アーク空間と特異点の不変量 II　　安田 健彦

復習

$X \rightsquigarrow X_n$ n-jet sch.　　$X_\infty = \varprojlim X_n$　arc space

explicit formula　X : smooth, D : SNC

$\int_{X_\infty} (\mathbb{L}^{\ldots})^{F_D}$

Birational transform

$$\begin{tikzcd} f : X' \arrow[r] & X \\ E \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \overline{E} \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

proper birational morph.

exc. set

$\rightsquigarrow f_\infty : X'_\infty \longrightarrow X_\infty$

<u>Lem</u>　$f_\infty : X'_\infty \setminus E_\infty \longrightarrow X_\infty \setminus \overline{E}_\infty$　is bijective

(∵) valuative criterion for properness

$$\begin{tikzcd} \mathrm{Spec}\, \mathbb{C}((t)) \arrow[r, dashed] \arrow[d] & X' \arrow[d] \\ \mathrm{Spec}\, \mathbb{C}[[t]] \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

<u>rmk</u>　$(X' \setminus E)_\infty \neq X'_\infty \setminus E_\infty$

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<u>Transformation rule</u>

(Change of variables formula)

Suppose X, X' smooth

$f : X' \longrightarrow X$ proper bir.

$K_f := K_{X'} - f^*K_X$, supported on the exc. locus. (→ uniquely determined)

<u>Thm</u> $F : X_\infty \supset A \longrightarrow \mathbb{Q}$

$$\int_A (u v)^F = \int_{f_\infty^{-1}(A)} (u v)^{F \circ f_\infty - F_{K_f}}$$

特に、$Y = \sum_i q_i Y_i$. $q_i \in \mathbb{Q}$, $Y_i \subset X$ closed subsch. に対して

$$\int_A (u v)^{F_Y} = \int_{f_\infty^{-1}(A)} (u v)^{F_{f^{-1}Y - K_f}}$$

<u>rmk</u> $f^{-1}Y = \sum_i q_i \underbrace{f^{-1}Y_i}$

sch. theoretic な ひきもどし.

$A = X_\infty$

$$E_{st}(X, Y) = E_{st}(X', f^{-1}Y - K_f)$$

これが 大もとの大切な thm.

mld (minimal log discrepancy)

For simplicity, suppose X normal ($\mathbb{Q}$-) Gorenstein var.

$$Y = \sum_i b_i Y_i \quad , \; Y_i \subset X$$

$f : X' \to X$ log resol.

$$\cancel{K_f =} \quad K_f - f^{-1}Y = \sum_i (a_i - 1) E_i$$

Def. $a(E_i ; X, Y) := a_i$ ← log discrepancy

$W \subset X$ closed subset

$$\mathrm{mld}(W; X, Y) := \inf\{ a(E; X, Y) \mid f : \underset{\substack{\cup \\ E}}{X'} \to X \;,\; f(E) \subset W \}$$

特に

$$(X, Y) \begin{array}{c} \text{klt} \\ \text{lc} \end{array} \underset{\text{def}}{\Longleftrightarrow} \mathrm{mld}(X; X, Y) \begin{array}{c} > 0 \\ \geq 0 \end{array}$$

Fact mld は 1つの log resol. で計算可.

$$f : X' \underset{a_i, E_i}{\longrightarrow} X$$

$$\mathrm{mld}(W; X, Y) = \min\{ a_i \mid f(E_i) \subset W \}$$

if $a_i \geq 0$ for $\forall i$.

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<u>Thm</u> X smooth, $Y = \sum_{i=1}^{r} q_i Y_i$, $\underline{Y} = (Y_1, \cdots, Y_r)$.

Then $\operatorname{mld}(X; Y) := \operatorname{mld}(X; X, Y) = \inf_{0 \neq \underline{m} \in \mathbb{Z}^r_{\geq 0}} \left\{ \operatorname{Codim}(F_{\underline{Y}}^{-1}(\underline{m})) - \sum q_i m_i \right\}$

∵ $f : X' \to X$ log resol.

$K_f - f^{-1}Y = D = \sum_{i=1}^{\ell} (a_i - 1) D_i$ とする.

~~$E = |f^{-1}Y|$~~ $E := |f^{-1}Y|$, $E = \bigcup_i D_i$

$$\int_{F_{\underline{Y}}^{-1}(>0)} (uv)^{F_Y} \overset{\text{trans. rule}}{=} \int_{F_E^{-1}(>0)} (uv)^{-F_D}$$

$$\overset{\text{explicit formula}}{=} \sum_{\phi \neq J \subset \{1, \cdots, \ell\}} E(D_J^\circ) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j} - 1} (uv)^{-d}$$

E と交わる arc だけを考える

deg $= 2 \max\{-a_j\}$

(簡単のため, $a_j > 0$ とする)

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一方,

$$\deg \int_{F_Y^{-1}(>0)} (uv)^{F_Y} = \max_{\underline{m}} \deg \int_{F_Y^{-1}(\underline{m})} (uv)^{F_Y}$$

各$\underline{m}$について, 最高次の係数が正.
よって, cancelは起きないので,
各$\underline{m}$についてのdegのmaxを考えて良い.

$$= \max_{\underline{m}} \deg \left(\mu(F_Y^{-1}(\underline{m}))(uv)^{\sum m_i q_i} \right)$$

$$= \max_{\underline{m}} \left\{ -2\,\mathrm{Codim}(F_Y^{-1}(\underline{m})) + 2\sum m_i q_i \right\}$$

$\boxtimes$

<u>lct (log canonical threshold)</u>

(X, Y) as before (X smooth)

Suppose $Y \geq 0$

<u>Def.</u> $\mathrm{lct}(X, Y) = \sup\{ c \mid (X, cY) \ \mathrm{lc} \}$

<u>Thm</u> X smooth, $Y \subset X$ closed subsch.

$$\Rightarrow \mathrm{lct}(X, Y) = \min_n \left\{ \frac{\mathrm{Codim}(Y_n, X_n)}{n+1} \right\}$$

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∵ $\operatorname{Codim}(Y_n, X_n) = \operatorname{Codim}(F_Y^{-1}(\geqslant n+1))$

$$\operatorname{mld}(X, cY) = \min_{m>0} \left\{ \operatorname{Codim}(F_Y^{-1}(m)) - cm \right\} \quad (\because \text{先の}\ \underline{\text{Thm}})$$

$$\therefore \operatorname{lct}(X, Y) = \min_{m>0} \left\{ \frac{\operatorname{Codim}(F_Y^{-1}(m))}{m} \right\}$$

$$\operatorname{Codim}(F_Y^{-1}(m)) \geqslant cm, \ \forall m$$

$$\Longleftrightarrow \operatorname{Codim}(F_Y^{-1}(\geqslant m)) \geqslant cm, \ \forall m$$

□

motivic integration over singular varieties

X^d singular var.

Prob. ~~X^n~~ $X_{n+1} \to X_n$ is not an $\mathbb{A}^d$-bdl.
not surjective

(cf) Grothendieck による smoothness の特徴づけ

Def. $C < X_\infty$ subset

C is stable if
- $\exists n$, $\pi_n(C)$ is constructible
- $C = \pi_n^{-1}(\pi_n(C))$
- $\forall m \geqslant n$ $\pi_{m+1}(C) \to \pi_m(C)$ is a piecewise trivial $\mathbb{A}^d$-bdl.

X_m loc. closed subsets による分割がある.

$\boxed{d = \dim X}$

For such C,

$$\mu(C) := E(\pi_n(C))\,(uv)^{-(n+1)d}$$

<u>Def.</u> $J_X < \mathcal{O}_X$ Jacobian ideal

J_X is locally generated by the $c \times c$ minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$,

$$X = (f_1(x) = \cdots = f_\ell(x) = 0) \subseteq A.$$

$$c = \operatorname{codim}(X, A)$$

$J_X = \mathrm{Fitt}_d(\Omega_X) \leftarrow d$-th Fitting ideal.

<u>Lem</u> $A < X_\infty$ cylinder $(A = \pi_n^{-1}(S),\ S < X_n$ constructible$)$

$\Rightarrow \forall n < \infty \quad A \cap F_{J_X}^{-1}(n)$ is stable.

<u>Def.</u> $F : X_\infty \supset A \longrightarrow \mathbb{Q}$ ~~is Cylindrical~~
Cylindrical

$$\int_A (uv)^F := \sum_{\substack{n \in \mathbb{Q}, \\ m \in \mathbb{Z}}} \mu\left(F^{-1}(n) \cap F_{J_X}^{-1}(m)\right) \cdot (uv)^n$$

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<u>Transformation rule</u>

$f : X' \longrightarrow X$ resol.

$J_f := \mathrm{Fitt}_0(\Omega_{Y/X}) \leftarrow$ 0-th Fitting ideal

$\mathrm{Im}(f^* \Omega_X^{\wedge d} \longrightarrow \Omega_{X'}^{\wedge d}) = J_f \Omega_{X'}^{\wedge d}$

(X smooth $\Rightarrow J_f$ is the defining ideal of K_f)

<u>Thm</u> $\displaystyle\int_A (u\nu)^F = \int_{f_\infty^{-1}(A)} (u\nu)^{F\circ f_\infty - F_{J_f}}$

<u>Key lem</u> $\gamma \in X_\infty$, $F_{J_f}(\gamma) = e < \infty$

$\Rightarrow \forall n \gg 0 \quad f_n^{-1} f_n \pi_n(\gamma) \cong \mathbb{A}^e \quad (f_n : X_n' \to X_n)$

~~dim~~ $X' = \mathbb{A}^1$ の場合

$\underline{f} : \mathbb{A}^1 \longrightarrow X \subset \mathbb{A}^n \qquad \underline{f} = (f_1, \cdots, f_n)$ def. poly. of $\underline{f}$

$\gamma \in \mathbb{A}^1_\infty = \mathbb{C}[\![t]\!]$

$\underline{f_\infty}(\gamma) = \underline{f}(\gamma)$

$\gamma' \in A'_\infty$ another arc

$\gamma' = \gamma + \varepsilon$

Taylor expansion

$$\underline{f}(\gamma') = \underline{f}(\gamma) + \mathrm{Jac}(\gamma)\cdot\varepsilon + \text{higher}$$

$$\mathrm{Jac} = \left(\frac{\partial f_1}{\partial x}, \dots, \frac{\partial f_n}{\partial x}\right)$$

$$e = \min_i \left\{ \mathrm{ord}\, \frac{\partial f_i}{\partial x}(\gamma) \right\}$$

$$f_n\pi_n(\gamma) = f_n\pi_n(\gamma')$$

$\Leftrightarrow \underline{f}(\gamma) = \underline{f}(\gamma') \quad \mathrm{mod}\ t^{n+1}$

$\Leftrightarrow \mathrm{Jac}(\gamma)\cdot\varepsilon + \text{higher} \equiv 0 \quad \mathrm{mod}\ t^{n+1}$

$\Leftrightarrow \mathrm{ord}\,\varepsilon \geqslant n+1-e$ ——— $n \gg 0$ とすれば higher order term が無視できる.

$$\therefore f_n^{-1} f_n \pi_n(\gamma) = \{\pi_n(\gamma') \mid f_n\pi_n(\gamma) = f_n\pi_n(\gamma')\}$$
$$\cong \{\pi_n(\varepsilon) \mid \mathrm{ord}\,\varepsilon \geqslant n+1-e\}$$
$$= \{\varepsilon \in \mathbb{C}[t]/(t^{n+1}) \mid \mathrm{ord}\,\varepsilon \geqslant n+1-e\}$$
$$\cong \mathbb{C}^e$$

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A variant of transformation rule

Suppose X Gorenstein

$\exists$ natural $\varphi : \Omega_X^{\wedge n} \longrightarrow \omega_X$

define $I_Z \subset \mathcal{O}_X$ by $\quad \mathrm{Im}\,\varphi = I_Z \omega_X$

$Z := \mathrm{Spec}(\mathcal{O}_X / I_Z) \subset X$ closed subsch.

$f : X' \to X$ resol.

$K_f \leftarrow \cdots \rightarrow \mathcal{K}$ fractional ideal

$\Rightarrow J_f = (f^{-1} I_Z)\mathcal{K}$

Cor $\displaystyle\int_A (uv)^{F_Z} = \int_{f_\infty^{-1}(A)} (uv)^{-F_{K_f}}$

$\because$ $\displaystyle\int (uv)^{F_Z} \underset{\text{trans. rule}}{=} \int (uv)^{F_{f^{-1}Z} - F_{J_f}} = \int (uv)^{-F_{K_f}}$ $\square$

Cor X, $Z \subset X$ as before

$W \subsetneq X$ closed subset. $Y = \sum q_i Y_i$, $Y_i \subset X$ closed subsch.

Then,

$$mld(W; X, Y) = \inf_{e,m} \left\{ \mathrm{Codim}\left(F_Y^{-1}(\underline{m}) \cap F_Z^{-1}(e) \cap F_W^{-1}(\geq 1) \right) - e - \sum q_i m_i \right\}$$

Inversion of adjunction

Thm X smooth $\supset D$ normal divisor

$Y = \sum q_i Y_i$, $D \not\subseteq \cup Y_i$, $W \subsetneq D$ closed

$\Rightarrow \mathrm{mld}(W; X, D+Y) = \cancel{\mathrm{mld}(W; Y, D)}\ \mathrm{mld}(W; D, Y|_D)$

☺ 左辺は X_∞の中で $F_D^{-1}(?)$や$F_Y^{-1}(?)$を見る.
一方,右辺は D_∞の中で $F_Z^{-1}(?)$や$F_Y^{-1}(?)$を見る.

直前の Lem を使う.

Key Lem $m > e$

- $\pi_m(F_Z^{-1}(e)) = \mathrm{Im}(D_{m+e} \to D_m) \cap F_{Z,m}^{-1}(e)$, $\subseteq D_m$

- $\pi_{X,m+e,m}^{-1}(\quad) \subset F_{Z \subset D, m+e}^{-1}(e)$

$\pi_{X,m+e,m} : X_{m+e} \to X_m$