What ideals are multiplier ideals?

1. Intro.
2. dim = 2 (Lipman − W)
3. syzygies of multiplier ideals
   (after Lazarsfeld & K. Lee)

§1. Intro

2002. Sept. MSRI
Lazarsfeld's lecture.

$I \subset A$, A : normal $\mathbb{Q}$-Gor. ("char = 0")

$f : X \to \mathrm{Spec}(A)$ log resol.

$I\mathcal{O}_X = \mathcal{O}(-F)$: invertible

$\underbrace{\mathcal{J}(I^c)}_{\text{Integrally closed}} := H^0(X, \mathcal{O}_X(\ulcorner K_{X/A} - cF \urcorner))$

converse?

Ex $I \subset k[x_1, \cdots, x_n]$: int. closed gen. by monomials

(Howald's Thm)

$$\underline{x}^m \in \mathcal{J}(I^c)$$

$$\Leftrightarrow m + \underset{(1,\cdots,1)}{\underset{\parallel}{\mathbb{1}}} \in \mathrm{Int}(c \cdot \underbrace{P_I}_{\text{Newton polygon}})$$

$$\Rightarrow \exists J,\ I = \mathcal{J}(\cancel{I}\, J^c)$$

§2. dim = 2.

Thm [J. Lipman – W (Favre – M. Jenssen)]

(A, m) : reg. local of $\dim A = 2$, A/m : alg. closed.

↳ log term.?

I : int. closed $\Rightarrow \exists J$, $I = \mathcal{J}(\not\exists J^c)$

証明の方法

$I \subset A$

$X \longrightarrow \mathrm{Spec}(A)$: log resol.

$\cup$

$E = \cup E_i$: excep. curves

$\boxed{E_i \cdot E_j : \text{int. number}}$

$Z = \sum n_i E_i^{>0}$ $(n_i \in \mathbb{Z})$

$I\mathcal{O}_X = \mathcal{O}_X(-Z) \Rightarrow Z$ anti-nef

$(\overset{\text{def}}{\Leftrightarrow} \forall i,\ Z \cdot E_i \leq 0)$

更に A : rational sing. のとき.

$\{X$上の anti-nef cycle$\} \overset{1:1}{\longleftrightarrow} \{I \subset A$: int. closed s/t $I\mathcal{O}_X$: inv.$\}$

(A, m) : reg. 2-dim. local

Zariski's Unique Factorization

2-dim. rat'l sing. のとき

I, J : int. closed $\Rightarrow IJ$: int. closed

I : int. closed (m-primary)

$\Rightarrow I = P_1^{a_1} \cdots P_s^{a_s}$ ($P_1, \cdots, P_s$: indecomp.)

説明

$X \to \mathrm{Spec}(A)$
1点の blow-up を繰返して得られる

$$I\mathcal{O}_X = \mathcal{O}_X(-Z)$$

$\{$ indecomp. $\mathfrak{p}_i$ s/t $\mathfrak{p}_i \mathcal{O}_X$: inv. $\}$

$\updownarrow$ 1:1

$\{ E_i \subset X \mid \text{exc} \}$

$M = (E_i \cdot E_j)_{i=1,\cdots,n}$: unimodular

$-M^{-1} = (n_{ij})$ 整係数

$W_i = \sum n_{ij} E_j$

(第 i 行) $\begin{cases} W_i \cdot E_i = -1 \\ W_j E_j = 0 \quad (\forall j \neq i) \end{cases}$

$\mathfrak{p}_i = H^0(X, \mathcal{O}_X(-W_i))$
$Z \cdot E_i = -n_i$ のとき $I = \prod_{i=1}^{n} \mathfrak{p}_i^{n_i}$

Ex I : indecomposable

$\boxed{I = \mathcal{J}(J^c)}$ J, $c = 1 + \varepsilon$ ε : 十分小さな正数

構成

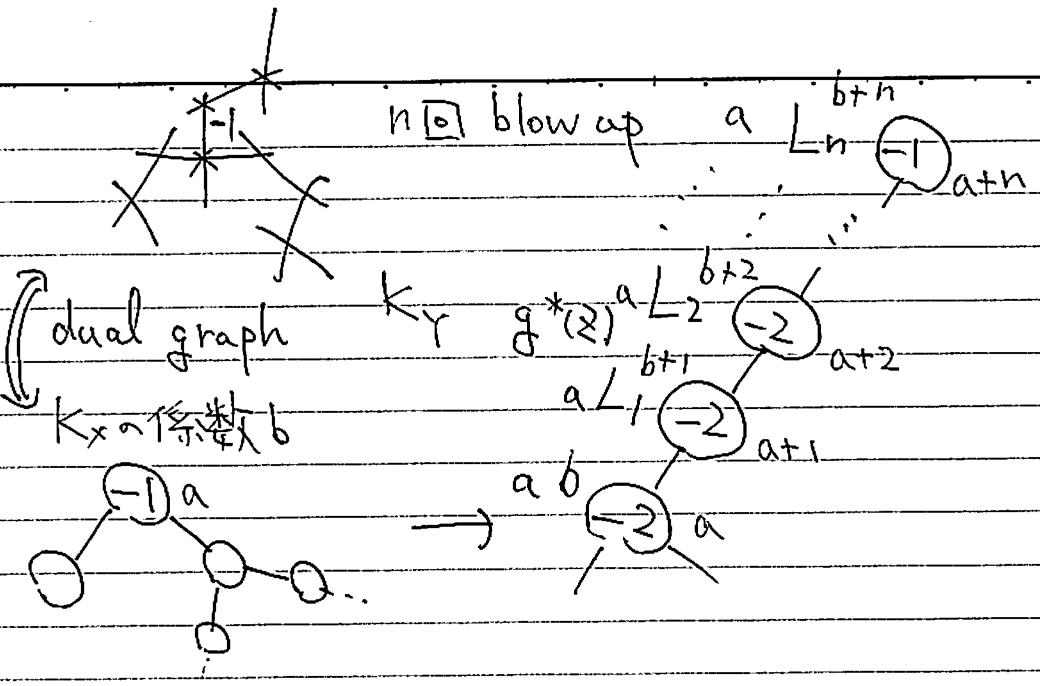
X : min log resol. of I

$\uparrow$

Z : cycle

$\Rightarrow \exists_1$ E : exc
$E^2 = -1$

$Z \cdot E = -1$, $Z \cdot E' = 0$ $(\forall E' \neq E)$



$g: Y \to X$: n回 blow-up

W :⟷ L_n に対応する indecomposable　　i.e. $\begin{cases} W \cdot L_n = -1 \\ W \cdot E' = 0 \quad (\forall E' \neq L_n \text{ on } Y) \end{cases}$

ZとWのちがい

$c = 1 + \varepsilon$を与える
→ nを決める
次のように決められる
① $\lfloor cW \rfloor - K_Y \leq g^*(Z)$
② 両辺のL_nの係数は同じ

書いてないcycle
W, Z同じ係数

$$H^0(Y, \mathcal{O}_Y(-\lfloor cW \rfloor + K_Y)) = I$$

anti nef. cycle
Z_1, Z_2 : anti nef
⇒ $\inf(Z_1, Z_2)$: anti nef　(easy ~~ex~~)

Z' : not anti nef
⇒ $\left[\begin{array}{l} (Z')^- : \text{min. anti nef cycle} \\ \qquad \geq Z' \end{array} \right.$
anti nef closure of Z'

①,② ⇒ $([CW]-K_Y)^- = g^*(z)$

⇒ $g(J^c) = I$

がいえる

残り given ε(十分小)

$\exists n$ s/t ① ②

$$[(1+\varepsilon)(a+n)] - (b+n) = a$$

$$(\cancel{a}+\cancel{n}) - (b+\cancel{n}) + [\varepsilon(a+n)] = \cancel{a}$$

$$1 > \varepsilon(a+n) - b \geq 0$$

General case

$I = H^0(X, \mathcal{O}_X(-z))$

$zE_i = -m_i \ (i=1,\cdots,s)$

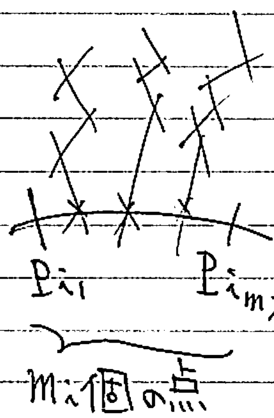
$(I = P_1^{m_1}\cdots P_s^{m_s})$

Y, J の構成

n_i回 blow up

最后の (-1) curve

$L_{1,n_i},\cdots, L_{m_i n_i}$



E_i → すべてのE_iに対して行う

W: $L_{1n_i},\cdots, L_{m_i n_i}$ とすべて

-1で交り 外はすべて0

$\Sigma m_i n_i$ 回の blow up $Y \xrightarrow{g} X$

与えた $\varepsilon > 0$ (十分小) に対し $c = 1+\varepsilon$

① $[cW] - K_Y \leq g^*(Z)$
② 両者のすべての (-1) curve ($\sum_{i=1}^{s} m_i$ 本) での係数が
等しくなるように $n_1, \cdots, n_s$ を決められる

$\Rightarrow ([cW] - K_Y) = g^*(Z)$ となり、

$J = H^0(Y, \mathcal{O}_Y(-W))$ とおけば $\boxed{\mathcal{J}(J^c) = I}$

以上が regular case.

2-dim. log terminal

$\pi : X_0 \to \operatorname{Spec}(A)$: min resol

$$K_{X_0} \underset{\text{num.}}{\equiv} \sum a_i E_i \qquad -1 < a_i < 0$$

$$K_{X_0} E_i = -2 - E_i^2 \geq 0$$

$$\ulcorner K_{X_0} \urcorner = 0$$

If $I\mathcal{O}_{X_0} = \mathcal{O}_{X_0}(-Z)$ のとき

$$\underset{\substack{\shortparallel \\ I}}{\mathcal{J}(I)} = H^0(X_0, \mathcal{O}_{X_0}(-Z - \underset{\substack{\shortparallel \\ 0}}{\ulcorner K_{X_0} \urcorner}))$$

in general $X \xrightarrow{f} X_0 \xrightarrow{\pi} \operatorname{Spec}(A)$

$I\mathcal{O}_X = \mathcal{O}_X(-Z)$

$Z = g^*(g_*(Z)) + Y$ where $g(Y)$: points

Zに対して、Yに対する regular case を適用できる。

$\forall \varepsilon > 0$ (十分小)

Yを reg. の場合の cycle と考えて、同じ操作の行ったものをW
($c = 1 + \varepsilon$)

$[cW] - \ulcorner K_{X'} \urcorner = \varphi^*(Z)$

なるように blow-up の回数を定める。

($\mathfrak{m}$-primary case)

How about gen. case in dim = 2?

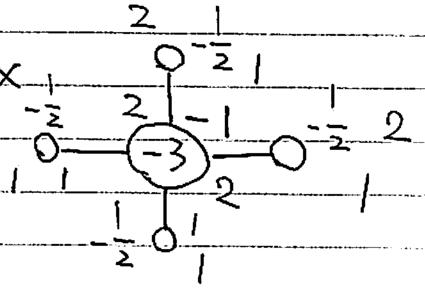
$(A, \mathfrak{m})$: not log terminal

$g(A) \subsetneq A$

虫の良い予想. $\forall I$: int. closed

$\subset g(A)$, $\exists J$, $\exists c$, $g(J^c) = I$?

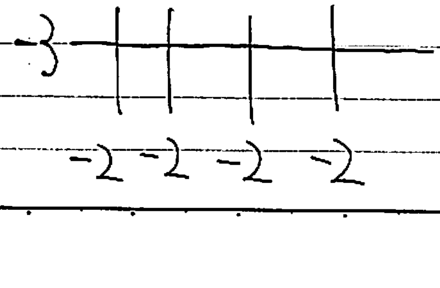
Ex



$\bigcirc = (-2)$

min. resol.

rational sing.



$I \neq \mathcal{J}(J^c)$

もしあったとすると、 $\exists g : X \to X_0$

$\exists$ anti-nef cycle W on X

s/t

$(\lfloor cW \rfloor - \lceil K_X \rceil)^- = Z$

§3. Syzygies of multiplier ideals [Lazarsfeld, Kyungyong Lee Invent 160 (2007)]

$(A, \mathfrak{m})$: reg. local "char = 0"

$I = \mathcal{J}(\mathfrak{b}^c)$

I の minimal free resol を考える.

$$0 \to F_{d-1} \to \cdots \to F_s \xrightarrow{u_s} F_{s-1} \to \cdots \to F_1 \to F_0 \to I \to 0$$

$e \longmapsto$

<u>Thm A</u> min. gen. of s-th syzygy

$\notin \mathfrak{m}^{d+1-s} F_{s-1}$

<u>Ex</u> $d \geq 3$ $I = (f, g)$: reg. seq.

radical ideal ⇒ int. closed

1-st syz $\begin{pmatrix} -g \\ f \end{pmatrix} \notin \mathfrak{m}^d$

$s=1$ If I is multiplier ideal

<u>Ex</u> $J \subset k[X_1, \cdots, X_d]$ homog. radical ideal

$J_k := J + \mathfrak{m}^k$

$\Rightarrow J_k$ is int. closed $(\forall k \geq 1)$

$(A, \mathfrak{m})$: 2-dim. reg.
I: int. closed $\mathfrak{m}$-primary
$\Rightarrow \operatorname{ord}(I) = \mu(I) - 1$

$\mu(I)$: 生成元の個数

$\operatorname{ord}(I)$: $I \subset \mathfrak{m}^s$ なる最小

$$0 \to A^{\mu-1} \longrightarrow A^{\mu} \longrightarrow I \to 0$$

$$M = (a_{ij})$$

Hilbert - Burch Thm
I is gen. by max. minors of M

(proof) $\mu(I) = \ell_A(A/\mathfrak{m}I) - \ell_A(A/I)$
$I = H^0(X, \mathcal{O}_X(-Z))$
$\mathfrak{m} = H^0(X, \mathcal{O}_X(-Z_0))$
$\ell_A(A/I) = -(Z^2 + K_X Z)/2$

$$\mu(I) = -\frac{1}{2}\left((Z+Z_0)^2 + K_X(Z+Z_0)\right) + \frac{1}{2}(Z^2 + K_X Z_0)$$

$$= \underbrace{-Z \cdot Z_0}_{\substack{\operatorname{ord} \\ 最初のblow\,up}} - \underbrace{\frac{Z_0^2 + K_X Z_0}{2}}_{=-1}$$

$$\boxed{\frac{Z_0^2 + K_X Z_0}{2} + 1 = 0}$$
$$\parallel$$
$$p_a(Z_0)$$

最初のblow up -1

証明のkey point

$$u_s(e) \notin \mathfrak{m}^{d-s+1}$$

Thm B $\mathfrak{a} = (h_1, \cdots, h_r)$

$K_\bullet = K_\bullet(h_1, \cdots, h_r)$: Koszul complex

$\Rightarrow 0 \leq {}^\forall s \leq r$. the natural map

$H_s(K_\bullet(\cancel{x}h_1, \cdots, h_r) \otimes \mathfrak{a}^{r-s} \mathcal{J}(\mathfrak{b}^c))$

$\to H_s(K_\bullet(h_1, \cdots, h_r) \otimes \mathcal{J}(\mathfrak{b}^c))$ は 0 map

$\mathfrak{a} = \mathfrak{m} = (z_1, \cdots, z_d)$: reg. parameters $\to$ A : reg. 要せ…

$K_\bullet(z_1, \cdots, z_d)$ は $A/\mathfrak{m}$ の min. free resol.

Cor. C The natural map

$\mathrm{Tor}_s(\mathfrak{m}^{d-s} I, A/\mathfrak{m}) \to \mathrm{Tor}_s(I, A/\mathfrak{m})$ is 0 ⌋

$I = \mathcal{J}(\mathfrak{b}^c)$

Thm D $n \geq 2$ s/t $u_s(e) \in \mathfrak{m}^n \cdot F_{s-1}$ (e, min free gen)

$\Rightarrow$ e は $\mathrm{Im}[\mathrm{Tor}_s(\mathfrak{m}^{n-1} I, A/\mathfrak{m}) \to \mathrm{Tor}_s(I, A/\mathfrak{m})]$ の元.

Thm D + Cor C $\Rightarrow$ Thm A