Motivic zeta functions in additive monoidal categories

Kimura – Takahashi

"Motivic zeta functions".

$\mathcal{C}$: ある種の圏
$X \in Ob(\mathcal{C})$, $S^n X$ "対称積"
$[X]$: ある環における類 (e.g., $K_0(Var)$)

$$\zeta_X(t) := \sum_{n=0}^{\infty} [S^n X] t^n \quad (\rightsquigarrow \text{合同ゼータ})$$

Q. 有理的?

今日: $\mathcal{C}$: pseudo-abelian add. mon. cat.
(e.g., pure Chow motive の圏)

§ Additive monoidal category.

Def. add. mon. cat. とは次のデータ.

- $\mathcal{C}$: additive category
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ bilinear functor
-
  - a : natural transformation for each triple U, V, W, $(U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$
  - c : $U \otimes V \xrightarrow{\sim} V \otimes U$
- $\mathbb{1} \in Ob(\mathcal{C})$, $\mathbb{1} \otimes U \xrightarrow{\sim} U \xleftarrow{\sim} U \otimes \mathbb{1}$

\+ axiom

例. Vec_K : K 上の有限次元ベクトル空間の圏

例. $sVec_K$: K 上の有限次元超ベクトル空間の圏

$Ob(sVec_K)$ $\mathbb{Z}/2\mathbb{Z}$-grading の与えられた f.d.v.s $V = V_0 \oplus V_1$

$$c((u_0+u_1) \otimes (v_0+v_1)) := v_0 \otimes u_0 + v_0 \otimes u_1 + v_1 \otimes u_0 - v_1 \otimes u_1$$

$V \in Ob(\mathcal{C})$

$a, c \rightsquigarrow \mathfrak{S}_n \curvearrowright V^{\otimes n} := \overbrace{V \otimes \cdots \otimes V}^{n\text{コ}}$

$\mathcal{C}$: $\mathbb{Q}$-linear のとき、$V^{\otimes n}$ は $\mathbb{Q}\mathfrak{S}_n$-module.

$\mathfrak{S}_n$ の既約表現 $(/\mathbb{Q}, \mathbb{C})$

$\longleftrightarrow$ Young 図形

$\longleftrightarrow$ $\mathbb{Q}\mathfrak{S}_n$ の原始べき等元 [冪等元が原始的 $\Leftrightarrow$ 非自明な直交する冪等元の和に分解しない.]

例. $\boxed{\ \cdots\ } \longleftrightarrow \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma$

$\boxed{\vdots} \longleftrightarrow \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) \cdot \sigma$

Young 図形 λ に対し, 冪等元 c_λ を選んでおく.

Def. $\mathcal{C}$: pseudo-abelian とは,

$f \in End(V),\ f \circ f = f \Rightarrow V \cong \mathrm{im}\, f \oplus \ker f$.

Def. $\mathcal{C}$: pseudo-abelian $\mathbb{Q}$-linear monoidal category

$S_\lambda(V) := c_\lambda(V^{\otimes n})$

$S_\lambda : \mathcal{C} \longrightarrow \mathcal{C}$ "Schur functor"

例. $V \in Vec_k$, $k \supseteq \mathbb{Q}$.

$S_{\boxed{\cdots}} V = S^n V$,

$S_{\boxed{\vdots}} V = \Lambda^n V$.

$\rightsquigarrow$ 一般の $\mathcal{C}$ でも

$S^n V := S_{\boxed{\cdots}} V$,

$\Lambda^n V := S_{\boxed{\vdots}} V$

と定義する.

$$V \otimes V \cong S^2 V \oplus \Lambda^2 V$$
$$V \otimes V \otimes V \cong S^3 V \oplus (S_{\boxplus} V)^{\oplus 2} \oplus \Lambda^3 V.$$

$V = V_0 \oplus V_1 \in sVec_k$

$$S^n V = \bigoplus_{i+j=n} (S^i V_0 \otimes \Lambda^j V_1)$$

← vector space としての wedge product.

<u>Def.</u> Grothendieck 環

$$K_0(\mathcal{C}) := \oplus \mathbb{Z}.(\text{isom. cl. in } \mathcal{C}) / (V - U - W \mid 0 \to U \to V \to W \to 0)$$

・$\otimes V$ が exact のとき, 環になる.

$$L(\mathcal{C}) := \oplus \mathbb{Z}.(\text{isom. cl. in } \mathcal{C}) / (V - U - W \mid V \cong U \oplus W)$$

X の類を $[X]$ と書く.

motivic zeta:

$$\zeta_X(t) := \sum_{n=0}^{\infty} [S^n X] t^n \in K_0(\mathcal{C})[[t]]$$
$$L(\mathcal{C})[[t]]$$

$$\zeta_{X \oplus Y}(t) = \zeta_X(t)\zeta_Y(t)$$

<u>例.</u> $sVec_k$ ($k \supseteq \mathbb{Q}$, i.e., char $k = 0$)

$$K_0(sVec_k) = L(sVec_k) \cong \mathbb{Z}[t]/(t^2 - 1)$$

$t \longleftrightarrow \mathbb{1}'$: odd, 1-dimensional

$X \cong m\mathbb{1} \oplus n\mathbb{1}'$

$$\zeta_X(t) = \frac{(1 + \mathbb{1}' \cdot t)^n}{(1 - \mathbb{1} \cdot t)^m} \quad \text{i.e.,} \quad (1 - \mathbb{1}t)^m \cdot \zeta_X(t) = (1 + \mathbb{1}'t)^n.$$

§ Chow motives and finiteness.

$\mathcal{C}$: $\mathbb{Q}$-linear pseudo-abelian monoidal category —— ⊛

<u>Def.</u> ◦ $X \in Ob(\mathcal{C})$ が Kimura finite とは,

$X \overset{\exists}{=} \exists X_0 \oplus \exists X_1$ s.t.

- X_0 : even i.e., $\exists N$ s.t. $(n > N \Rightarrow \Lambda^n X_0 = 0)$
- X_1 : odd i.e., $\exists N$ s.t. $(n > N \Rightarrow S^n X_1 = 0)$

◦ Schur finite とは, $\exists \lambda$ s.t. $S_\lambda X = 0$.

<u>Remark.</u> Kimura finite $\Rightarrow$ Schur finite

<u>Conjecture.</u> 任意の (pure) Chow motive は Kimura finite.
(このとき, N は $\dim H^*(X_0)$, $\dim H^*(X_1)$ でよい.)

~~(Chow motive) Bloch's conjecture ⇐ $X \cong X_0 \oplus X_1$ Kimura finite, ⇒ $\zeta_X(t)$: rational ⇐? Y. André~~
~~(mixed motive で反例 $GL(1|1)$ の自然な表現) ⇓ ⇑✕ Schur finite~~

$$\zeta_X(t) = \frac{1 + [X_1]t + [S^2 X_1]t^2 + \cdots}{1 - [X_0]t + [\Lambda^2 X_0]t^2 + \cdots}$$

$X \cong X_0 \oplus X_1$ Kimura finite $\overset{\Longrightarrow}{\underset{?}{\Longleftarrow}}$ $\zeta_X(t)$: rational
Y. André

(Chow motive) ⇙ Bloch's conjecture

⇓ ⇑✕ (mixed motive で反例 ; $GL(1|1)$ の自然な表現)

Bloch's conjecture ⇐✕ "多分" Schur finite ⟹ ? $\zeta_X(t)$: rational

<u>結論</u> 有理性の定義による.

§ 有理性の定義

(Larsen-Lunts)

A: 可換環, $f(t) = \sum_{n=0}^{\infty} a_n t^n \in A[\![t]\!]$

Def. (1) $f(t)$: uniformly rational if

$\Downarrow$ $\quad {}^{\exists} p(t) \in 1 + tA[t]$ s.t. $p(t)f(t) \in A[t]$

(2) $f(t)$: globally rational if

$\Downarrow$ $\quad {}^{\exists} P(t), Q(t) \in A[t]$ s.t. $f(t)$ は $P(t)f(t) = Q(t)$ の一意解

(3) $f(t)$: determinantally rational if

$\Downarrow$ $\quad \exists n_0, m$ s.t. ${}^{\forall} n \geq n_0$, $\det(a_{n+i+j})_{i,j=0}^{m-1} = 0$

(4) $f(t)$: pointwise rational if

${}^{\forall}\varphi : A \to k$, k a field, $\varphi(t) := \sum \varphi(a_n) t^n$ is rational.

(1), (3), (4) は係数環の hom. で保たれる.

Thm. (1) X Schur finite $\Longrightarrow \zeta_X(t) \in L(\mathcal{C})[\![t]\!]$ determinantally rational

田で関係 (Jacobi-Trudy)

(2) $\exists \mathcal{C}$: abelian, semisimple

$\exists X \in \mathcal{C}$ Schur finite ($S_{田} X = 0$)

s.t. $\zeta_X(t)$ not uniformly rational

$\in L(\mathcal{C})[\![t]\!]$

$\| $

$K_0(\mathcal{C})$

$\rightsquigarrow$ ⊛ (cf. p.4) の範囲での一般論からは unif. rat. は出ない.

<u>Proof</u> 1つの object で生成される普遍的な monoidal category を考える:

$\mathcal{U}_0$: $Ob(\mathcal{U}_0) = \{0, 1, 2, \ldots\}$
$\mathbb{1}, V, V^{\otimes 2}, \ldots$ と書く.

$$\mathrm{Hom}_{\mathcal{U}_0}(i,j) = \begin{cases} 0 & (i \neq j) \\ \mathbb{Q}\mathfrak{S}_i & (i = j) \end{cases}$$

$i \otimes j := i + j$

$\mathrm{End}(i) \times \mathrm{End}(j) \longrightarrow \mathrm{End}(i+j)$ "自然に"

$\mathbb{Q}\mathfrak{S}_i \times \mathbb{Q}\mathfrak{S}_j \longrightarrow \mathbb{Q}\mathfrak{S}_{i+j}$

$\mathcal{U}_1$: $\mathcal{U}_0$ の objects の direct sum を考えたもの.

$$Ob(\mathcal{U}_1) = \bigoplus_{i \geq 0} \mathbb{Z}_{\geq 0}$$

$\cancel{(n_0, n_1, \ldots) \longleftrightarrow \bigoplus_i n_i V^{\otimes n_i}}$

$(n_0, n_1, n_2, \ldots) \longleftrightarrow \bigoplus_i n_i V^{\otimes i}$

Hom : たとえば

$$\mathrm{Hom}_{\mathcal{U}_1}(n V^{\otimes i}, m V^{\otimes i}) = M_{\mathbb{Q}\mathfrak{S}_i}(m, n)$$

$\mathcal{U}$: $\mathcal{U}_1$ の pseudo-abelian envelope

$$Ob(\mathcal{U}) = \{(X, p) \mid X \in Ob(\mathcal{U}_1),\ p \in \mathrm{End}(X),\ p \circ p = p\}$$

$$\mathrm{Hom}_{\mathcal{U}}((X,p),(Y,q)) = q \circ \mathrm{Hom}_{\mathcal{U}_1}(X, Y) \circ p$$

λ : Young 図形.

$\mathcal{U}_\lambda$: $Ob(\mathcal{U}_\lambda) = Ob(\mathcal{U})$

$$\mathrm{Hom}_{\mathcal{U}_\lambda}(X,Y) := \mathrm{Hom}_{\mathcal{U}}(X,Y)/\underset{\substack{\| \\ \{\text{factor through } \bigoplus_{\mu_i \supseteq \lambda} S_{\mu_i} V\}}}{I(X,Y)}$$

<u>Proposition.</u> ∘ $\mathcal{U}$: abelian, semisimple, (monoidal, ...)

simple objets は $S_\mu V$ たち

$$K_0(\mathcal{U}) = L(\mathcal{U}) = \underset{\text{polynomial ring}}{\mathbb{Z}[[V], [S^2V], [S^3V], \dots]}$$

∘ $\mathcal{U}_\lambda$: abelian, semisimple, (...)

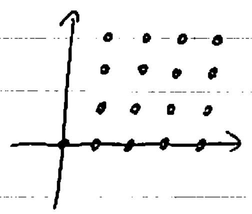
simple objects は $S_\mu V$ ($\mu \not\supseteq \lambda$)

($\mu \supseteq \lambda$ のとき $S_\mu V = 0$)

($S_\lambda X = 0$ となる X に対する universality)

$$K_0(\mathcal{U}_{\boxplus}) \otimes \mathbb{Q} \cong \mathbb{Q}[xy^i \mid i \geq 0]$$

$$xy^i \updownarrow S^{(i+1)}V$$



$$\Rightarrow \zeta_V(t) \in (K_0(\mathcal{U}_{\boxplus}) \otimes \mathbb{Q})[[t]]$$

not uniformly rational.

<u>Problem.</u> · rigid のとき? (GL(1|1) のとき?)

(dual がある)