Log canonical singlarities と Du Bois singlatities について

Th A. (Kovács, Schwede, Smith '08)

X: normal alg. var. / $\mathbb{C}$

(X, Δ) lc

X: Cohen-Macauley

$\Rightarrow$ X has only Du Bois singlarities

Th B.

X: normal CM alg var / $\mathbb{C}$

$\rho: X' \to X$ resol. s.t. $\mathrm{Exc}(\rho) = G$

G: s.n.c div on X' reduced

X: Du Bois $\Leftrightarrow$ $\rho_* \omega_{X'}(G) \simeq \omega_X$

Rem

X: rational sing

$\Leftrightarrow$ $\rho_* \omega_{X'} \simeq \omega_X$

Th B $\Rightarrow$ Th A excercise.

§ Du Bois singularities.

X : alg. var. $/\mathbb{C}$

$\Rightarrow \exists\ \underline{\Omega}^{\bullet}_X \in Ob(D\,\mathrm{filt}(X))$

Du Bois complex of X

s.t.

(1) $\phi : Y \to X$

$\Rightarrow \phi^* : \underline{\Omega}^{\bullet}_X \to R\phi_* \underline{\Omega}^{\bullet}_Y$

$\underline{\Omega}^{\bullet}_X \in Ob(D^b_{\mathrm{filt},\,\mathrm{coh}}(X))$

If ϕ : proper $\Rightarrow \phi^*$: morphism in $D^b_{\mathrm{filt}}\,\mathrm{coh}(X)$

(2)

$\Omega^{\bullet}_X$: de Rham complex of Kähler differential

with "filtration bête" = stupid

$\Rightarrow \exists\ \Omega^{\bullet}_X \to \underline{\Omega}^{\bullet}_X$ natural map of filtered complex.

If X : smooth, it is quasi isom.

(3) $U \subset X$: open $\Rightarrow \underline{\Omega}^{\bullet}_X|_U \simeq \underline{\Omega}^{\bullet}_U$

(4) X : proper

$\exists\ E_1^{p,q} = \mathbb{H}^q(X, Gr^p_{\mathrm{filt}}\,\underline{\Omega}^{\bullet}_X[p])$

$\Rightarrow H^{p+q}(X, \mathbb{C})$

degenerates at E_1

Rem

$\underline{\Omega}^\cdot_X$: resol of $\mathbb{C}_X$

(5) $\dim \operatorname{Supp} h^i(Gr^p_{filt}\, \underline{\Omega}^\cdot_X) \leq \dim X - i + p$

(6) $\underline{\Omega}^0_X := Gr^0_{filt}\, \underline{\Omega}^\cdot_X$

$\Sigma \supset \operatorname{Sing} X$

$f : Y \to X$ proper bir

$E = f^{-1}(\Sigma)$ f : isom over $X \setminus \Sigma$

$\Rightarrow$ $\exists$ triangle

$$\underline{\Omega}^0_X \to \underline{\Omega}^0_\Sigma \oplus Rf_* \underline{\Omega}^0_Y \to Rf_* \underline{\Omega}^0_E \xrightarrow{+1}$$

(7) $L \subset X$ general hyperplane section.

$$\underline{\Omega}^\cdot_L \simeq \underline{\Omega}^\cdot_X \otimes \mathcal{O}_L$$

Def X : Du Bois

$\overset{def}{\Longleftrightarrow}$ $\mathcal{O}_X \to \underline{\Omega}^0_X$ quasi isom.

$$\begin{array}{ccc} \mathcal{O}_X & \to & \underline{\Omega}^0_X \\ \| & & \| \\ Gr^0_p \underline{\Omega}^\cdot_X & & Gr^0_{filt} \underline{\Omega}^\cdot_X \end{array}$$

Rem

X : proper DuBois

$\Rightarrow$ $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$: surj $\forall i$

§ Lemmas

Lem (Kóvacs) X : var. $/\mathbb{C}$

P : finite set of paints s.t. $X \setminus P$: Du Bois

$\Rightarrow H^i_p(X, \mathcal{O}_X) \to \mathbb{H}^i_p(X, \underline{\Omega}^0_X)$ surj for $\forall i$

proof

X : affine.

$$\mathcal{O}_X \to \underline{\Omega}^0_X \to F^\cdot \xrightarrow{+1}$$

$\mathrm{Sing}_{DB} X := \bigcup \operatorname{supp} h^i(F^\cdot)$

$P \supset \mathrm{Sing}_{DB} X$

$\overline{X}$: proj closure of X

$Q = \overline{X} \setminus X \quad Z = P \cup Q$

$\Rightarrow \overline{X} \setminus Z \simeq X \setminus P$ Du Bois

$\Rightarrow \underline{\Omega}^0_{\overline{X}\setminus Z} \simeq \mathcal{O}_{\overline{X}\setminus Z}$

$H^i(\overline{X}, \mathbb{C}) \to H^i(\overline{X}, \mathcal{O}_{\overline{X}}) \to \mathbb{H}^i(\overline{X}, \underline{\Omega}^0_{\overline{X}})$ surj for $\forall i$

$$\Rightarrow \begin{array}{ccccccc} H^{i-1}(\overline{X}\setminus Z, \mathcal{O}_{\overline{X}\setminus Z}) & \to & H^i_Z(\overline{X}, \mathcal{O}_{\overline{X}}) & \to & H^i(\overline{X}, \mathcal{O}_{\overline{X}}) & \to & \cdots \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \delta \downarrow \\ \mathbb{H}^{i-1}(\overline{X}\setminus Z, \underline{\Omega}^0_{\overline{X}\setminus Z}) & \to & \mathbb{H}^i_Z(\overline{X}, \underline{\Omega}^0_X) & \to & \mathbb{H}^i(\overline{X}, \underline{\Omega}^0_X) & \to & \cdots \end{array}$$

α, δ : isom. γ : surj

$\Rightarrow \beta$: surj $\therefore H^i_p(X, \mathcal{O}_X) \to \mathbb{H}^i_p(X, \underline{\Omega}^0_X)$ surj $\forall i$

Th (Schwede)

$\pi : \widehat{X} \to X$ proper bir $\widehat{X}, X$: smooth

Σ : subvar $\subset X$ π : isom over $X \setminus \Sigma$

$\pi^{-1}(\Sigma) = E$: Du Bois

$\Rightarrow \underline{\Omega}^0_\Sigma \simeq R\pi_* \mathcal{O}_E$

(∵)

$$\underline{\Omega}^0_X \to R\pi_*\mathcal{O}_{\widehat{X}} \oplus \underline{\Omega}^0_\Sigma \to R\pi_*\mathcal{O}_E \xrightarrow{+1}$$

$$R\pi_*\mathcal{I}_E \to R\pi_*\mathcal{O}_{\widehat{X}} \longrightarrow R\pi_*\mathcal{O}_E \xrightarrow{+1}$$

八面体公理 $\Rightarrow$

$$\begin{tikzcd} \exists\, R\pi_*\mathcal{I}_E \arrow[r] \arrow[d, equal] & \underline{\Omega}^0_X \arrow[r] \arrow[d] & \underline{\Omega}^0_\Sigma \arrow[r, "+1"] \arrow[d] & {} \\ R\pi_*\mathcal{I}_E \arrow[r] & R\pi_*\mathcal{O}_{\widehat{X}} \arrow[r] & R\pi_*\mathcal{O}_E \arrow[r, "+1"] & {} \end{tikzcd}$$

X : smooth $\Rightarrow \mathcal{O}_X \simeq \underline{\Omega}^0_X$

$\Rightarrow \underline{\Omega}^0_\Sigma \simeq R\pi_*\mathcal{O}_E$

§ proof of Thm B

Set up

X : var / $\mathbb{C}$

$X \hookrightarrow Y$ Y: smooth.

$\Sigma = \mathrm{Sing}\, X$ $\quad \mathrm{codim}_Y X \geq 2$

$\pi : \widetilde{Y} \to Y$ resol. of X in Y

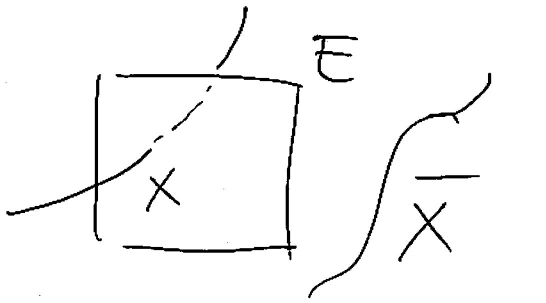
$\widetilde{X}$: strict trans of X s.t.

(i) π : isom over $X \setminus \Sigma$

(ii) $\underset{\overset{\shortparallel}{E}}{\pi^{-1}(\Sigma)}$: s.n.c div on $\widetilde{Y}$

$\pi^{-1}(\Sigma)$ intersecs $\widetilde{X}$ in a s.n.c div of $\widetilde{X}$

$\overline{X} := \pi^{-1}(X) = \widetilde{X} \cup E$



$\Rightarrow$ $R\pi_* \mathcal{O}_E \simeq \underline{\Omega}^0_\Sigma$

$R\pi_* \mathcal{O}_{\overline{X}} \simeq \underline{\Omega}^0_X$

Lem X : Du Bois

$\Leftrightarrow$ $R\pi_* \omega^\circ_X \to \omega^\circ_X$ quasi-isom.

(!) X : Du Bois $\Leftrightarrow$ $\mathcal{O}_X \to R\pi_* \mathcal{O}_X$: quasi isom.

G-duality

$\Longleftrightarrow$

$$R\mathcal{H}om(R\pi_*\mathcal{O}_{\bar{X}}, \omega^\bullet_X) \to \omega^\bullet_X \quad : \text{q-isom.}$$

$$\| $$

$$R\pi_*\omega^\bullet_{\bar{X}}$$

Prop X : CM $\dim X = d$

X : Du Bois

$\Longleftrightarrow$ $R^{-d}\pi_*\omega^\bullet_{\bar{X}} \to \omega_X$ is surj ($\Longleftrightarrow$ isom)

(∵) X : Du Bois

($\Rightarrow$) $\omega^\bullet_X = \omega_X[d]$

$\Leftarrow$ we can assume X : affine

By taking general hyperplane cuts

we can assume that $X \setminus P$: Du Bois
(P : closed pt)

$\times$ (X : not Du Bois at P)

we can localize everything at P

$$\mathcal{O}_{X_P} \to R\pi_*\mathcal{O}_{\bar{X}_P} \qquad I : \text{inj. hull of } \mathcal{O}_{X_P}/m_P$$

$$R\Gamma_P\mathcal{O}_{X_P} \to R\Gamma_P R\pi_*\mathcal{O}_{\bar{X}_P}$$

$\wr$ local duality $\qquad\qquad \wr$

$$Hom(R\mathcal{H}om(\mathcal{O}_{X_P}, \omega^\bullet_{X_P}), I) \longrightarrow Hom(R\mathcal{H}om(\overset{R\pi_*}{\check{\mathcal{O}}_{\bar{X}_P}}, \omega^\bullet_{X_P}), I)$$

$$\wr$$

$$R\pi_*\omega^\bullet_{\bar{X}_P}$$

Kovács' Lem.

$\Rightarrow \ (R^i\pi_* \omega_{\tilde{X}}^{\bullet})_p \to h^i(\omega_X^{\bullet})_p$ inj

$\Rightarrow \ R^i\pi_* \omega_{\tilde{X}}^{\bullet} = 0 \quad i \neq -d$

X: CM

$\Rightarrow \ R^{-d}\pi_* \omega_{\tilde{X}}^{\bullet} \simeq \omega_X$

$\Rightarrow \ R\pi_* \omega_{\tilde{X}}^{\bullet} \simeq \omega_X^{\bullet} \quad X$: Du Bois.

Lem

Z : reduced closed subscheme of Y

$\Rightarrow \ h^i(R\mathcal{H}om_Y(\underline{\Omega}_Z^0, \omega_Y^{\bullet})) = 0$ for $i < -\dim Z$

(:) Z, Y : affine.

$z \in Z$: $^\forall$ closed pt.

By local duality it is suff to see

$\mathbb{H}_z^j(Y, \underline{\Omega}_Z^0) = \mathbb{H}_z^j(Z, \underline{\Omega}_Z^0) = 0 \quad j > \dim Z$

$H_z^p(Z, h^q(\underline{\Omega}_Z^0)) \Rightarrow \mathbb{H}_z^{p+q}(Z, \underline{\Omega}_Z^0)$

$\dim \operatorname{Supp} h^q(\underline{\Omega}_Z^0) \leq \dim Z - q$

$\Rightarrow \ H_z^p(Z, h^q(\underline{\Omega}_Z^0)) = 0$

$p > \dim Z - q$

$\Rightarrow$ O.K.

Cor.

$$R^i\pi_* W^\cdot_E = 0 \quad \text{for } i < -\dim \Sigma$$

(∵) $R\pi_* \mathcal{O}_E \simeq \Omega^\circ_\Sigma$

$\updownarrow$ G-duality

$$R\pi_* W^\cdot_E \simeq R\mathcal{H}om(\Omega^\circ_\Sigma, \omega^\cdot_Y)$$

$\Rightarrow$ O.K. by Lem.

Th

$\operatorname{codim}_X \Sigma \geq 2$ (仮定) X: CM

$$\Rightarrow R^{-d}\pi_* W^\cdot_{\bar{X}} \simeq \pi_* \omega_{\tilde{X}}(E|\tilde{X})$$

(∵)

$$0 \to \mathcal{O}_{\tilde{X}}(-E|\tilde{X}) \to \mathcal{O}_{\bar{X}} \to \mathcal{O}_E \to 0$$

$\downarrow$ $\bar{X} = \tilde{X} \cup E$

$$W^\cdot_E \to \omega^\cdot_{\bar{X}} \to \omega^\cdot_{\tilde{X}} \otimes \mathcal{O}_{\tilde{Y}}(-E) \xrightarrow{+1}$$

$\downarrow$ taking cohomologies

$$R^{-d}\pi_* W^\cdot_E \to R^{-d}\pi_* W^\cdot_{\bar{X}}$$
$$\to R^{-d}\pi_*(\omega^\circ_{\tilde{X}} \otimes \mathcal{O}_{\tilde{Y}}(E))$$
$$\to R^{-d+1}\pi_* W^\cdot_E \quad \text{by Cor.}$$
$$\Rightarrow R^{-d}\pi_* W^\cdot_{\bar{X}} \simeq \pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{Y}}(E))$$