アーク空間と特異点の不変量 I　　安田 健彦

- Kontsevich
- Denef-Loeser
- Batyrev
- Mustaţă
- Ein, Yasuda, Lazarsfeld
- Kawakita

jets & arcs

X : cpx alg var

Def: $n \in \mathbb{Z}_{\geq 0}$　n-jet on X := a morphism $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \longrightarrow X$

n-jet scheme $X_n := \{\text{n-jets on } X\}$ a separated scheme of finite type $/\mathbb{C}$

ex　$X_0 = X$, $X_1 = TX$

$n' \geq n$　$\mathbb{C}[t]/(t^{n'+1}) \twoheadrightarrow \mathbb{C}[t]/(t^{n+1})$　truncation map

$\operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \hookrightarrow \operatorname{Spec} \mathbb{C}[t]/(t^{n'+1})$　closed imm　より

natural morphism $X_{n'} \longrightarrow X_n$ が定まる

Def　arc on X := a morphism $\operatorname{Spec} \mathbb{C}[[t]] \longrightarrow X$

arc space $X_\infty := \{\text{arcs on } X\}$

$X_\infty = \varprojlim X_n$ になる　※　$\mathbb{C}[[t]] = \varprojlim \mathbb{C}[t]/(t^{n+1})$

Defining eqs of X_n

$X = (f = 0) \subset \mathbb{C}^m$　　$f = (f_1, \dots, f_r)$　　$x = (x_1, \dots, x_m)$

i.e. $X = \operatorname{Spec} \mathbb{C}[x]/(f)$

n-jet　$\gamma : \mathbb{C}[x]/(f) \longrightarrow \mathbb{C}[t]/(t^{n+1})$

$\Leftrightarrow$ $\gamma : \mathbb{C}[x] \longrightarrow \mathbb{C}[t]/(t^{n+1})$ s.t. $\gamma(f) = 0$

$\Leftrightarrow$ $\gamma = (\gamma_1, \dots, \gamma_m) \in (\mathbb{C}[t]/(t^{n+1}))^m = \mathbb{C}^{m(n+1)}$ s.t. $f(\gamma) = 0$

$f_i(\gamma) = \sum_{j=0}^{n} a_j t^j$　a_j : polynomial in coeff of γ と書ける　$a_j = 0$.

ex $\quad X = \{ y^2 - x^3 = 0 \} \subset \mathbb{C}^2$

$X_1 = \{ (\gamma_1, \gamma_2) \in (\mathbb{C}[t]/(t^2))^2 \mid \gamma_2^2 - \gamma_1^3 = 0 \}$

$\gamma_1 = a + bt \quad \gamma_2 = c + dt$ とおくと、

$X_1 = \{ (a,b,c,d) \mid a^2 - c^3 = 0, \ 2ab - 3c^2 d = 0 \}$

特に、 $X = \mathbb{A}^d \Rightarrow X_n = \mathbb{A}^{(n+1)d}$

lem $\quad Y \to X$ étale $\Rightarrow$

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X_n \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

: cartesian diagram

Cor $\quad X$: smooth $\Rightarrow$ $X_{n+1} \to X_n$ is a Zariski loc. trivial $\mathbb{A}^d$-bdle

Rem "Xの特異点が悪い $\longleftrightarrow$ X_nが大きい"

Th (Mustata, Ein-Mustata-Y, Ein-Mustata)

X : normal l.c.i, then X : terminal / canonical / lc $\iff$ $\forall m$ X_m normal / irreducible / equidim

motivic integration on smooth varieties

E-polynomial (Hodge-Deligne polynomial)

V: cpx var. $\quad E(V) := \sum_{i,p,q} (-1)^i h^{pq}(H^i_c(V, \mathbb{Q}))\, u^p v^q$

Basic properties

1. $E(\phi) = 0$
2. $E(V \times W) = E(V) E(W)$
3. $V \underset{\text{closed}}{\subset} W \quad E(W) = E(V) + E(W \setminus V)$
4. $\deg E(V) = 2 \dim V$
5. $E(\mathbb{A}^n) = (uv)^n$

3 $\rightsquigarrow$ $C \subset V$ constructible set $\quad C = \sqcup C_i \quad C_i \subset V$ loc. closed

$E(C) := \sum_i E(C_i)$

motivic measure

Suppose X smooth $\dim X = d$.

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \qquad \pi_n : X_\infty \longrightarrow X_n$$

$\mathbb{A}^d$-bdle

Def $C \subset X_\infty$ subset

C : cylinder if $\exists A \subset X_n$ constructive subset s.t. $C = \pi_n^{-1}(A)$

C : cylinder $\rightsquigarrow$ $\mu(C) = E(A)(uv)^{-(n+1)d} \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$

Prop μ is a finite additivity

i.e. $C = \sqcup C_i$ finite disjoint C, C_i cylinder $\quad \mu(C) = \sum_i \mu(C_i)$

Why "motivic"?

本来 $\mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ ではなく $K_0(Var)$ に値をとる

$K_0(Var) := \bigoplus_{[V]:\ \text{variety 同型類}} \mathbb{Z}[V] \,/\,$ relation $V \underset{\text{closed}}{\subset} W \quad [W] = [V] + [W \setminus V]$: Grothendieck ring of varieties

$$[V]\cdot[W] = [V \times W]$$

$K_0(Var)$ the set of poor man's motives (by Drinfeld)

Integration

$F : X_\infty \supset A \longrightarrow \mathbb{Z}$ function on A s.t. $\forall n \in \mathbb{Z}$ $F^{-1}(n)$ cylinder

$$\int_A (uv)^F := \sum_{n \in \mathbb{Z}} \mu(F^{-1}(n))(uv)^n \quad \text{not allways well-defined}$$

order function & contact locus

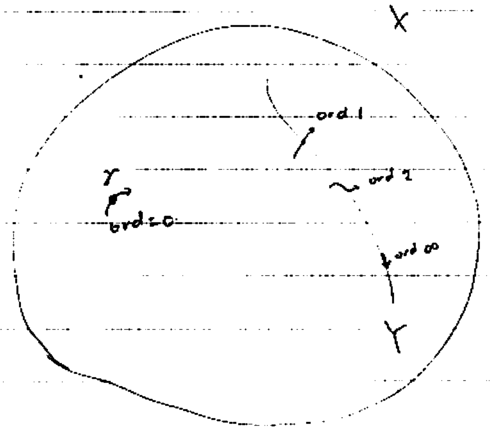
$Y \subset X$ closed subscheme defined by $I \subset \mathcal{O}_X$

$\gamma : \operatorname{Spec} \mathbb{C}[\![t]\!] \longrightarrow X \quad \in X_\infty$

$\mathbb{C}[\![t]\!] \supset \gamma^{-1} I = (t^m) \quad \exists m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, ただし $t^\infty = 0$

Def $\operatorname{ord}_\gamma(I) = m$ order of I at γ

Notation $F_Y(\gamma) = \operatorname{ord}_\gamma(I) \qquad F_Y : X_\infty \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$



lem $F_Y^{-1}(\infty) = Y_\infty \subset X_\infty$

∵) $X = \operatorname{Spec} R$ $\quad \gamma : R \to \mathbb{C}[[t]]$

$$\gamma \in Y_\infty \iff \gamma(I) = 0 \iff F_Y(\gamma) = \infty$$

lem $m < \infty$ $\quad F_Y^{-1}(m)$ is a cylinder

(∵ contact locus)

∵) 同様に $F_{Y_n} : X_n \to \{0, 1, 2, \dots, n, \infty\}$ が定まる

$$\begin{array}{ccc} X_\infty & \longrightarrow & X_{m-1} \\ \cup & & \cup \\ F_Y^{-1}(\geq m) & \longrightarrow & Y_{m-1} \end{array}$$

∴ $F_Y^{-1}(\geq m)$ cylinder.

∴ $F_Y^{-1}(m) = F_Y^{-1}(\geq m) \setminus F_Y^{-1}(\geq m+1)$ cylinder

従って $\int_{X_\infty} (uv)^{F_Y}$ を考えられる.

ex $X = \mathbb{A}^1 \ni 0 = Y$

$$X_\infty = \mathbb{C}[[t]] \qquad X_n = \mathbb{C}[t]/(t^{n+1})$$

$$F_0^{-1}(n) = \{\text{power series of order } n\} \subset \mathbb{C}[[t]]$$

$$\mu(F_0^{-1}(n)) = E(\underbrace{\pi_n(F_0^{-1}(n))}_{\cong\, \mathbb{C}^*})(uv)^{-n-1} = (uv-1)(uv)^{-n-1}$$

$$\int_{X_\infty} (uv)^{-F_0} = \sum_{n \geq 0} \mu(F_0^{-1}(n))(uv)^{-n}$$

$$= \sum_{n \geq 0} (uv-1)(uv)^{-2n-1}$$

$$= (uv-1)\frac{uv}{(uv)^2-1} = \frac{uv}{uv+1}$$

codim & degree

Def X : smooth var

$C \subset X_\infty$ cylinder $\qquad \operatorname{codim}(C) := \operatorname{codim}(A, X_n)$

$\quad \overset{\shortparallel}{\pi_n^{-1}(A)}$

lem $C \subset X_\infty$ cylinder

$$\operatorname{codim}(C) = -\frac{1}{2} \deg \mu(C)$$

$Y = \sum_i a_i Y_i, \quad a_i \in \mathbb{Q} \qquad Y_i \subset X$ closed subscheme

$\rightsquigarrow \quad F_Y = \sum a_i F_{Y_i} : X_\infty \dashrightarrow \mathbb{Q}$

defined over $X_\infty \setminus \bigcup Y_{i,\infty}$

$\rightsquigarrow \quad \int (uv)^{F_Y}$

explicit formula

$D = \sum_{i=1}^r a_i D_i$ SNC div on $X \qquad \underline{D} = (D_1, \dots, D_r)$

$F_{\underline{D}} = (F_{D_1}, \dots, F_{D_r}) : X_\infty \longrightarrow (\mathbb{Z}_{\geq 0} \cup \{\infty\})^r$

$\underline{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$

Prop $$\mu(F_{\underline{D}}^{-1}(\underline{m})) = E(D_J^\circ)(uv-1)^{|J|}(uv)^{-\sum m_i - d}$$

ここで $J = \{i \mid m_i > 0\} \qquad D_J^\circ = \bigcap_{i \in J} D_i \setminus \bigcup_{j \notin J} D_j$

Cor $\operatorname{codim}(F_{\underline{D}}^{-1}(\underline{m})) = \sum m_i$.

Cor $D = \sum a_i D_i \quad (a_i < 1)$

$$\int_{X_\infty} (uv)^{F_D} = \sum_{J \subseteq \{1, \dots, r\}} E(D_J^\circ) \prod_{j \in J} \frac{uv - 1}{(uv)^{1+a_j} - 1} (uv)^{-d} \quad \left(=: E_{st}(X, D) \right)$$

これの係数 = stringy Hodge number