8/11 原. 伸生「F-singularities and F-pure thresholds」

- F-singularities
- F-pure thresholds

char $p>0$	char 0
(strongly) F-regular	Kawamata lt
(sharp) F-pure	lc
gen. test ideal $\tau(\mathfrak{a}^t)$	multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$
F-pure threshold	lc threshold
F-pure center	lc center

1. Splitting of Frobenius

$/k$: 完全体, char $k = p > 0$.
$X = \mathrm{Spec}\, R$ (normal)
R : k上有限生成整域 又は. その局所化 / 完備化.

$F : X \to X$ Frobenius mor. id or sp(X).

$$\mathcal{O}_X = F_*\mathcal{O}_X \xleftarrow{F} \mathcal{O}_X$$
$$a^p \longleftarrow\!\shortmid\ a$$

$q = p^e$

$F^e : \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X$

$$
\begin{array}{ccccc}
 & a & \longmapsto & a^q & \\
F^e : & R & \longrightarrow & R & \\
 & \| & \circlearrowright & \wr\| & \updownarrow \\
 & R & \hookrightarrow & R^{1/q} & \\
 & a & \longmapsto & a = (a^q)^{1/q} &
\end{array}
$$

D eff. div. on X.

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \hookrightarrow F^e_* (\mathcal{O}_X(D))$$

$D = \mathrm{div}_X(c)$, $c \in R$.

$$
\begin{array}{ccccc}
R & \hookrightarrow & R^{1/q} & \longrightarrow & \left(\frac{1}{c}R\right)^{1/q} \\
\| & & \circlearrowright & & \wr\| \\
R & & \xrightarrow{\quad c^{1/q} \quad} & & R^{1/q}
\end{array}
$$

(例) $R = k[x, y]$

$$R^{1/q} = k[x^{1/q}, y^{1/q}] = \bigoplus_{0 \le i,j \le q-1} \underbrace{R x^{i/q} y^{j/q}}_{\cong R}$$

$0 \le i, j \le q-1$ のとき,

$$R \xrightarrow{(x^i y^j)^{1/q}} R^{1/q}$$

R lin. map として分裂.

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Def Δ: eff. $\mathbb{R}$-div on $X = \operatorname{Spec} R$.

1) (X, Δ) : <u>F-pure</u>

$$\Leftrightarrow {}^\forall q = p^e, \quad F^e : \mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X(\lfloor (q-1)\Delta \rfloor)$$

splits on $\mathcal{O}_X$-lin. map

2) (X, Δ) : <u>sherp F-pure</u>

$$\Leftrightarrow {}^\exists q = p^e \text{ s.t. } F^e : \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil (q-1)\Delta \rceil)$$

splits

3) (X, Δ) : <u>strongly F-regular</u>

$\Leftrightarrow$ $0 \neq {}^\forall c \in R$.

$${}^\exists q = p^e \text{ s.t. } cF^e : \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil (q-1)\Delta \rceil) \text{ splits.}$$

$$F^e : \mathcal{O}_X \searrow F^e_* \mathcal{O}_X(\lceil (q-1)\Delta \rceil + \operatorname{div}_X(c)) \cong F^e_* \mathcal{O}_X(\lceil (q-1)\Delta \rceil)$$

Rmk

- str. F-reg. $\Rightarrow$ sh. F-pure $\Rightarrow$ F-pure.
- (X, Δ) : F-pure (resp. ⋯)
  $\Rightarrow$ ${}^\forall \Delta' \leq \Delta$, $\Delta' \geq 0$, (X, Δ') : F-pure (resp. ⋯).
- (X, Δ) : sh. F-pure
  $\Rightarrow {}^\exists q = p^e, \ {}^\forall n \in \mathbb{N}$ s.t. $F^{en} : \mathcal{O}_X \longrightarrow F^{en}_* \mathcal{O}_X(\lceil (q^n - 1)\Delta \rceil)$ split

$$\lceil (qq'-1)\Delta \rceil \leqq q \lceil (q'-1)\Delta \rceil + \lceil (q-1)\Delta \rceil$$

- 根元で split したら 先の方でも split.

- (X, Δ) : str. F-reg.

  $\Rightarrow 0 \neq {}^\forall c \in R, \quad {}^\exists q_0 \quad {}^\forall q = p^e \gg q_0$

  $cF^e : \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil q\Delta \rceil)$ split.

D: prime div.

- (X, Δ) : F-pure (resp. str. F-reg).

  $\Rightarrow \mathrm{coeff}_D(\Delta) \leq 1 \quad (\text{resp. } < 1)$.

  (∵) もし $\mathrm{coeff}_D(\Delta) > 1$.

  ${}^\exists q = p^e$ s.t. $qD \leq \lfloor (q-1)\Delta \rfloor$

$$
\begin{array}{ccc}
F^e : \mathcal{O}_X & \longrightarrow & F^e_* \mathcal{O}_X(qD) \subset F^e_* \mathcal{O}_X(\lfloor \text{"} \rfloor) \\
\downarrow & & \uparrow \\
\mathcal{O}_X(D) & \xrightarrow[F^e]{} & \mathcal{O}_X(D) \otimes F^e_* \mathcal{O}_X
\end{array}
$$

non split.

例1) $X = \mathrm{Spec}\, k[x, y]$

$\Delta = s \cdot \mathrm{div}_X(x) + t \cdot \mathrm{div}_X(y) \quad 0 \leq s, t.$

(X, Δ) : F-pure (⇔ sh. F-pure)
⇔ $s \leq 1$ & $t \leq 1$.

(X, Δ) : str. F-reg. ⇔ $s < 1$ & $t < 1$.

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Fedder の判定法

(R, m): regular local ring

$0 \neq g \in m$

- $R \xrightarrow{g^{1/q}} R^{1/q}$ splits $\Leftrightarrow$ $g \in m^{[q]} = (a^q \mid a \in m)$

- 例えば $\Delta = t \cdot \mathrm{div}_X(f)$

  1. (X, Δ): F-pure $\Leftrightarrow$ $\forall q = p^e$. $f^{\lfloor t(q-1) \rfloor} \notin m^{[q]}$

例2) $X = \mathrm{Spec}\, k[[x, y]]$, $\Delta = t \cdot \mathrm{div}_X(f)$.

$f = y^2 - x^3$

$p \equiv 1 \pmod 3$

(X, Δ): F-pure $\Leftrightarrow$ (X, Δ) sh. F-pure

$\Leftrightarrow$ $t \leq \frac{5}{6}$ $(= \mathrm{lct})$.

(X, Δ): str. F-reg. $\Leftrightarrow$ $t < \frac{5}{6}$

$p \equiv 2 \pmod 3$

(X, Δ): F-pure $\Leftrightarrow$ $t \leq \frac{5p-1}{6p}$

(X, Δ): sh. F-pure $\Leftrightarrow$ str. F-reg. $\Leftrightarrow$ $t < \frac{5p-1}{6p}$.

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<u>Thm</u> /$\mathbb{Q}$

(X, Δ) : pair /$\mathbb{Q}$ s.t. $K_X + \Delta$: $\mathbb{Q}$-Cartier
(X: normal)

$\updownarrow$

(X_p, Δ_p) : red mod p

① $^\exists\infty$ p s.t. (X_p, Δ_p) : F-pure (str. F-reg)

② (X, Δ) : klt $\Rightarrow$ $^\forall p \gg 0$. (X_p, Δ_p) : str. F-reg.

char p, $f: Y \longrightarrow X$: resol.

- (X, Δ) : F-pure

$$F^e : \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X$$

$$\phi \in Hom(F^e_* \mathcal{O}_X, \mathcal{O}_X) = H^0(X, (1-q)K_X)$$

$$\mathcal{H}om(F^e_* \mathcal{O}_X, \omega_X) \otimes \omega_X$$
$$\cong F^e_*(\omega_X) \otimes \omega_X^{-1}$$
$$\cong F^e_*(\omega_X^{\otimes(1-q)})$$

$U = Y \setminus Supp(\phi)_\infty$

$D = (\phi)_0 - (\phi)_\infty \sim (1-q) K_Y$

ϕはU上のsplitting $\Rightarrow$ $D|_U$の各係数 $\leq q-1$

$0 \leq f_* D \sim (1-q) K_X$

$0 \leq f^* f_* D \sim (1-q) f^* K_X$

$$K_{Y/X} = K_Y - f^* K_X = \frac{1}{q-1} (\underbrace{f^* f_* D}_{\geq 0} - \underbrace{D}_{\leq q-1})$$

係数 ≥ -1.

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§2.

$\mathfrak{a} \subset R^{\hat{}\,I}$

$0 \le t \in \mathbb{R}$

$\mathfrak{a}^t$ - tight closure.

$I \subset I^{*\mathfrak{a}^t}$

↓

$\tau(\mathfrak{a}^t)$: gen. test ideal.

$\mathfrak{a} = (f)$, $\Delta = t \cdot \mathrm{div}_X(f)$

(X, Δ) : str. F-reg $\Leftrightarrow$ $\tau(f^t) = \tau(X, \Delta) = R$.

§3. F-pure threshold (char $p > 0$).

$x \in X = \mathrm{Spec}\, R$: str. F-reg

D : eff Cartier div on X.

$x \in D = \mathrm{div}_X(f)$

$$\mathrm{fpt}_x(X, D) = \mathrm{fpt}(f) := \sup\{t \in \mathbb{R}_{\ge 0} \mid (X, tD) : \text{F-pure at } x\}$$

$$= \sup\{ \text{〃} \mid (X, tD) \text{ s.t. F-reg at } x\}$$

$$= \sup\{ \text{〃} \mid \tau(f^t) = R\}$$

Rmk

- $x \in D$: integral div $\Rightarrow$ $\mathrm{fpt}_x(X, D) \le 1$

- $\lambda = \mathrm{fpt}_x(X, D)$ のとき.

$$\begin{cases} (X, tD) : \text{F-pure} \Leftrightarrow 0 \le t \le \lambda \\ \quad \text{〃} \quad \text{str. F-reg.} \Leftrightarrow 0 \le t \lneq \lambda. \end{cases}$$

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(例) $f = x^3 - y^2 \in k[[x, y]]$

$p \equiv 2 \pmod 3$

$\mathrm{fpt}(f) = \frac{5p-1}{6p} = \lambda$

(R, f^{λ}) : Non sh. F-pure.

Thm /$\mathbb{Q}$ $X = \mathrm{Spec}\, R$, (R, m) : $\mathbb{Q}$-Gor. /$\mathbb{Q}$

$0 \neq f \in m$

このとき. $\mathrm{lct}(f) = \lim_{p \to \infty} \mathrm{fpt}(f_p)$

Thm (Blickle - Mustata - Smith)

⊛ ((R, m) : regular loc. ring /k, char $k = p$
$0 \neq f \in m \Rightarrow \mathrm{fpt}(f) \in \mathbb{Q}$

Schwede によれば ⊛ の状況で $\lambda = \mathrm{fpt}(f) \in \mathbb{Q}$ とするとき

(R, f^{λ}) : sh. F-pure $\Leftrightarrow$ λ の分母が p で割れない.

§4. Global geometry in char p

1° Global F-splitting

X : proj. var. /k, char $= p > 0$

X : F-split $\Leftrightarrow$ $F : \mathcal{O}_X \longrightarrow F_* \mathcal{O}_X$ $\Rightarrow K^{-1}(X) \geq 0$

φ : splits as $\mathcal{O}_X$-mod.

$\phi \in H^0(X, (1-p)K_X)$

$\wedge$

X : glob. F-reg

$\Downarrow$

$K^{-1}(X) = \mathrm{div}\, X$.

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obs $\mathcal{L}$: ample l.b. on X

X : F-split $\Rightarrow H^i(X, \mathcal{L})$

$= H^i(X, \omega_X \otimes \mathcal{L}) = 0 \quad (i>0)$

X : glob. F-reg $\Rightarrow$ " $= 0 \quad (i>0)$

$\Rightarrow K^{-1}(X) \geq 0 \quad \mathcal{L}$: nef big で ok.

$\otimes \mathcal{L} \; (\mathcal{O}_X \xrightarrow{\oplus} F^e_* \mathcal{O}_X)$

$\mathcal{L} \oplus F^e_*(\mathcal{L}^{\otimes p^e})$

$H^i(\mathcal{L}) \oplus H^i(\mathcal{L}^{\otimes p^e}) = 0$

$\mathcal{H}om(\ , \omega_X)$ をとると. $\omega_X \overset{\dashrightarrow}{\underset{\otimes}{\longleftarrow}} F^e_* \omega_X$.

2° **prop** (Smith)

X : proj. var. $/k$, char $p > 0$

с only F-pure sing.

$\mathcal{L}$: ample l.b. gen. by glob. sect.s

$\Rightarrow \omega_X \otimes \mathcal{L}^{\otimes \dim X + 1}$: gen. by g.b. sects.

pf $f^e : \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X$ loc. split

$\omega_X \overset{\dashrightarrow}{\longleftarrow\!\!\!\!\!\leftarrow} F^e_* \omega_X$

$n = \dim X$

$\underline{\omega_X \otimes \mathcal{L}^{n+1}} \twoheadleftarrow \underline{F^e_*(\omega_X \otimes \mathcal{L}^{p^e(n+1)})}$

$\shortparallel \; \mathcal{F}_e$: g.g ??

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Lem (Mumford)

$\mathcal{L}$: ample glob. gen. l.b.

$H^i(X, \mathcal{F} \otimes \mathcal{L}^{-i}) = 0$, $i > 0$.

$\Rightarrow \mathcal{F}$: gen. glob sect.

$0 < i \leq n$

$$H^i(X, \mathcal{F}_e \otimes \mathcal{L}^{-i}) = H^i(X, \omega_X \otimes \mathcal{L}^{p^e \overbrace{(n+1-i)}^{\ell}}) = 0$$

3° 失敗例

X : smooth proj. surf char $p > 0$

$\underset{x}{\cup}$

L : ample l.b. $L^2 > 4$

$LC \geq 2$, $x \in {}^\forall C \overset{?}{\Rightarrow} K_X + L$: free at x ?

§5. Center of F-purity

char 0 (X, Δ)

$Z \underset{\text{subvar.}}{\subset} X$ が (X, Δ) の lc center.

$\Leftrightarrow$ (X, Δ) に関して discrepancy ≤ -1 の div over X.

により Z が dominate される。

$\Leftrightarrow$ ${}^\forall G$: eff. $\mathbb{Q}$-div. $\mathrm{Supp}(G) \supseteq Z$.

$(X, \Delta + G)$: non lc.

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Char p (Schwede)

$X = \mathrm{Spec}\, R \ni \xi = Q$.

Q (または $Z = \overline{\{\xi\}}$) が (X, Δ) の F-pure center

$\Leftrightarrow$ ${}^\forall c \in Q$ ${}^\forall q = p^e$ に対して.

$cF^e : \mathcal{O}_{X,\xi} \longrightarrow F^e_* \mathcal{O}_X(\ulcorner (q-1)\Delta \urcorner)_\xi$ が non-split.

$\Delta = t \cdot \mathrm{div}_X(f)$ のとき.

$\Leftrightarrow$ ${}^\forall c \in Q$, ${}^\forall q = p^e$, $R_Q \xrightarrow{(c f^{\ulcorner t(q-1) \urcorner})^{1/q}} R_Q^{1/q}$: non-split.

(R, f^t)

$CFP(X, \Delta) = \{(X, \Delta)$ の F-pure center$\}$

Thm (Schwede)

- (X, Δ) sh. F-pure $\Rightarrow$ $CFP(X, \Delta) \neq \phi$ 有限.
- (X, Δ) の F-pure center の有限個の共通部分は. F-pure center の和集合.
- $P \in CFP(X, \Delta)$: min. center $\Rightarrow \mathcal{O}_X/P$: str. F-reg.

(例) $f = y^2 - x^3 \in k[[x, y]]$

$p \equiv 2 \pmod 3$

$\lambda = \frac{5p-1}{6p}$ で (R, f^λ) : Not sh. F-pure.

$m \in CFP(R, f^\lambda)$

(R, f^λ) : F-pure

$CFP = \{(f)\}$.

① $\lambda = \mathrm{fpt}\,(X, D)$

$\Rightarrow (X, \lambda D)$: F-pure

② $(X, \lambda D)$ の min. F-pure center Z
が定義できて. Z : loc. str. F-reg
sh. F-pure.

<u>Reicher 型</u>

$D \sim tL \quad \mathrm{ord}_x D = 2$

$(t < 1)$

$\lambda = \mathrm{fpt}_x (X, D)$

$(X, \lambda D)$ min ℓ center $= \begin{cases} \{x\} \\ C \end{cases}$

$$0 \to m_x \otimes \mathcal{O}(K_x + L) \xrightarrow{H^0} \overset{M}{\overset{\wedge}{\mathcal{O}}}(K_x + L) \xrightarrow{H_0} \mathcal{O}_x(K_x + L) \otimes k(x) \to 0$$

$$\exists \uparrow \qquad \phi \uparrow ① \qquad \uparrow H^0$$

$$0 \to F^e_* (m_x \otimes \mathcal{O}_x(\text{''} \;)) \xrightarrow{H^0} F^e_* \mathcal{O}_x(K_x + qL - \langle \lambda(q-1) \rangle) \to \mathrm{Coker} \to 0$$

$\overset{\cup}{m}^{1/q}$

Coker: supp $= \{x\}$, $H^1 = 0$.

$$\mathcal{O}_x \longrightarrow F^e_* \mathcal{O}_x(\langle \lambda(q-1) D \rangle)$$