookrightarrow D(Y)$ inclusion

このとき i は right adjoint p をもつ

$p : D(Y) \to \mathrm{Res}(X)$

$S_{\mathrm{Res}(X)}$: $\mathrm{Res}(X)$ の Serre functor

一方 $S_{\mathrm{Res}(X)} = p \circ S_{D(Y)} \circ i$, $S_{D(Y)} = (\cdot) \overset{\mathbb{L}}{\otimes} \omega_Y[n+1]$

Lemma $\forall a \in D(Y)$, $p(a \otimes \omega_Y) = p(a)$

$\Rightarrow$ $b \in \mathrm{Res}(X)$

$$S_{\mathrm{Res}(X)}(b) = p(S_{D(Y)}(b)) = p(b \otimes \omega_Y[n+1])$$

$$= p(b \otimes \omega_Y)[n+1] = p(b)[n+1]$$

$$= b[n+1] \to S_{\mathrm{Res}(X)} \cong^{\exists} [n+1]$$

Veronese Cone

$\mathcal{O}(d) : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ $\mathrm{Im} = V_d$

X : cone over V_d

$$\begin{tikzcd} Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(d)) \arrow[r] & X \\ E \simeq \mathbb{P}^n \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

- ODP

$Q_n \hookrightarrow \mathbb{P}^N$

X : cone over Q_n

$$\begin{tikzcd} Y \arrow[r] & X \\ E \simeq Q_n \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

- X : cone over Fano homogenous spin

$\Rightarrow$ $^{\exists}\mathrm{Res}(X)$: crepant catspace resolution. ?

$$D(\mathbb{P}^n) = \langle \underbrace{\mathcal{O}(-n),\ \mathcal{O}(-n+1),}_{\mathcal{B}} \underbrace{\cdots,\ \mathcal{O}_{\mathbb{P}^n}}_{\mathcal{C}} \rangle$$

$$\mathcal{C} = \langle \mathcal{O}(-d+1), \cdots, \mathcal{O}_{\mathbb{P}^n} \rangle \subset D(\mathbb{P}^n)$$

$$D(\mathbb{P}^n) = \langle \mathcal{C},\ N\otimes\mathcal{C},\ N^2\otimes\mathcal{C}, \cdots, N^q\otimes\mathcal{C} \rangle$$

$$\shortparallel$$

$$D(E) = (1+N+N^2+\cdots+N^q)\mathcal{C}, \quad N = \mathcal{O}_{\mathbb{P}^n}(-d)$$

$$\mathcal{C} = \frac{D(E)}{1+N+N^2+\cdots+N^q}$$

$$\mathrm{Res}(X) \doteqdot D(Y\setminus E) + \mathcal{C}$$

$$= D(Y\setminus E) + \frac{D(E)}{1+N+\cdots+N^q}$$

E_{st} : stringy E function $= \sum h^{p,q}_{st}(X)\, u^p v^q$

$$E_{s.t}(X) = E(Y\setminus E) + \frac{E(E)}{1+(uv)+\cdots+(uv)^q}$$

問：$E_{s.t}$, $h^{p,q}_{s.t}$ を $\mathrm{Res}(X)$ から定義することが可能か？

On Fano indices of $\mathbb{Q}$-Fano 3-folds.

No.

Date '02 . 9 . 19

鈴木香織(東大)

X: $\mathbb{Q}$-Fano 3-folds. $\rho(X) = 1$.

$\left\{ \begin{array}{l} \text{at most term. sing.} \\ \mathbb{Q}\text{-Factorial} \\ -K_X : \text{ample} \end{array} \right.$

[2つの index]

① singular index r $\overset{d}{\Longleftrightarrow}$ rK_X が Cartier になる. $r \in \mathbb{Z}_{>0}$

② Fano index f $\overset{d}{\Longleftrightarrow}$ $^{\exists} D \in \mathrm{Div}X$ $-K_X \sim fD$ $f \in \mathbb{Z}_{>0}$

History

1980 Iskovskikh
smooth case

1993 Borisov - Borisov
toric $\mathbb{Q}$-Fano n-folds

1996 Sano
$f \geqq r$ のとき 8 type
"$f \geqq d-2$ のときの $\mathbb{Q}$-Fano d-fold について"
$f < r$ のときを考える.

- Graded Ring Method
  Reid, Corti, Brown, Altinok, Fletcher

- Birational Geometry (MMP)
  Mori, Mukai, Takagi etc.

No.

Date

§ 主定理

Th. X

$f(X) = f$: X の ($\mathbb{Q}$-) Fano index $-K_X \sim fD$.　　$f < 210$!

1) $\operatorname{Max} f = 19$, $f(\mathbb{P}(3,4,5,7)) = 19$　　< 24　　conj. $X \simeq \mathbb{P}(3,4,5,7)$

2) $f(X) = 19 \Rightarrow$ Hilbert series of X = Hilbert series of $\mathbb{P}(3,4,5,7)$

$$= \frac{1}{(1-t^3)(1-t^4)(1-t^5)(1-t^7)}$$

3) $f \in \{1, 2, 3, \cdots, \underline{10}, 11, 13, 17, 19\}$

Def Graded Ring　　$-K_X \sim fD$　D = Weil

$$R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$$

$X = \operatorname{Proj} R(X, D)$.

Def Hilbert series of X.

$$P(X, t) := \sum_{n \geq 0}^{\infty} P_n(X) t^n \in \mathbb{C}[[t]]$$

実際の計算方法

Pn　　Sing R-R [YPG]

$$\chi(\mathcal{O}_X(nD)) = \chi(\mathcal{O}_X) + \frac{n(n+f)(n+2f)}{12} D^3 + \frac{n}{12} D \cdot c_2(X)$$

$$+ \sum_{\beta} \left\{ -\frac{r^2-1}{12r} + \sum_{j=1} \frac{\overline{b_j}(r - \overline{b_j})}{2r} \right\}$$

$\overline{x}$: smallest residue of mod n

B : 特異点の Basket $\{ \frac{1}{r_j}(1, a_j, r_j - a) \}$
$\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad \{ [r_j, a_j] \}$

i : $nD \sim iK_X$ をみたす最小の正整数 .

$$P(X,t) = \frac{1}{1-t} + \frac{\{ (f^3+3f+2)t + (2f^2+8)t^2 + (f^2-3f+2)t^3 \} D^3}{12(1-t)^4}$$

$$+ \frac{t}{12(1-t^2)} \cdot \frac{D \cdot C_2(X)}{12} + \sum_B \sum_{n=1}^{r-1} \frac{1}{1-t^n} \left\{ -i \frac{r^2-1}{12r} + \sum_{j=1}^{i-1} \frac{\overline{bj}(1-\overline{bj})}{2r} \right\}$$

$$\boxed{\text{以下 } A = D} \qquad \underset{\displaystyle \frac{1}{f}\left(2 - \sum_B \frac{r^2-1}{12r}\right)}{\|}$$

$f \geq 3$ のみ考える.
$P_{-1}(t) = 0$

$$D^3 = \frac{12}{(f-1)(f-2)} \left\{ 1 - \frac{1}{12} D C_2 + \sum_B \sum_{j=1}^{i-1} \frac{\overline{bj}(r-\overline{bj})}{2r} \right\}$$

Ex. $f = 9$. $\{ [\overset{r}{2}, \overset{a}{1}], [4, 1], [5, 2] \}$

$A^3 = \frac{1}{20} \qquad \frac{1}{12} A C_2 = \frac{31}{240}$

$P_1(X) = 1 \quad P_2(X) = 2, \quad P_3(X) = 3, \quad P_4(X) = 5, \quad P_5(X) = 3, \quad P_6(X) =$

$H^0(\mathcal{O}_X(nD)) \quad H^0(X, \mathcal{O}_X(2D))$

$X_6 \subset \mathbb{P}(1, 2, 3, 4, 5)$
$1+2+3+4+5-6 = 9$.

$n=1 \quad \textcircled{x}$
$2 \quad x^2, \textcircled{y}$
$3 \quad x^3, xy, \textcircled{z}$
$4 \quad x^4, x^2y, xz, y^2, \textcircled{w}$
$5 \quad x^5, x^3y, x^2z, xy^2, xw, yz, \textcircled{t}$.
6

$$\frac{(1-t^6)(1-t^4)}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^5)}$$ ε Mirage.

codim 0 type $\mathbb{P}(a_1, a_2, a_3, a_4)$ $X_{4,6} \subset \mathbb{P}(1,2,3,4,4,5)$
〃 1 〃 $X_d \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5)$
〃 2 〃 $X_{d,d} \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5, a_6)$
〃 3 〃 $X_{pf} \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$
(pfaffian) $\underset{\bot}{}$

5×5 skew symmetric 行列
Aから作られる. 4つの定義式の共通解.

Thm. $f \geq 2$. codim 3 type の X は 存在しない.
☺ 1つ1つ確かめる.

§ Outline of the proof.

Key Thm. (Kawamata's Boundness Thm. '92)

X : as above.
$\mathcal{E} = \Omega_X^{1\,\wedge\wedge}$

(1) $\mathcal{E}$: semistable.
$$(-K_X)^3 \leq 3(-K_X)\cdot c_2(X).$$

(2) $\mathcal{E}$: not semistable.
$\mathcal{F}$: $\mathcal{E}$ の max destabilizing sheaf
$r = \mathrm{rk}\,\mathcal{F}$ $(= 1, 2)$
$c_1(\mathcal{F}) \equiv tK_X$ となる t をとる。 $(0 < t < \frac{r}{3},\ t \in \frac{\mathbb{Z}}{f})$

a) if $s=1$ $(1-t)(1+3t)(-K_X)^3 \leqq 4(-K_X)\cdot c_2(X)$

b) if $s=2$ $\underbrace{t(4-3t)}(-K_X)^3 \leqq 4(-K_X)\cdot c_2(X)$

〰〰は $t=\frac{1}{f}$ のとき最小値をとる。

(2). b) の不等式は

$$\frac{4f^2-3f}{R} < 4\,\frac{(-K_X)\cdot c_2(X)}{}$$

$$\overset{\parallel}{24-\sum_{B}\left(r-\frac{1}{r}\right)} \leqq 2487$$

$\therefore f \leq 50.2 \cdots$

Bについての条件.

1) $\sum_{1}^{m}\left(r-\frac{1}{r}\right)<24$ # $m\leq 15$

2) $A^3>0$

3) $P_{-i}=0$ $i=1,\cdots,f-1$

4) Kawamata's Condition 2. b).

List.

f	X	B
1	$X_5 \subset \mathbb{P}(1,1,1,1,2)$	$[2,1]$
2	$X_{10,12} \subset \mathbb{P}(1,2,3,5,6,7)$	$\{[3,1], [3,1], [7,3]\}$
3	$X_{12,15} \subset \mathbb{P}(3,3,4,5,7,8)$	$\{[4,1], [7,1], [8,1]\}$
4	$X_6 \subset \mathbb{P}(1,1,2,3,3)$	$\{[3,1], [3,1]\}$
5	$\mathbb{P}(1,1,1,2)$	$\{[2,1]\}$
6	$X_6 \subset \mathbb{P}(1,1,2,3,5)$	$\{[5,2]\}$
7	$\mathbb{P}(1,1,2,3)$	$\{[2,1], [3,1]\}$

f		X	B
8		$X_6 \subset \mathbb{P}(1,2,3,3,5)$	$\{[3,1], [3,1], [5,1]\}$
9	①	$X_6 \subset \mathbb{P}(1,2,3,4,5)$	$\{[2,1], [4,1], [5,2]\}$
	②	$X_{12} \subset \mathbb{P}(2,3,4,5,7)$	$\{[2,1]\times 3, [5,2], [7,2]\}$
10		$\operatorname{codim} \geq 7$	$\{[3,1], [5,2], [13,6]\}$
11	①	$\mathbb{P}(1,2,3,5)$	$\{[2,1], [3,1], [5,2]\}$
	②	$X_{12} \subset \mathbb{P}(1,4,5,6,7)$	$\{[2,1], [5,1], [7,2]\}$
	③	$X_{10} \subset \mathbb{P}(2,3,4,5,7)$	$\{[2,1], [3,1], [4,1], [7,3]\}$
13		$\mathbb{P}(1,3,4,5)$	$\{[2,1]\times 2, [3,1], [7,37]\}$
17		$\mathbb{P}(2,3,5,7)$	$\{[2,1], [2,1], [5,1], [7,3]\}$
19		$\mathbb{P}(3,4,5,7)$	$\{[3,1], [4,1], [5,2], [7,3]\}$

これからやりたいこと.

Conj. $f \geq 2$ の場合は.
$\operatorname{codim} \geq 4$ type の X は 少ない 例外を除いて存在しない.

$f=2 \Rightarrow$ Reid 氏と共著 である.

$f \geq 3$ なし と予想~

No.

Construction of Spin(7) metric based on 3-Sasakian manifold

Date 2002 · 9 · 20

講演者 安井 幸則 (大阪市立大学)

Introduction

* なぜやるか — 例があまりない.

{ Bryant-Salamon (1989) Spin(7) 1コ
{ Gibbons-Page-Pope (1990) G_2 3コ

↓

(2001) ㊕ 物理

{ Cretic-Gibbons-Lü-Pope
{ Brandhuber-Gomis-Gubser-Gukov
{ Gukov-Sparks
{ Kanno-Y.Y.

* One-parameter family of G_2 and Spin(7) metric

Spin(7) $\mathbb{B}$, $\mathbb{C}^8$, and Spin^c bundle over $\mathbb{CP}(2)$

G_2 2コ or 3コ.

* Cohomogeity one metric on $M^{(n)} \simeq \mathbb{R}^+ \times G/H$ $(n=7,8)$

(1) Spin(7)

$$G/H = Sp(2)/Sp(1),\ SU(3)/U(1) \quad \longleftarrow \text{weak } G_2$$

(2) G_2

$$G/H\cdot U(1) = Sp(2)/Sp(1)\cdot U(1),\ SU(3)/U(1)^2 \quad (S^3\times S^3)$$

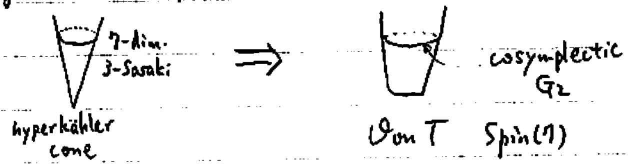
nearly Kähler

$$\begin{tikzcd}
 & & \mathbb{R}^+\times G/H \ \text{⑧} \arrow[dl] & \\
 & \text{⑥}\ G/H\cdot U(1) \arrow[dr] & & G/H\ \text{⑦} \arrow[ul] \arrow[ll] \arrow[dl] \\
\Uparrow & & G/H\cdot Sp(1)\ \text{④}\ \text{(Wolf space)} & \\
 & & C(S)\ (4n+4) \arrow[dl] & \\
 & (4n+2)\ \boxed{\text{twistor}}\ T \arrow[dr, "S^2"'] & & S\ (4n+3)\ \underline{\text{3-Sacaki}} \arrow[ul, "i"'] \arrow[dl, "S^3"] \quad \cdots\ (*) \\
 & & (4n)\ \mathcal{O} &
\end{tikzcd}$$

ℚ-factorial Fano contact variety with Kähler Einstein

(4n) $\mathcal{O}$ quaternion Kähler orbifold (n=1) ⇒ Self-dual Einstein.

[1] Inhomogeneous manifold への拡張 in the sense of (*)



Remark. ϑ = Kähler self-dual Einstein $(\mathbb{C}P(2))$
"3-Sasaki construction" 以外の構成ができる.

[2] Ordinary differential equation の解.
[3] BPS solution

Def.
- (S, g, ξ) : odd dim. Riemannian with unit Killing vector ξ $(L_\xi g = 0)$
- type (1,1) tensor $\Phi \equiv \nabla\xi$
  $(\nabla_X \Phi)(Y) = \eta(Y)X - g(X,Y)\xi$ η : dual 1-form of ξ

(S, g, ξ) Sasaki

⊛ $\Phi^2 - \xi \otimes \eta = -1$ etc.

Def. 3-Sasaki
① three killing ξ_k $(k=1,2,3)$ s.t. $g(\xi_k, \xi_l) = \delta_{kl}$, $[\xi_k, \xi_l] = 2\epsilon_{klm}\xi_m$
② each killing は Sasaki and
$\Phi^k \circ \Phi^l - \xi_k \otimes \eta^l = -\epsilon_{klm}\Phi^m - \delta_{kl}\,\mathrm{id}.$

以下, 7-dim. 3 Sasaki $(S, g, \underbrace{\xi_k)}_{\text{complete}}$

Konishi (1975)
(O, g_0) : 4-dim self-dual Einstein with positive scalar curvature
⟹ there is an SO(3) principal bdl s.t. the total space is a 3-Sasakian.

$$TS = V \oplus H$$

(ξ_k) (X_μ) : local orthogonal

$\updownarrow$ dual $\updownarrow$ frame $(\mu = 1, 2, 3)$

η_k θ^μ (1-form)

$(k = 1, 2, 3)$

3-Sasaki metric

$$g_S = \sum_{\mu=0}^{3} \theta^\mu \otimes \theta^\mu + \sum_{k=1}^{3} \eta^k \otimes \eta^k$$

$$\begin{tikzcd} S \arrow[d, "\pi\ SO(3)"] \\ (g_{\mathcal{O}}, \mathcal{O}) \end{tikzcd}$$

$$\sum_{\mu=0}^{3} \theta^\mu \otimes \theta^\mu = \pi^* g_{\mathcal{O}}$$

$$\eta = i\eta' + j\eta^2 + k\eta^k \in \Omega^1(S, \mathrm{Im}\,\mathbb{H})$$

$$g_S(X, Y) = g_{\mathcal{O}}(\pi_* X, \pi_* Y) + \langle \eta(X), \eta(Y) \rangle_{\mathbb{H}}.$$

3-Sasakian の definition より

$$\boxed{\begin{aligned} d\eta + \eta \wedge \eta &= 2\Theta \\ d\Theta &= -\eta \wedge \Theta + \Theta \wedge \eta \end{aligned}}$$ self-dual equations.

$$\theta \equiv \theta^0 + i\theta^1 + j\theta^2 + k\theta^3 \in \Omega^1(S, \mathbb{H})$$

$$\Theta \equiv \frac{1}{2}\bar{\theta} \wedge \theta = i\Theta^1 + j\Theta^2 + k\Theta^3 \in \Omega^2(S, \mathrm{Im}\,\mathbb{H})$$

where $\Theta^1 = \theta^0 \wedge \theta^1 - \theta^2 \wedge \theta^3$, $\Theta^2 = \theta^0 \wedge \theta^2 - \theta^3 \wedge \theta^1$, $\Theta^3 = \theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2$

◎ $\mathcal{O}$: 4-mfd with $g_{\mathcal{O}}$

Riemannian curvature on $T\mathcal{O}$

$$R : \Lambda^2 T\mathcal{O} \hookrightarrow \Lambda^2 T\mathcal{O}$$ (symmetric linear)

$$\Lambda^2 T\mathcal{O} = \Lambda^2_+ T\mathcal{O} \oplus \Lambda^2_- T\mathcal{O}$$

matrix (6×6) Weyl Ricci scalar

$$R = \left(\begin{array}{c|c} W_+ & 0 \\ \hline 0 & W_- \end{array}\right) + \left(\begin{array}{c|c} 0 & K^+ \\ \hline K^- & 0 \end{array}\right) + \frac{\sigma}{12}\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

$(\mathcal{O}, g_{\mathcal{O}})$ が self-dual Einstein

$$\Longleftrightarrow \begin{cases} K^+ = 0 & \text{Einstein condition} \\ W_- = 0 & \text{self-dual} \end{cases}$$

Frame bundle

$$\begin{tikzcd} \mathcal{F} = \{ u : T_x\mathcal{O} \xrightarrow{\text{isi.}} \mathbb{R}^4 = \mathbb{H} \} \arrow[d, "SO(4)"] \\ \mathcal{O} \end{tikzcd}$$

$SO(4) = Sp(1) \oplus Sp(1) = \mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H}$

Levi-Civita connection $\equiv \omega = \omega_+ + \omega_- \in \Omega^1(\mathcal{F}, \mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H})$.

canonical 1-form $\theta \in \Omega^1(\mathcal{F}, \mathbb{H})$

$\Omega_{-} = d\omega_{-} + \omega_{-} \wedge \hat{\omega}_{-} \in \Omega^2(\mathcal{F}, \mathrm{Im}\,\mathbb{H})$

$\boxed{\Omega_{1} = \text{const. } Ⓗ}$ $\omega_{-} = \eta$ と同一視.

$\boxed{\text{Cone metric on } \mathbb{R}^+ \times S}$

$\bar{g} = dt^2 + t^2 g_S$

$= dt^2 + t^2 \pi^{\dagger} g_0 + t^2 \sum_{k=1}^{3} \eta^k \otimes \eta^k$

hyperkähler form

$\beta^k \equiv -t^2 Ⓗ^k + \frac{t^2}{2} \epsilon_{klm} \eta^l \wedge \eta^m + t \eta^k \wedge dt$ $k=1,2,3$

$\Rightarrow d\beta^k = 0$ ∴ hyperkähler ($Sp(2)$ holonomy)

Prop. g : metric on $\mathbb{R}^+ \times S$

$$g \equiv dt^2 + b^2(t) \overset{\pi^{\dagger} g_0 \; /\!/}{\boxed{\sum_{\mu=0}^{3} \theta^\mu \otimes \theta^\mu}} + \sum_{k=1}^{3} a_k^2(t)\, \eta^k \otimes \eta^k$$

Let $\{b(H), a_k(t)\}$ be a solution of the following equations :

$$\begin{cases} \dfrac{\dot{b}}{b} = \dfrac{a_1 + a_2 + a_3}{b^2} \\ \dfrac{\dot{a}_1}{a_1} = \dfrac{a_1^3 - (a_2 + a_3)^2}{a_1 a_2 a_3} - 2\dfrac{a_1}{b^2} \end{cases}$$

Then there exist solutions with $\boxed{\text{a certain B.C.}}$ [illegible] and $Hol(g)$ is contained in $Spin(7)$.

Remark. ① 解はある.

$S = S^4$ and $\mathbb{C}P(2)$ $\Rightarrow$ complete $Spin(7)$ metric ⊗

② $\mathbb{R}^+ \times S$ 上の metric

global にどこの metric に lift されるか?

⊗ の case ... Spinor bundle over S^4.

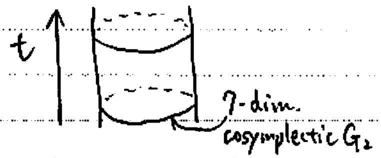
一般に ⁇ $S \times_{SO(3)} F$ $F = \mathbb{H}$ が ⊗ になる.

$$\begin{tikzcd} S \times_{SO(3)} F \arrow[d] \\ O \end{tikzcd}$$

F : cohomogeneity one mfd with $SO(3)$ action

$F \approx \mathbb{R}^+ \times SU(2)/\Gamma \leftarrow$ 離散群

(証明) $\boxed{\text{Hitchin DG/010710}}$



$M^7 \times I$

7-dim. M の時間発展方程式

= grad. flow.

NO

① "3-Sasaki" に cosymplectic G_2 を入れる.

$\forall 4$ parameters が入る

$\Updownarrow$

$a_k \, (k=1,2,3), \, b.$

<u>B.C.</u> 3タイプ

$$\begin{tikzcd}
 & \mathbb{R}^+ \times S \equiv M^{(7)} \arrow[dl] \arrow[dd] \arrow[dr] & \\
T \arrow[dr, "S^2"'] & & S \arrow[ll, "\text{②}\ S^1"'] \arrow[dl, "\text{①}\ S^3 \leftrightarrow (a_1, a_2, a_3)"] \\
 & \mathcal{O} \leftrightarrow b &
\end{tikzcd}$$

type (I) "bolt"

$$M^{(8)} \xrightarrow[\text{①}]{t \mapsto 0} \mathcal{O}$$

Gibbons : ガウスの超幾何

$$a_1, a_2, a_3 \longrightarrow 0$$

$$\begin{cases} a_k^2 \longmapsto t^2 \quad (k=1,2,3) \\ b^2 \longmapsto {}^{\forall}\mathrm{const.} \end{cases}$$

type (II) "bolt"

$$M^{(8)} \xrightarrow{t \mapsto 0} T$$

解析解なし

$$\begin{cases} a_1^2 \longmapsto 16t^2 \\ a_2, a_3, b \longmapsto {}^{\forall}\mathrm{const.} \end{cases}$$

type (III) "NUT"

$$M^{(8)} \longrightarrow \mathrm{p.t.}$$

解析解あり

$$a_k^2, \, b^2 \longmapsto t^2$$