F-singularities 入門 　　　　高木 俊輔

§ No boundary case

Defn (tight closure, Frobenius closure)

R: noeth. domain of char $p>0$

$I \subseteq R$

$x \in I^F \Leftrightarrow {}^\exists q = p^e$, $x^q \in I^{[q]} = (a^q \mid a \in I)$ $({}^\forall q = p^e \gg 0)$

$x \in I^* \Leftrightarrow {}^\exists c \neq 0 \in R$ $c x^q \in I^{[q]}$ for ${}^\forall q = p^e \gg 0$

$$I \subseteq I^F \subseteq I^* (\subseteq \bar{I})$$

M: R-module (not nec. f.g.)

$(0)^F_M \subseteq M$

$z \in (0)^F \Leftrightarrow z \otimes 1 = 0 \in M \otimes_R R^{\frac{1}{q}}$ for ${}^\forall q = p^e \gg 0$

$z \in (0)^* \Leftrightarrow {}^\exists c \neq 0 \in R$

$z \otimes c^{\frac{1}{q}} = 0 \in M \otimes_R R^{\frac{1}{q}}$ for ${}^\forall q = p^e \gg 0$

rmk $I^*/I = (0)^*_{R/I}$, $I^F/I = (0)^F_{R/I}$

Defn-Prop

$(R, \mathfrak{m})$: F-finite local domain of char $p>0$

i.e. $F: R \to R$ $x \mapsto x^p$ is finite.

Assume $R = {}^{RLR}/\sim$

~~$F: R \to R \ x \mapsto x^p$~~

(1) F-pure

R: F-pure $\Leftrightarrow R \hookrightarrow R^{\frac{1}{p}}$ splits as R-mod

$\Leftrightarrow {}^\forall M$: R-module $M \to M \otimes R^{\frac{1}{p}}$: inj

$\Leftrightarrow H^d_{\mathfrak{m}}(\omega_R) \to \underline{H^d_{\mathfrak{m}}(\omega_R) \otimes R^{\frac{1}{p}}}$: inj

$H^d_{\mathfrak{m}}(\omega_R) \updownarrow K_X$, $\quad H^d_{\mathfrak{m}}(\omega_R) \otimes R^{\frac{1}{p}} \cong H^d_{\mathfrak{m}}(\omega_R^{(p)}) \updownarrow pK_X$

$\underset{\sim\sim\sim\sim}{\Leftrightarrow} I^F = I$ for ${}^\forall I \subseteq R$

(☆) $R \hookrightarrow R^{\frac{1}{p}}$ splits

$\Leftrightarrow \mathrm{Hom}(R^{\frac{1}{p}}, R) \to \mathrm{Hom}(R, R) = R$: surj

$\underset{\text{local duality}}{\Leftrightarrow} H^d_{\mathfrak{m}}(\omega_R) \to H^d_{\mathfrak{m}}(\omega_R) \otimes R^{\frac{1}{p}}$: inj

(2) strongly F-regular

R: strongly F-regular

$\Leftrightarrow {}^\exists c \neq 0 \in R \quad \underbrace{{}^\exists q = p^e}$ $\leftarrow \left(\begin{array}{c} {}^\exists q_0 \in \mathbb{N} \\ {}^\forall q \geq q_0 \end{array} \right)$

s.t. $R \hookrightarrow R^{\frac{1}{q}}$ $\ 1 \mapsto c^{\frac{1}{q}}$ splits as an R-mod

$\Leftrightarrow (0)^* = (0)$ in $H^d_{\mathfrak{m}}(\omega_R)$

$\Rightarrow I = I^*$ for ${}^\forall I \subseteq R$

逆は conj (If R is $\mathbb{Q}$-Goren $\Rightarrow$ o.K.)

・non $\mathbb{Q}$-Goren locus が isolated
・R: $\mathbb{N}$-graded ring
なら o.K.

(☆) $x \in I^*$

$^{\exists}c \neq 0 \in R \quad cx^q \in I^{[q]}$ for $^{\forall}q = p^e \gg 0 \rightsquigarrow c^{\frac{1}{q}} x \in IR^{\frac{1}{q}}$

R: str. F-reg なので $^{\exists}\varphi : R^{\frac{1}{q}} \to R \quad c^{\frac{1}{q}} \mapsto 1$

$x = \varphi(c^{\frac{1}{q}} x) \in \varphi(IR^{\frac{1}{q}}) \subseteq I$

//

(3) F-rational

R: F-rat'l $\Leftrightarrow I = I^*$ for $^{\forall (\exists)} I \subseteq R$ parameter ideal

$\Leftrightarrow R$: CM かつ $(0)^* = (0)$ in $H^d_m(R)$

Cohen capturing

$x_1, \cdots, x_i, x_{i+1}$: part of s.o.p. for R

$(x_1, \cdots, x_i) : x_{i+1} \subseteq (x_1, \cdots, x_i)^*$

② $H^d_m(R) = \varinjlim R/(x_1^t, \cdots, x_d^t)$ $\quad (x_1, \cdots, x_d)$: s.o.p for R

str. F-reg $\Rightarrow$ F-pure

$\Uparrow$ Gor $\Downarrow$

F-rat'l

$H^i_m(R) \to H^i_m(R) \otimes R^{\frac{1}{p}}$

$\cong$

(4) F-injective

R: F-inj $\Leftrightarrow \boxed{F : H^i_m(R) \to H^i_m(R)}$: inj for $^{\forall}i$

If R is C.M.

$\Leftrightarrow I^F = I$ for $^{\forall (\exists)} I \subseteq R$: parameter ideal

$$
\begin{tikzcd}
\text{Str. F-reg} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{F-pure} \arrow[d, Rightarrow] \\
\text{F-rat'l} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[u, bend left, "\mathbb{Q}\text{-Gor}"] & \text{F-injective} \arrow[d, Rightarrow] \arrow[u, bend right, "\mathbb{Q}\text{-Gor}"'] \\
\text{CM} & \cancel{\text{weakly normal}}
\end{tikzcd}
$$

($R \cong \omega_R$)

normal

semi normal

X: normal

<u>char. 0 の 特異点</u>

$$
\begin{tikzcd}
\text{lt} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{lc} \arrow[d, dashed] \\
\text{rat'l} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[u, bend left, "1\text{-Gor}"] & \text{Du Bois} \arrow[u, bend left, "1\text{-Gor}"] \arrow[d, Rightarrow] \\
\text{CM} & \text{seminormal (weakly normal)}
\end{tikzcd}
$$

conj (Kollar) (If X is CM $\Rightarrow$ O.K)

Kovacs

<u>Thm</u> (Hara, Schwede, Smith, Watanabe)

R: reduced local ring ess of finite type $/\mathbb{C}$

R_p: reduction of R to char $p > 0$

(1) If R_p is F-inj for infinitely many $p \Rightarrow \operatorname{Spec} R$: Du Bois

If R_p is F-rat'l for infinitely many $p \Rightarrow \operatorname{Spec} R$: rat'l sing

(2) Assume R is $\mathbb{Q}$-Goren + normal

If R_p is F-pure ——————— $\Rightarrow \operatorname{Spec} R$: lc

If R_p is str.F-reg ——————— $\Rightarrow \operatorname{Spec} R$: lt

Thm (Hara-Mehta-Srinivas)

R : rat'l sing $\Leftrightarrow$ R_p is F-rat'l for $\forall p \gg 0$

R : lt $\Leftrightarrow$ $\begin{cases} R_p \text{ is str.F-reg for } \forall p \gg 0 \\ R : \mathbb{Q}\text{-Goren} \end{cases}$

Conj

(1) R : lc $\Leftrightarrow$ R : $\mathbb{Q}$-Goren かつ R_p is F-pure for infinitely many p

R : DuBois $\Leftrightarrow$ R_p is F-inj for infinitely many p

(2) If R is not $\mathbb{Q}$-Goren

R : lt in the sense of de Fernex-Hacon $\Leftrightarrow$ R_p is str. F-reg for $\forall p \gg 0$

Nakayama $\Rightarrow$ O.K.

X : lt $\Leftrightarrow$ $^{\exists}\Delta \geq 0$: $\mathbb{Q}$-divisor $K_X + \Delta$: $\mathbb{Q}$-Cartier

(X, Δ) : KLT

$\Downarrow$

(X_p, Δ_p) : str. F-reg

$\Downarrow$

X_p : str. F-reg

$K_X + \Delta$: $\mathbb{Q}$-Cartier

str. F-reg = klt

R : str. F-reg

$^{\exists}\Delta \geq 0$ $\mathbb{Q}$-div s.t. $K_X + \Delta$: $\mathbb{Q}$-Cartier

(R, Δ) : str. F-reg

<u>ex</u>. $R = \mathbb{F}_p[x.y.z]/x^3+y^3+z^3 \quad p \neq 3$

R: F-pure $\Leftrightarrow p \equiv 1 \mod 3$

$R = \mathbb{F}_p[x.y.z]/x^2+y^3+z^5$

R: str. F-reg $\Leftrightarrow p \geq 7$

<u>rmk</u> If conj(1) (lc case) holds true

$\Rightarrow \exists$ infinitely many ordinary primes for an abelian variety

(dim = 2 まで OK)

§ boundary case

R: F-finite local normal domain of char $p > 0$

$X = \mathrm{Spec} R$. $\Delta \geq 0$: $\mathbb{R}$-divisor

(1) $(X.\Delta)$: F-pure

$\Leftrightarrow \mathcal{O}_X \to F^e_*(\mathcal{O}_X(\lfloor (q-1)\Delta \rfloor))$: splits as an $\mathcal{O}_X$-mod for $\forall q = p^e$

(2) $(X.\Delta)$: strongly F-reg

$\Leftrightarrow \forall D \geq 0$: Cartier divisor on X

$\exists q = p^e \leftarrow (\exists q_0 \in \mathbb{N}, \forall q \geq q_0)$

$\mathcal{O}_X \to F^e_*(\mathcal{O}_X(\lfloor (q-1)\Delta \rfloor + D))$ splits

(3) (X,Δ) : divisorially F-regular

$\Leftrightarrow$ $\forall D \geq 0$ Cartier divisor on X which has no common comp with Δ

$\exists q = p^e$ $\mathcal{O}_X \to F^e_*(\mathcal{O}_X(\lfloor (q-1)\Delta \rfloor + D))$ splits

Q str. F-reg $\Rightarrow$ div. F-reg $\Rightarrow$ F-pure

char 0 の特異点

klt $\Rightarrow$ plt $\Rightarrow$ lc ($\Leftrightarrow$ $\mathrm{mld}(X,\Delta) \geq 0$)

klt $\Updownarrow$ ($\mathrm{mld}(X,\Delta) > 0$, $\lfloor \Delta \rfloor = 0$)

plt $\Updownarrow$ ($\mathrm{mld}(X,\Delta) > 0$)

Thm (Hara-Watanabe-T—)

R : normal local ring ess of f.t. / $\mathbb{C}$

$\Delta \geq 0$ $\mathbb{R}$-divisor on $X = \operatorname{Spec} R$ s.t. $K_X + \Delta$: $\mathbb{R}$-Cartier

(X_p, Δ_p) : modulo p reduction of (X,Δ)

If (X_p, Δ_p) is str F-reg for infinitely many p

$\Rightarrow$ (X,Δ) : klt

If (X_p, Δ_p) is F-pure

$\Rightarrow$ (X,Δ) : lc

If (X_p, Δ_p) : div F-reg

$\Rightarrow$ (X,Δ) : plt $\leftarrow$ dlt ではない

(X,Δ): plt $\Leftrightarrow$ (X_p, Δ_p): div F-reg for $\forall p \gg 0$

klt $\Leftrightarrow$ —— str. F-reg

§ test ideals

<u>Defn</u> $d = \dim R$

R: F-finite normal local domain of char $p>0$

$\Delta \geq 0$: $\mathbb{R}$-divisor on $\operatorname{Spec} R = X$

(1) $(0)^{*\Delta} \subseteq H^d_m(\omega_X)$

$z \in (0)^{*\Delta} \Leftrightarrow \exists D \geq 0$ Cartier divisor on X

$F^e_D(z) = 0 \quad \forall e \gg 0$

$F^e_D : H^d_m(\omega_X) \to H^d_m(\omega_X^{(q)}(\lfloor (q-1)\Delta \rfloor + D))$

$(q = p^e)$

$\tau(X,\Delta) := \operatorname{Ann}(0)^{*\Delta} \subseteq \mathcal{O}_X$

Δ: boundary $(\Delta = \sum d_i D_i \quad 0 \leq d_i \leq 1)$

(2) $(0)^{\mathrm{div}*\Delta} \subseteq H^d_m(\omega_X)$

$z \in (0)^{\mathrm{div}*\Delta} \Leftrightarrow \exists D \geq 0$ Cartier div on X

which has no common comp with $\lfloor \Delta \rfloor$

$F^e_D(z) = 0 \quad \forall e \gg 0$

$\tau^{\mathrm{div}}(X,\Delta) := \operatorname{Ann}(0)^{\mathrm{div}*\Delta} \subseteq \mathcal{O}_X$

(3) $(0)^{F.\Delta} \subseteq H^d_m(\omega_X)$

$z \in (0)^{F.\Delta} \iff \exists q = p^e \quad F^e(z) = 0$

$$F^e : H^d_m(\omega_X) \to H^d_m(\omega_X^{(q)}(\lfloor (q-1)\Delta \rfloor))$$

$\tau^F(X,\Delta) := \mathrm{Ann}(0)^{F.\Delta} \subseteq \mathcal{O}_X$

<u>char. 0</u>

(1) multiplier ideal $\mathcal{J}(X,\Delta)$

$\pi : \tilde{X} \to X$: log resol of (X,Δ)

$\mathcal{J}(X,\Delta) := \pi_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \lfloor \pi^*(K_X+\Delta) \rfloor)$

(2) adjoint ideal $\mathrm{adj}(X;\Delta)$

Assume $\pi^{-1}_* \lfloor \Delta \rfloor$ is smooth (disconnected)

$\mathrm{adj}(X,\Delta) = \pi_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \lfloor \pi^*(K_X+\Delta) \rfloor + \pi^{-1}_* \lfloor \Delta \rfloor)$

(3) non-lc ideal ?

$K_{\tilde{X}} + \Delta_{\tilde{X}} = \pi^*(K_X+\Delta)$

$\mathcal{J}'(X,\Delta) := \pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}} - \pi^*(K_X+\Delta) + \varepsilon \Delta_{\tilde{X}}^{\geq 1} \rceil) \quad 0 < \varepsilon \ll 1$

$\Delta_{\tilde{X}}^{\geq 1} = \Delta_X$の係数が$\geq 1$の部分

同様に $\mathfrak{a} \subseteq \mathcal{O}_X$ に対し

$\mathcal{J}(X, \mathfrak{a}^t)$, $J(X, \mathfrak{a}^t)$ も定義できる.

ex. $X = \mathbb{A}^d$, $\mathfrak{a} \subseteq \mathcal{O}_X$: mono ideal

$$\mathcal{J}(X, \mathfrak{a}^t) = \langle \underline{x}^{\underline{\nu}} \,;\, \underline{\nu} + \mathbb{1} \in \mathrm{Int}(t \cdot \underbrace{P(\mathfrak{a})}_{\phi}) \rangle$$

$\mathfrak{a}$ の Newton polygon

$$J(X, \mathfrak{a}^t) = \langle \underline{x}^{\underline{\nu}} \,;\, \underline{\nu} + \mathbb{1} \in t \cdot P(\mathfrak{a}) \rangle$$

rmk · char $p > 0$

(X, Δ) : str. F-reg $\iff \tau(X, \Delta) = \mathcal{O}_X$

div. F-reg $\iff \tau^{\mathrm{div}}(X, \Delta) = \mathcal{O}_X$

F-pure $\iff \tau^F(X, \Delta) = \mathcal{O}_X$

· char 0

(X, Δ) : klt $\iff \mathcal{J}(X, \Delta) = \mathcal{O}_X$

plt $\iff \mathrm{adj}(X ; \Delta) = \mathcal{O}_X$

lc $\iff J(X, \Delta) = \mathcal{O}_X$

Thm R : normal local ring ess of fin.type / $\mathbb{C}$

$\Delta \geq 0$ $\mathbb{R}$-divisor on $X = \mathrm{Spec}\, R$, $K_X + \Delta$: $\mathbb{R}$-Cartier

(X_p, Δ_p) : reduction of (X, Δ) to char $p \gg 0$

(1) $\mathcal{J}(X, \Delta)_p = \tau(X_p, \Delta_p)$ $(\forall p \gg 0)$

(これ $\forall p$ で o.k.!)

(2). $\mathrm{adj}(X, \Delta)_p = \tau^{\mathrm{div}}(X_p, \Delta_p)$ $(\forall p \gg 0)$

(3) $J(X, \Delta)_p \supseteq \tau^F(X_p, \Delta_p)$ $(\forall p)$

(X, Δ): klt — (X, Δ) str. F-reg

$c \vdash \tau(X, \Delta) \Leftarrow$— splitting みたいな言葉で表現

$\Downarrow$

$K + \Delta$ の section を与える

F-pure　Hochster - Roberts

str. F-reg　Hochster - Huneke

F-inj　Fedder

F-rat'l　Fedder　Watanabe

[illegible] $G \curvearrowright k[X_1, \cdots, X_n]$　str F-reg $\Rightarrow$ CM.