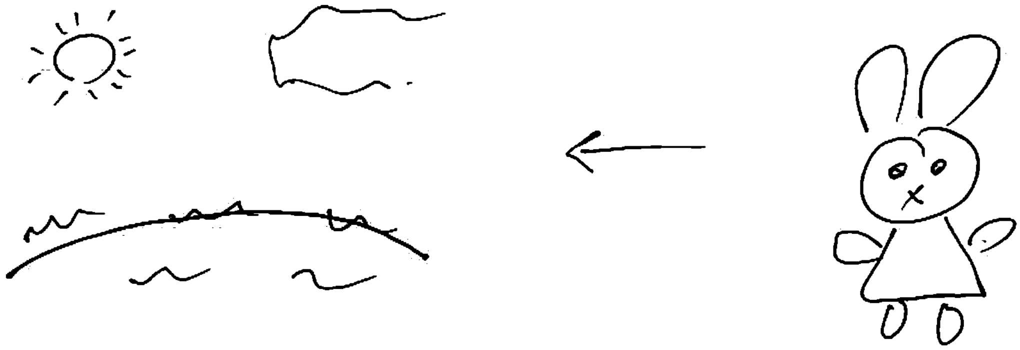
若気のいたり？

⇓

On noncommutative algebraic spaces

土基 善文（高知大学）。

§ Intro



世界 $\longleftrightarrow$ みている人

（わからないもの）

非可換なもの，例えば環の Spec.

空間 $\longleftrightarrow$ 多様体

粒子 $\longleftrightarrow$ 点

§ Weyl algebra / R .　（R：可換環）

$A_n(R) := R\langle r_1, \cdots, r_{2n}\rangle / CCR$

CCR : canonical commutative relation

$[r_i, r_j] = r_i r_j - r_j r_i = i\hbar_{ij}$

としたとき、

$$(\hbar_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & -1 & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -1 \\ 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & 1 & 0 & \cdots & 0 \end{pmatrix}$$

No. 2 Date

<u>$n=1$.</u> $A_n(R) = R\langle r_1, r_2\rangle / [r_2, r_1] = 1$

when R : field of char $= 0$

$\Rightarrow$ $A_n(R)$: simple algebra
(i.e. has no two-sided ideal)

$\Rightarrow$ $A_n(R)$ has no finite dimensional rep.

Heisenberg :「位置と運動量を両方とも測るのは
虫が良すぎるやんけ。」

---

§ $\boxed{\text{Guiding Problem}}$

- Diximer conjecture (D_n)

k : field of char $= 0$

$\forall \varphi : A_n(k) \longrightarrow A_n(k)$
k-algebra endmorphism

$\Rightarrow$ φ : invertible ⇔ つまり isomorphism.

● Jacobian conjecture (J_n)

k: field of char $= 0$

$$k[X_1, \cdots, X_n] \xrightarrow{\varphi} k[X_1, \cdots, X_n]$$

k: algebra hom. s.t. Jacobian $= 1$

$\Rightarrow$ φ : invertible

<u>Fact</u> $D_n \Rightarrow J_n$

<u>if char $k = p \neq 0$</u>

$$\begin{array}{ccc} A_1(k) & \xrightarrow{\alpha} & A_1(k) \\ \xi & \longmapsto & \xi \\ \eta & \longmapsto & \eta + \eta^p \end{array} \qquad \begin{array}{ccc} k[X_1, X_2] & \xrightarrow{\beta} & k[X_1, X_2] \\ X_1 & \longmapsto & X_1 \\ X_2 & \longmapsto & X_2 + X_2^p \end{array}$$

α, β give non invertible alg. endmorphisms.

この様に char $k = p \neq 0$ の場合には、D_n, J_n は不成立であるが、char $k = p$ の場合に還元して、D_n, J_n にアタックできる。

If k is a field of char $k = p \neq 0$

$$Z(A_n(k)) = k[r_1^p, r_2^p, \cdots, r_{2n}^p]$$

■ $A_n(k)$ is a finite algebra over its center $Z(A_n(k))$

$$\begin{array}{ccc} Z & & \\ \cup\!\!\!\!\cup & & \\ Z(A_n(k)) & \longrightarrow & k[T_1, \cdots, T_{2n}] =S \\ \cup\!\!\!\!\cup & & \\ r_i^p & \longmapsto & T_i^p \end{array}$$

$$\Longrightarrow \quad S \otimes_Z A_n(k) \simeq M_{p^n}(S).$$

■ (char$(k) = p \neq 0$ の場合)

$$\begin{array}{ccccl} A_n(k) & \xrightarrow{\varphi} & A_n(k) & ; & k\text{-alg hom} \\ \cup & \circlearrowright & \cup & & \\ Z & \dashrightarrow & Z & & \end{array}$$

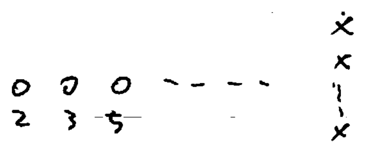
$$\text{Jacobian}(\varphi) = 1 \quad \text{if} \quad p >> \deg(\varphi).$$

§ ⓪ ultrafilter

$$\text{char} = p \underset{\text{"ultrafilter"}}{\overset{\text{mod } p}{\rightleftarrows}} \text{char} = 0$$

P : set.

e.g. $P = \{\text{primes}\}$



maximal cpt化 の boundary が ultrafilter

$\mathcal{F} \subset 2^{P}$.

Def $\mathcal{F}$ is filter

$\underset{\text{def}}{\Longleftrightarrow}$ ① $\phi \notin \mathcal{F}$

② $A \in \mathcal{F}$ $A \subset B \subset P \Rightarrow B \in \mathcal{F}$

③ $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

$\mathcal{F}$ is ultrafilter

$\Longleftrightarrow$ maximal filter.

(which exists because of Zorn's lemma) example

① fix $p \in P$

then

$F_p := \{ U \subset P \mid U \ni p \}$

$\Rightarrow$ F_p : ultrafilter.

(principal ultrafilter)

Ⅱ assume $\# P = +\infty$

$F_{cofinite} := \{ S \subset P \quad \#(P \setminus S) < +\infty \}$

: a filter on P.

$\exists \mathcal{U}$: ultrafilter which is finer than $F_{cofinite}$.

this is a non-principal ultrafilter.

◎ exercise

$\mathcal{U}$: non principal ultrafilter on P.

f : bounded $\mathbb{C}$-valued function on P.

$\Rightarrow$ $\exists 1\, c \in \mathbb{C}$, s.t. $\lim_{p \to \mathcal{U}} f(p) = c$

i.e. $\forall \varepsilon > 0$, ~~$\exists U \in \mathcal{U}$~~

s.t. $\{ p \cancel{\in U} \mid |f(p) - c| < \varepsilon \} \in \mathcal{U}$

lemma

$\mathcal{F}$: filter on $\mathbb{P}$.

$\mathcal{F}$ is an ultrafilter

$\Longleftrightarrow$ $\forall A \subset \mathbb{P}$.

「$A \in \mathcal{F}$ or $CA \in \mathcal{F}$」

◎ limit using ultrafilter

$\mathcal{U}$: ultrafilter

$\forall p \in \mathbb{P}$ に対し module M_p が（バラバラに）与えられているとき.

$$\lim_{p \to \mathcal{U}} M_p := \prod_{p \in \mathbb{P}} M_p \Big/ \left\{ (m_p)_{p \in \mathbb{P}} \;\middle|\; \begin{array}{c} \exists U \in \mathcal{U} \\ m_p = 0 \\ \forall p \in U \end{array} \right\}$$

$$M_p : \text{ring} \implies \lim_{p \to \mathcal{U}} M_p \quad \text{ring}$$

<u>From now, we assume</u> $\underline{P} = \{\text{primes}\}$.

$\mathcal{U}$: non-principal ultrafilter on $\underline{P}$.

$\lim_{p \to \mathcal{U}} \mathbb{F}_p \ni (a_2, a_3, a_5, \cdots)$

is a field of char $= 0$.

- $\lim \mathbb{F}_p$ is field

(!) $(a_2, a_3, \cdots)$ non zero in $\lim_{p \to \mathcal{U}} \mathbb{F}_p$

$\Rightarrow \{p \mid a_p = 0\} \notin \mathcal{U}$

$\Rightarrow \underbrace{\{p \mid a_p \neq 0\}}_{U} \in \mathcal{U}$

put $$b_p = \begin{cases} a_p^{-1} & p \in U \\ 1 & p \notin U \end{cases}$$

$$\Rightarrow (a_p)(b_p) = (a_p b_p) = \begin{cases} 1 & p \in U \\ 0 & p \notin U \end{cases}$$

$$= 1 \quad \text{in } \lim_{p \to \mathcal{U}} \mathbb{F}_p$$

$\lrcorner$

$$\text{char} \lim_{p\to\mathcal{U}} \mathbb{F}_p =: \mathbb{F}_{\mathcal{U}} \quad = 0$$

(∵)

$k \in \mathbb{Z}$. $k = 0$ in $\mathbb{F}_{\mathcal{U}}$

$\Rightarrow (k, k, \cdots) = 0$

$\Rightarrow \{ p \quad k = 0 \text{ in } \mathbb{F}_p \} \in \mathcal{U}$.

contradict to $\mathcal{U}$: non principal. $\lrcorner$

$$\mathbb{C}_{\mathcal{U}} := \lim_{p\subset\mathcal{U}} \overline{\mathbb{F}_p}$$

$\Longrightarrow$ $\mathbb{C}_{\mathcal{U}}$: algebraically closed field of char $= 0$.

A_p : $\mathbb{F}_p$-algebra $\quad p \in \mathbb{P}$

$$A_p = \bigcup_{j\geq 0} A_{p,j}$$

$$\sup_p \dim_{\mathbb{F}_p} A_{p,j} < +\infty \qquad (\forall j)$$

$$\Longrightarrow \bigcup_{j\geq 0} \lim_{p\to\mathcal{U}} (A_{p,j}) \subset A_{\mathcal{U}}.$$

$\varphi : A_n(\mathbb{Z}) \longrightarrow A_n(\mathbb{Z})$ $\quad$ $\mathbb{Z}$-alg. hom.

$$\Rightarrow \varphi_p = \varphi \bmod p : \begin{array}{ccc} A_n(\mathbb{F}_p) & \longrightarrow & A_n(\mathbb{F}_p) \\ \cup & & \cup \\ Z & \longrightarrow & Z \end{array}$$

Jacobian $= 1$ if $p \gg \deg \varphi$.

☆ degree of φ_p は bound されている。

$\Rightarrow$ shadow of φ

$\psi_u : \mathbb{F}_u[T_1, \cdots, T_{2n}] \longrightarrow \mathbb{F}_u[T_1, \cdots, T_{2n}]$

Jacobian $\psi_u = 1$.

<u>Proposition</u>

$\exists$ shadow map : $\mathrm{End}_{\mathbb{C}_u\text{-alg}}(A_n(\mathbb{C}_u)) \longrightarrow \underset{\text{self morphism}}{\mathrm{Symplectic}}(\mathbb{C}_u[T_1, \cdots, T_{2n}])$

(monoid hom).

§ NAS (noncommutative algebraic spaces)

LNAS / k : field. ---- (little NAS)

$X = (\mathcal{C}, |X|, \mathcal{A})$ is an LNAS / k.
with a shadow $|X|$ if.

1 : $\mathcal{C}$: k-linear abelian category
2 : $|X|$ is an algebraic space / k (<---- 「stack とした方が よいかも。」)
3 : $\mathcal{A}$ is a sheaf of algebras over $|X|$
which contains $\mathcal{O}_{|X|}$ as a central algebra.
4 : $\mathcal{A}$ is $\mathcal{O}_{|X|}$-coherent.
5 : $\mathcal{C} \simeq \mathcal{A}\text{-Mod}$

NAS

R : commutative Dedekind domain (e.g. $\mathbb{Z}$).

A NAS X over R consists of the following data.

1. $\mathcal{C}$: R linear abelian category.
which is augmented, flat over R.

2. LNAS $X_{(\mathfrak{p})}$ over $R/\mathfrak{p}$
for each $\mathfrak{p} \in \mathrm{Spm} R$.

3. an isomorphism
$\mathcal{C}/\mathfrak{p} \simeq X_{(\mathfrak{p})}$ of $R/\mathfrak{p}$-linear abelian categories for each $\mathfrak{p} \in \mathrm{Spm}(R)$

$A_n(\mathbb{Z})$ の外にも. $U(\mathfrak{g})$ なんかも考えられる。