

伊藤

Seshadri constants and Okounkov bodies

§0. Intro

$$X = \text{proj } \mathbb{C}P^2 / \mathbb{C}$$

$$L = \text{ample line bdl}$$

Seshadri const.

Okounkov bodies $\subset \mathbb{R}^n$

の 2 つの 定数。

(toric geom = 多面体
polytope $\alpha \in \mathbb{R}^n$
に $\in \mathbb{R}_{\geq 0}$)

Then The Okounkov body of L has some information of the Seshadri constant of L .

§1. Seshadri constants

Def $\left(\begin{array}{l} X = \text{proj } \mathbb{C}P^2 \\ L = \text{ample} \end{array} \right), p \in X$

$$\epsilon(X, L, p) \stackrel{\text{def}}{=} \inf_c \frac{L \cdot c}{m_p c} \quad (> 0)$$

(m_p multiplicity at p)

$$= \max \{ \epsilon > 0 \mid \pi^* L - \epsilon E = \text{wpf} \}$$

$$\begin{array}{ccc} \pi: X & \rightarrow & X \\ \downarrow & & \downarrow \\ E & & p \end{array}$$

(blow up at p)

$$= \lim_{R \rightarrow \infty} \frac{j(RL, p)}{R}$$

$$j(RL, p) = \max \{ j \geq 0 \mid H^0(\mathbb{C}P^2, L^{\otimes j}) \neq 0 \}$$

$$\varepsilon(x, L, 1) = \varepsilon(x, L, p)$$

for general it $p \in X$.

Lemma L : big implies $\varepsilon(x, L, 1)$ can be defined.

§2. Okounkov bodies

X : proj var

$p \in X$: smooth

L : big

$Z = (z_1, \dots, z_n)$: local coords at p

$$U = \mathcal{O}_{X,p} / \mathfrak{o}_p \xrightarrow{\sim} \mathbb{N}^n$$

$$f = \sum_{i \in \mathbb{N}^n} a_i z^i \mapsto \min \{i \mid a_i \neq 0\}$$

$$L_p \simeq \mathcal{O}_p \text{ , big}$$

$$H^0(\mathbb{P}^1, L) \otimes \mathfrak{o}_p \rightarrow L_p \otimes \mathfrak{o}_p \xrightarrow{\sim} \mathcal{O}_p \otimes \mathfrak{o}_p \xrightarrow{\sim} \mathbb{N}^n$$

$$\text{Image } \varepsilon(L, \mathbb{P}^1) \simeq \mathbb{N}^n.$$

$$\Delta_{\mathbb{R}}(L) := \bigcup_{\mathbb{R}^n} \frac{1}{\mathbb{R}} \varepsilon(L, \mathbb{P}^1) \subseteq \mathbb{R}^n$$

§3. Result

Def $\Delta \subseteq \mathbb{R}^n$ closed convex body

we define $s(\Delta) > 0$ as follows: (note)

When $\Delta =$ rational polytope

$$S(\Delta) = \sum (X_{\Delta_i} \cdot L_{\Delta_i} \cdot 1) \\ \uparrow \quad \uparrow \text{Q-dim} \\ (\text{toric val associated to } \Delta.)$$

general case

$$\Delta_1 \subset \Delta_2 \subset \dots \quad \Delta = \overline{\cup \Delta_i}$$

↑ ↑
rational rational

$$S(\Delta) := \lim_{i \rightarrow \infty} S(\Delta_i) > 0.$$

Then $X =$ proj val
 $L =$ big
 $R, P =$ as above

$$\rightsquigarrow \sum (X \cdot L \cdot 1) \geq S(\Delta_2(L))$$

Idea of proof

$$P := \bigcup_{k \geq 0} (kR) \times V(kL) \subset \mathbb{N} \times \mathbb{N}^n$$

if P is f.g.

$$(X \cdot L) \rightsquigarrow (X_{\Delta_2(L)} \cdot L_{\Delta_2(L)}) \text{ (f.g.)} \\ \uparrow \quad \uparrow \\ \sum (X \cdot L \cdot 1) \quad S(\Delta_2(L))$$

$$\exists \text{ } z \text{ for case (2) } \quad \lim_{R \rightarrow \infty} \frac{\#(R \cdot P)}{R} \in \mathbb{R}.$$

Graded category and its applications.

Thm (Quillen)

$A =$ Noether ring

$\text{mod } A =$ the cat of f.g modules

$$K(\text{mod } A) \cong K(\text{mod } A[x])$$

$$\text{mod } A \hookrightarrow \text{mod } A[x]$$

$$M \mapsto M \oplus_A A[x]$$

$\text{mod } A \ni$ abel cat \xrightarrow{A} $\text{mod } A[x] \ni$ abel cat \cong $\text{mod } A[x]$

$\text{mod } A[x] \cong$ $\text{mod } A[x]$ polynomial category

Thm 1 $\mathcal{A} =$ Noeth. abelian cat with projectives

$$K(\mathcal{A}) \cong K(\mathcal{A}[x])$$

- $\mathcal{A} =$ abel cat
- $\text{End } \mathcal{A} =$ the cat of end morphism
(X, φ)
- $\text{Lex } \mathcal{A} =$ the cat of left exact fun

(\mathcal{A} is a Noetherian ring and \mathcal{A} is a Noetherian abelian cat)

$\text{Noeth } \mathcal{A} =$ the full sub cat of noeth obj in \mathcal{A}

Def $\bullet A = \text{noeth} \stackrel{\text{def}}{\iff} \text{noeth } A = A$
 $\bullet A[x] \stackrel{\text{def}}{=} \text{noeth } \text{End } \text{Lex } A$

base change $\iff \exists \tilde{A} \xrightarrow{\sim} A$ functor id

$$A \iff \text{Lex } A \longrightarrow \text{End } \text{Lex } A$$

$$a \longmapsto \text{Hom}(-, a) \longmapsto (\oplus_{i \in \mathbb{N}} t^i, \epsilon)$$

(Note object id Note obj $\iff \exists \tilde{A} \xrightarrow{\sim} A$ functor id)

$$A = \text{Noe } \text{Lex } A \iff \text{mod } A$$

$$\rightsquigarrow A[x] = \text{mod } A[x]$$

$$(\bar{M}, \bar{\epsilon}) \longleftarrow M$$

$$\epsilon: M \rightarrow M$$

graded cat $(\iff \exists \tilde{A} \xrightarrow{\sim} A \text{ functor } \text{id})$

Def $\langle n \rangle : \text{Obj } \langle n \rangle = \mathbb{N}$

$$\begin{array}{ccccccc}
 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & \dots \\
 \cdot & \downarrow & \cdot & \downarrow & \cdot & \downarrow & \dots \\
 & \parallel & & \parallel & & \parallel & \\
 & \xrightarrow{\alpha} & & \xrightarrow{\alpha} & & \xrightarrow{\alpha} & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 (x_1^1, \dots, x_n^1) & & (x_1^2, \dots, x_n^2) & & (x_1^3, \dots, x_n^3) & & \dots
 \end{array}$$

(α is invertible $\iff \exists \beta: \text{relator } \epsilon \lambda \alpha \beta$)

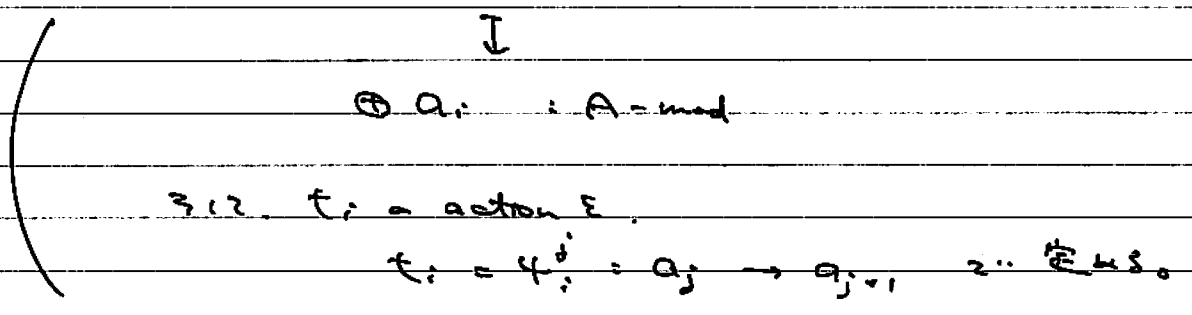
$$(x_1^1 \dots x_n^1) \xrightarrow{\alpha} (x_1^2 \dots x_n^2) \xrightarrow{\alpha} (x_1^3 \dots x_n^3) \quad (x_1^1 \cdot x_2^1 = x_2^1 \cdot x_1^1)$$

$$\mathcal{A}_{\text{gr}} \langle n \rangle := \text{Hom}(\langle n \rangle, \mathcal{A})$$

$$\hat{\mathcal{A}}_{\text{gr}} \langle n \rangle := \text{noeth } \mathcal{A}_{\text{gr}} \langle n \rangle$$

- $A = \text{noeth ring}$
 $\mathcal{A} = \text{mod } A$
 $\mathcal{A}_{gr}[n] = \{ \text{graded } A[t_1, \dots, t_n] \text{ mod } \}$

(\because) $a_1 \begin{matrix} \rightrightarrows \\ \rightleftharpoons \\ \leftleftarrows \end{matrix} a_2 \begin{matrix} \rightrightarrows \\ \rightleftharpoons \\ \leftleftarrows \end{matrix} \dots \quad a_i \in \text{mod } A$



Thm 2 $\mathcal{A} = \text{abelian}$

$\mathbb{Z}[0] \otimes K(A) \cong K(\mathcal{A}_{gr}[n])$
 $\uparrow \quad \cong$
 (不能定, 二次條件に注意)

Prop $\frac{\mathcal{A}_{gr}[2]}{\mathcal{A}_{gr.\text{nil}}[2]} \cong \mathcal{A}(A)$

\cong

$0 \rightarrow K(\mathcal{A}_{gr.\text{nil}}[2]) \rightarrow K(\mathcal{A}_{gr}[2]) \rightarrow K(\mathcal{A}(A)) \rightarrow 0$

$\parallel \quad \quad \quad \parallel \quad \quad \quad \parallel$

$K(\mathcal{A}_{gr}[1]) \quad \quad \quad \parallel \quad \quad \quad \parallel$

\parallel

$0 \rightarrow \mathbb{Z}[0] \otimes K(A) \rightarrow \mathbb{Z}[0] \otimes K(A) \rightarrow K(A) \rightarrow 0$

$1 \otimes \alpha \quad \quad \quad (1-\alpha) \otimes \alpha$

2011

7/8 玉原 14:00 ~ 權業

"Remarks on the non-vanishing Conjecture"

Conj Nd: (X, Δ) : d -dim proj lc pair s.t. $K_X + \Delta$ = p.eff. $\Delta \in \mathbb{R}$
 $\Rightarrow \exists D = \mathbb{R}$ -div ≥ 0 s.t. $K_X + \Delta \sim_{\mathbb{R}} D$

この話は "Global ACC" & "ACC for lct" の仮定だ。 (announced by Hacon-McKernan Xu)

Thm1: Conj Nd for smooth var with no boundary & abundance in $\leq d-1$
 \Rightarrow Conj Nd

Thm2: X : rationally conn proj. (X, Δ) lct s.t. $K_X + \Delta$ = p.eff.
 $\Rightarrow \exists D \geq 0$ s.t. $K_X + \Delta \sim_{\mathbb{R}} D$.

key lemma: (cf. [DHP §8]) (X, Δ) : \mathbb{Q} -fac dlt pair $K_X + \Delta$ p.eff. $\Delta_i \geq 0$ s.t.
 $\lim_{i \rightarrow \infty} \Delta_i = \Delta$ & $\Delta_i \leq \Delta_{i+1}$ $K_X + \Delta_i$: not p.eff.
 $\Rightarrow \exists \varphi: X \dashrightarrow X' \xrightarrow{f} Y'$ s.t. $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0$ (X', Δ') : lc
 $i \rightarrow \infty$ (X, Δ_i) の Mori fibre sp.

Rem: when $K_X + \Delta$: p.eff we need ACC conj's when $K_X + \Delta$: nef we just use "length of ext"

proof of Thm2:

Induction on dimension We may assume $K_X + (1-\epsilon)\Delta$: not p.eff.
 for $\forall \epsilon > 0$ Key lemma $\rightarrow \exists \varphi: X \dashrightarrow X' \xrightarrow{f} Y'$
 s.t. $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0$

F : gen. fibre of g .

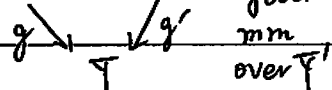
$v(\beta^*(K_X + \Delta')) = 0$

$\alpha^*(K_X + \Delta) + E = \beta^*(K_{X'} + \Delta') + G$

$\rightsquigarrow v(\alpha^*(K_X + \Delta)|_F) = 0$

$K_W + \Gamma_W = \alpha^*(K_X + \Delta) + E' \rightsquigarrow v((K_W + \Gamma_W)|_F) = 0$

$\xrightarrow{L_{\Delta_i}} W \dashrightarrow W'$



$K_{W'} + \Gamma_{W'} \sim_{\mathbb{Q}} g'^* 0$
 $\rightsquigarrow K_{W'} + \Gamma_{W'} \sim_{\mathbb{Q}} g'^*(K_{Y'} + \Delta_{Y'})$
 Ambro's
 can bdd.

$$\Upsilon: \mathbb{R}^d \rightsquigarrow K_{\Upsilon'} + \Delta_{\Upsilon'} \geq 0 \rightsquigarrow K_{\overline{\Upsilon}} + \overline{\Delta}_{\overline{\Upsilon}} \geq 0 \rightsquigarrow K_{\mathcal{X}} + \Delta \geq 0 \quad \square$$

Global ACC

$d \in \mathbb{N}$ & $I \subseteq [0, 1]$ DCC. $\exists I_0 \subseteq I$ finite s.t. $(\mathcal{X}, \Delta) = d$ -dim proj pc

$\cdot K_{\mathcal{X}} + \Delta \geq 0 \quad \cdot \Delta \in I$

$\Rightarrow \Delta \in I_0$

ACC for lct

$d \in \mathbb{N}$ $T \subseteq [0, 1)$ DCC $S \subseteq \mathbb{R}_{>0}$ finite $\{lct(\mathcal{X}, \Delta) \mid (\mathcal{X}, \Delta) \text{ l.c. dim } \mathcal{X} = d, \Delta \in P, D \in S\}$

$= \text{Acc}$

14:30 ~ 大川

problem: X : sm proj var / \mathbb{C} $|K_X|$ base point free.

$\Rightarrow D^b(\text{coh}(X)) = D(X)$ has no nontrivial semi-orthogonal decomposition (SOD)

Motivation: Conj: $f: X \dashrightarrow Y$ extremal contr. induces an SOD of $D(X)$

SOD: T : triangulated cat. $T = \langle \mathcal{A}, \mathcal{B} \rangle$ is an SOD if

- $\mathcal{A}, \mathcal{B} \subseteq T$ full subtriangulated cat.
- $\text{Hom}_T(\mathcal{B}, \mathcal{A}) = 0$
- $\forall X \in T \exists \Delta [f \rightarrow X \rightarrow a \rightarrow b[1]] \quad a \in \mathcal{A}, b \in \mathcal{B}$.

example: $f: X \dashrightarrow Y \quad Z \subseteq Y, Z, Y$: sm $X = B|_Z Y \quad Z$ of codim c in Y

$i: E \hookrightarrow X \quad B$ exc $f_E: E \rightarrow Z$

$\Rightarrow D(X) = \langle D_{-c+1}, \dots, D_{-1}, f^* D(Y) \rangle \quad D_i = i_* f_E^* D(Z) \otimes \mathcal{O}_X(-iE)$

Cor: K_X not nef $\Rightarrow D(X)$ has an SOD.

Rem!!: Converse is NOT true! i.e. K_X : even if nef, $D(X)$ may decompose!

example: ① X : Enriques ($g=0, K_X \neq \mathcal{O}_X, K_X^{\otimes 2} \cong \mathcal{O}_X$)

$\rightarrow h^1(\mathcal{O}_X) = 0, h^2(\mathcal{O}_X) \xrightarrow{\text{some duality}} h^0(K_X) = 0 \rightarrow \mathcal{O}_X$ is an exceptional obj.

Def: $E \in T$: Δ -cat. is exceptional if $\text{Hom}(E, E[i]) \cong H^i(\text{pt}, \mathbb{C}) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i \neq 0 \end{cases}$

E : exceptional $\rightarrow T = \langle \langle E \rangle_{\text{tr}}^\perp, \langle E \rangle_{\text{tr}} \rangle$ is an SOD.

For $\mathcal{A} \subseteq T, \mathcal{A}^\perp = \{x \in T \mid \text{Hom}(a, x) = 0, \forall a \in \mathcal{A}\}$

② X : Godeaux surf. (K_X : ample, $g=0, h^0(K_X) = 0$)

$\rightarrow \mathcal{O}_X$: exc.

Fact: $K_X \cong \mathcal{O}_X \rightarrow$ exists no SOD.

Evidence:

prop1: X curve, $g \geq 1$ ($\Leftrightarrow |K_X|$ base point free)

$\Rightarrow D(X)$ has no SOD.

prop 2: In the same situation of the problem then $D(X)$ has no except obj.

proof of prop 2: $0 \neq E \in D(X)$ $\text{Hom}(E, E[\dim X]) \cong \text{Hom}(E, E \otimes k_X)^\vee \neq 0$ if k_X is base point free. \rightarrow exists no exceptional obj.

Lemma (1), \otimes : $0 \neq E \in D^*(\text{Coh } X)$ L : line bundle base point free
 $\Rightarrow \text{Hom}(E, E \otimes L) \neq 0$

pf: E : sheaf \rightarrow trivial

in general $m := \inf \{i \mid h^i(E) \neq 0\}$

$$\exists \Delta: \tau_{\geq m+1} E[-1] \rightarrow \tau_{\leq m} E \rightarrow E \rightarrow \tau_{\geq m+1} E$$

$$\otimes \downarrow \quad \text{sheaf} \downarrow \otimes \quad \downarrow \otimes \neq 0 \quad \downarrow \otimes$$

$$\tau_{\geq m+1} E[-1] \otimes L \rightarrow (\tau_{\leq m} E) \otimes L \rightarrow E \otimes L \rightarrow \tau_{\geq m+1} E \otimes L$$

14:00 ~ 真瀬

Families of K3 hypersurfaces in toric and non-toric Fano 3-fold

§1. Introduction, §2 Main Theorem, §3 Idea of proof

§1.
fix $\mathbb{P}^3 \supset \ell$ line
 $\cup C$ sm plane cubic

$\sigma: X' \xrightarrow{E} \mathbb{P}^3$ blow up along ℓ . $\sim X'$ sm toric Fano 3-fold

$\pi: X \xrightarrow{D} \mathbb{P}^3$ blow up along C . X : sm non-toric Fano 3-fold

$H = \mathcal{O}_{\mathbb{P}^3}(1)$

$S' \in -K_{X'} $	generic	$S \in -K_X $
$\text{Pic}(S') = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$	isometry \cong	$\text{Pic}(S) = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}$

$\{\sigma^*H, E\}$	\longrightarrow	$\{\pi^*H, D\}$
σ^*H	\longmapsto	π^*H
E	\longmapsto	$\pi^*H - D$

§2

Main Theorem For any $S' \in |-K_{X'}|$, at most ADE subj's.

$\exists S' \in |-K_{X'}|$ s.t. $S' \sim S$ and vice versa

<u>Rank</u>	(i)	$\dim -K_{X'} = 2g$	$\dim -K_X = 22$
	(ii)	" $S' \sim S$ " means	$\{\tau(\bar{S}')\} = \{\tau(\bar{S})\} \subseteq \Omega$

§ 3.

Step 1 Lem Let $\varphi: Y \rightarrow \mathbb{P}^2$ blow up along sm. plane curve Σ of degree 1, 2, 3.

If $M \in |-K_Y|$ has at most ADE sing's, then

(i) $\varphi(M) \in |-K_{\mathbb{P}^2} - \Sigma|$

(ii) $M, \varphi(M) : sm$

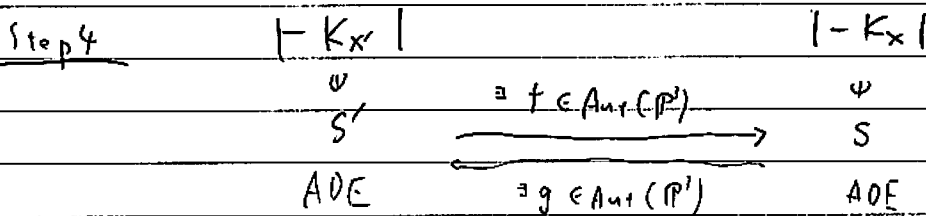
$\Rightarrow \varphi|_M : M \rightarrow \varphi(M) \quad 1:1 \text{ onto}$

$S \in |-K_Y| \quad sm \quad \rightsquigarrow \quad \pi(S) \supset C$
 sm

$\mathcal{S} := \{ S \in |-K_{\mathbb{P}^2} - C| \mid S : sm \overset{\text{elli. curve}}{\ni} C' \neq C \text{ s.t. } C' \subseteq S \}$

Step 2. \mathcal{S} is open dense in $|-K_{\mathbb{P}^2} - C|$

Step 3 $\mathcal{S} = \{ S \in |-K_{\mathbb{P}^2} - C| : sm \}$

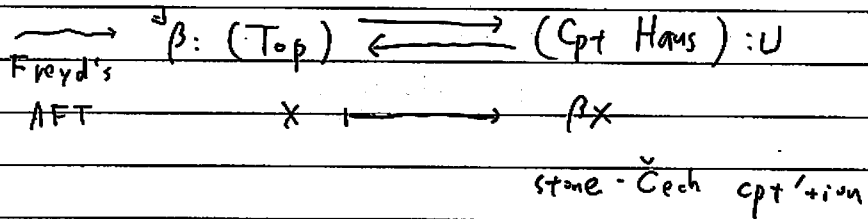


14:30 ~ 高木

A-schemes & its applications

§0 Intro

Topology (Cpt Haus) : complete (Tychonoff)
(\Rightarrow limit)



Compare (Sch) : not complete! e.g. $\prod_{\mathbb{N}} \mathbb{R}$

pl: Main results

X : top space is coherent (\Leftrightarrow) X : sober, g -cpt, g -sep
 $\Rightarrow g$ -cpt open basis
 (e.g. underlying sp of noeth schemes)

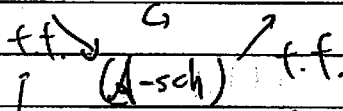
[Coh] := cat of coh sp & g -cpt morph

Thm (Stone duality)

(DLat) = cat of distributive lattices
 $\sim \xrightarrow{\text{Spec}} (\text{Coh}) \cong (\text{DLat})^{\text{op}}$

$X \mapsto C(X)_{\text{cpt}} = \{ Z \subset X : \text{closed, } X \setminus Z = g\text{-cpt} \}$

Thm [T-] (Coh.sch) $\xleftrightarrow{\text{f.f.}}$ (Coh.LRS)



preserves fiber product.
 finite patching via g -cpt open

$$\begin{array}{ccc} \text{Spec} : (\text{Ring}) & \xrightleftharpoons{\Gamma} & (\text{Coh Sch})^{\text{op}} : \Gamma \\ & \searrow \text{spec}^2 & \downarrow \text{f.t.} \\ & \Gamma & (\mathcal{A}\text{-sch})^{\text{op}} \end{array}$$

- $(\mathcal{A}\text{-sch})$
 - complete
 - co-complete
 - separatedness
 - properness

Thm [T-] S : base $\mathcal{A}\text{-sch}$

$$\mathbb{Z}R_S : (\mathcal{A}\text{-sch}/S) \xrightleftharpoons{\Gamma} (\text{proper } \mathcal{A}\text{-sch}/S) : U$$

$$X \longmapsto \mathbb{Z}R_S(X) : \text{zariski-Riemann sp.}$$

$$X \xrightarrow{\varepsilon} \mathbb{Z}R_S(X) \quad \text{If } X, S : \text{coh sch, } X \rightarrow S \text{ sp. of f.t.}$$

$\leadsto \varepsilon$: open imm

Cor $X \rightarrow S$: sep of f.t. of coh sch

(Nagata embedding) $\rightarrow X \xrightarrow{\varepsilon} Y$: open imm proper/S

§2 construction of \mathcal{A} -schemes

Def $X \in (\text{coh}) \leadsto$ "canonical" (DLat) -valued sheaf τ_X
 $U \mapsto \varinjlim_{V \subset U} C(V)_{\text{opt}}$

Def $\alpha_i : (\text{Ring}) \rightarrow (\text{DLat})$

$$R \mapsto \{ \text{f.g. ideal}/R \} / (\alpha_i = \alpha)$$

1/2 本

$$\begin{array}{ccc} \alpha_2^{\text{nat}} : R & \longrightarrow & \alpha_1 R \\ \psi & & \psi \\ a & \longmapsto & (a) \end{array}$$

Remark $\text{Spec } R = \text{Spec } \alpha_1 R$

Def An \mathcal{A} -scheme is a triple $X = (|X|, \mathcal{O}_X, \beta_X)$

- $|X|$: coh sp
- \mathcal{O}_X : (Ring)-valued sheaf / X
- $\beta_X : \alpha_1 \mathcal{O}_X \longrightarrow \tau_X$ "support morph"

s.t. $V \subset U$ g -cpt, open / X ($Z = U \setminus V$)

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\text{res}} & \mathcal{O}_X(V) \\ \downarrow & \searrow \cong & \uparrow \\ \mathcal{O}_X(U)_Z & : \text{localization alg} & \\ & \{ a \in \mathcal{O}_X(U) \mid \beta_X \alpha_2(a) \in Z \} & \end{array}$$

\mathcal{A} -morphism $X \rightarrow Y$ of \mathcal{A} -schemes is a pair $(f, f^\#)$

$$f : |X| \rightarrow |Y| : g\text{-cpt}$$

$$f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X \quad \text{s.t.} \quad \begin{array}{ccc} \alpha_1 \mathcal{O}_Y & \longrightarrow & \alpha_1 f_* \mathcal{O}_X \\ \downarrow \cong & & \downarrow \cong \\ \tau_Y & \xrightarrow{f_1} & f_* \tau_X \end{array}$$