A SPECIAL LAGRANGIAN 3-TORUS AS A REAL SLICE

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A concrete example of Calabi-Yau threefolds which have a special Lagrangian 3-torus as a real slice is constructed.

1 Introduction

Special Lagrangian submanifolds in Calabi-Yau manifolds are expected to have a beautiful moduli theory similar to that of analytic subschemes through mirror symmetry. In general, it is not easy to construct special Lagrangian submanifolds, since one cannot usually describe Kähler-Einstein metrics. One way to circumvent that difficulty is Bryant's idea of taking a real slice; the set of real points of a Calabi-Yau manifolds defined over $\boldsymbol{R}$ is, if not empty, a special Lagrangian submanifold [2]. We remark that local deformation of special Lagrangian sumbanifolds are unobstructed and the dimension of the deformation space coincides with the first Betti number of the submanifold [4].

Special Lagrangian 3-tori are of special interest [5], and one may expect a differentiable fibration structure on Calabi-Yau manifold by mid-dimensional real tori. But up to now, only few concrete examples of such a construction are known [3]. We give here an example of Calabi-Yau threefold whose real point set is a smooth real 3-torus. A merit of this method is that one can describe special Lagrangian manifolds as real algebraic varieties.

2 A toy model

Take an elliptic curve $E = \boldsymbol{C}/\boldsymbol{Z} \oplus \boldsymbol{Z}\tau$, where τ belongs to the upper half plane as usual. There is an embedding ι of E into $\boldsymbol{P}^2 = \{(X : Y : Z)\}$ as

$$z \mapsto (\wp(z) : \wp'(z) : 1), \tag{1}$$

whose image is a cubic curve

$$C = \{Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3\}. \tag{2}$$

Let us assume that the coefficients g_2, g_3 are in $\boldsymbol{R}$ and that the discriminant is negative. Then the set of real points of C is a special Lagrangian submanifold

in C, which is diffeomorphic to S^1 since the left hand side of the defining cubic equation (2) has only one real root for X/Z.

Since the coefficients of Laurent power series of $\wp$-function are in the ring $\boldsymbol{Q}[g_2, g_3]$, the image of the set of real z's is contained in, and in fact coincides with, the set of real points of C.

Take a translation T_t of E by $t\tau$ where $0 \leq t \leq 1$. Then the composition $\iota \circ T_t : E \to \boldsymbol{P}^2$ is also embedding with a constant image C. As t varies, the inverse images of the real point set of C define a smooth foliation on E, whose leaves are all S^1. We have a smooth map from E to $S^1 = \{z \in \boldsymbol{C} \mid |z| = 1\}$ by assigning $e^{2\pi it}$.

We remark that in the case that the discriminant is positive, we have a fiber bundle of two disjoint S^1's, which also reduces to an S^1-bundle.

In this way, by changing the map, which may not always be an embedding, one may have a special Lagrangian real slice fibration on original Calabi-Yau manifold.

3 Construction of real 3-torus

Let us take a weighted projective space $\boldsymbol{P}(1,1,1,1,4,6)$. We denote the coordinates by T_0, T_1, T_2, T_3, X and Y. Let $P(T)$ be a general positive real polynomial in T_i's of degree 12. Our choice here[a] is

$$P(T) = \sum_{i=0}^{3} \alpha_i T_i^{12} \tag{3}$$

where α_i are positive real numbers satisfying

$$\alpha_0 \alpha_3 \neq \alpha_1 \alpha_2. \tag{4}$$

Consider the following system of equations

$$\begin{cases} F := X^3 - Y^2 + P(T) + 6\lambda XY T_0 T_3 = 0 \\ G := T_0 T_3 - T_1 T_2 = 0 \end{cases} \tag{5}$$

where $\lambda \in \boldsymbol{C}$ is a parameter which does not satisfy any of the equalities

$$\sqrt{\alpha_0 \alpha_3} \pm \sqrt{\alpha_1 \alpha_2} = \pm 54\lambda^6. \tag{6}$$

The two equations (5) define a variety $\bar{M}$.

[a] We thank for the referee who has pointed out that the equation on the first draft was not general enough.

Theorem 1 *$\bar{M}$ has a unique canonical Gorenstein singularity, which can be resolved by a single blowing up $M \to \bar{M}$. M is a smooth Calabi-Yau threefold which has a structure of elliptic fibration over a smooth quadric surface.*

When all the variables are real valued, $M(\boldsymbol{R})$ is diffeomorphic to T^3.

Remark. If $G = \sum T_i^2$, $M(\boldsymbol{R})$ is vacant. If $G = T_0^2 + T_1^2 + T_2^2 - T_3^2$, $M(\boldsymbol{R})$ is $S^2 \times S^1$. If $G = T_0^2 + T_1^2 - T_2^2 - T_3^2$, $M(\boldsymbol{R})$ is T^3.

Proof. The Jacobian matrix $\partial(F,G)/\partial(T_0, \ldots, T_3, X, Y)$ is

$$\begin{pmatrix} 12\alpha_0 T_0^{11} + 6\lambda XYT_3 & 12\alpha_1 T_1^{11} & 12\alpha_2 T_2^{11} & 12\alpha_3 T_3^{11} + 6\lambda XYT_0 \\ T_3 & -T_2 & -T_1 & T_0 \\ & & 3X^2 + 6\lambda YT_0T_3 & -2Y + 6\lambda XT_0T_3 \\ & & 0 & 0 \end{pmatrix}. \quad (7)$$

Using the assumptions (4) and (6), one can show that the Jacobian criterion fails only at a point $(0:0:0:0:1:1)$, which is the unique common point with the singular locus $\{T_0 = \ldots = T_3 = 0\}$ of the ambient weighted projective space. The first equation $F = 0$ is transverse to the singular locus of the ambient space, and the section is a quotient singularity of type $\frac{1}{2}(1,1,1,1)$. Thus the singularity of $\bar{M}$ is canonical Gorenstein. By blowing up the point, the second row of the Jacobian matrix is nonzero and we have a smooth variety, which has a trivial canonical bundle by the adjunction formula.

This blowing up, at the same time, resolves the indeterminancy of the rational projection map from $\bar{M}$ to $\boldsymbol{P}^3 = \{(T_0 : T_1 : T_2 : T_3)\}$. The image is the quadric surface Q defined by G whose real slice $Q(\boldsymbol{R})$ is diffeomorphic to $S^1 \times S^1$, and general fibers are elliptic curves. Next we shall check the real slice of the fibers.

We denote $\lambda T_0 T_3$ by μ. $F = 0$ is equivalent to

$$(Y - 3\mu X)^2 = X^3 + 9\mu^2 X^2 + P(T). \quad (8)$$

The right hand side equals

$$\begin{aligned} &(X + 3\mu^2)^3 - 27\mu^4 X + P(T) - 27\mu^6 \\ &= (X + 3\mu^2)^3 - 27\mu^4(X + 3\mu^2) + P(T) + 54\mu^6. \end{aligned} \quad (9)$$

The discriminant is

$$-4(-27\mu^4)^3 - 27(P(T) + 54\mu^6)^2 = -27(P(T)^2 + 108\mu^6 P(T)), \quad (10)$$

which is negative for all real value of T_i's, λ and positive values of the α_i. That means all the fibers of the real slice are diffeomorphic to S^1, for any $\lambda \in \boldsymbol{R}$.

Since this real fibration is trivial, the total space is diffeomorphic to a real 3-torus.

A similar argument works for other choices of G.

Remark. The threefold $\bar{M}$ can be represented by a toric hypersurface in a toric fourfold corresponding to a polytope Δ, and in fact the hypersurface is Δ-regular.

Δ is given by the convex hull in $\boldsymbol{R}^4$ of the following six points:

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} -3 \\ -2 \\ 6 \\ 6 \end{pmatrix}, \\ v_4 &= \begin{pmatrix} -3 \\ -2 \\ 6 \\ -6 \end{pmatrix}, v_5 = \begin{pmatrix} -3 \\ -2 \\ -6 \\ 6 \end{pmatrix}, v_6 = \begin{pmatrix} -3 \\ -2 \\ -6 \\ -6 \end{pmatrix}. \end{aligned} \tag{11}$$

When the parameter λ tends to infinity, the variety goes to the large complex structure limit.

We can compute the toric mirror variety using this fact[1]. The vertices of the dual polytope Δ^*, that is, the facets of Δ are:

$$\begin{aligned} w_{1234} &= \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, w_{1235} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, w_{1246} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \\ w_{1256} &= \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, w_{13456} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, w_{23456} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{12}$$

where the subscripts refers the vertices on that facet, that is, the inner product with the vertices are -1.

The mirror toric hypersurfaces determined by the polytope Δ^* have again a structure of elliptic fibration over a surface $\{T_0^2 = T_1T_2 = T_3T_4\}$ in $\boldsymbol{P}^4 = \{(T_0 : \cdots : T_4)\}$, which has four A_1 singularities.

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References

1. V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Alg. Geom. **3**, 493–535 (1994).
2. R. Bryant, *Minimal Lagrangian submanifolds of Kähler-Einstein manifolds*, in *Lecture Note in Math. 1255* (Springer, 1985).
3. M. Gross and P. M. H. Wilson, *Mirror symmetry via 3-tori for a class of Calabi-Yau threefolds*, alg-geom/9608004, to appear in Math. Ann.
4. R. C. McLean, *Deformation of calibrated submanifolds*, Duke University preprint (1996).
5. A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* B **479**, 243–259 (1996).