Deformation quantization in algebraic geometry II

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§4 Formality thm in algebraic case

X : smooth alg var $/\mathbb{C}$ on C^∞-mfd

$T^\cdot[1]$, $D_X^\cdot[1]$ ⋯⋯ sheaves of DGLA

$T_X^\cdot[1]^p = \overset{p+1}{\Lambda} T_X \qquad D_X^\cdot[1]^p \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X^{\otimes p+1}, \mathcal{O}_X)$

Formality thm : X : C^∞-mfd $\Rightarrow {}^\exists L_\infty$-q.i.s

$$\Gamma(X, T_X^\cdot[1]) \xrightarrow{\ \Psi\ } \Gamma(X, D_X^\cdot[1])$$

Idea (i) construct Ψ for $X = \mathbb{R}^d$ explicity

depends on coordinate system on $\mathbb{R}^d$

(ii) $\pi : X^{coor} \longrightarrow X$ fiber of π

$\simeq$ {choice of coordinate system}

$d = \dim X \Rightarrow GL(d, \mathbb{C}) \curvearrowright X^{coor}$

$X^{aff} := X^{coor} / GL(d, \mathbb{C}) \longrightarrow X$ use the assumption that X ⋯ C^∞-mfd

${}^\exists s$ ⋯ C^∞-section

(Roughly use (i), to construct some L_∞-q.i.s. between two DGLA on X^{aff}, pull-back by s.)

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Idea (i) is applied for $X = \mathbb{A}^d$

$\Rightarrow$ Formality thm holds for $\mathbb{A}^d$

Global case : (From here, X ⋯ sm var / $\mathbb{C}$)

HKR - isom : $T_X^\bullet[1] \longrightarrow D_X^\bullet[1]$: q.is

doesn't necessary indue $\Gamma(X, T_X^\bullet[1]) \longrightarrow \Gamma(X, D_X^\bullet[1])$

need to replace $\Gamma(X, T_X^\bullet[1])$, $\Gamma(X, D_X^\bullet[1])$
by $R\Gamma(X, T_X^\bullet[1])$, $R\Gamma(X, D_X^\bullet[1])$

$R\Gamma(-) : D(\mathcal{O}_X\text{-mod}) \longrightarrow D(\mathbb{C}\text{-mod})$

derived categories

$\mathcal{U} = \{U_i\}_{1 \le i \le N}$ $U_i \subset X$ affin open $\mathcal{U}$ --- open cover

$\mathcal{F} \in \mathcal{O}_X\text{-mod}$ $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 \cdots i_p} j_{i_0 \cdots i_p *} j_{i_0 \cdots i_p}^* \mathcal{F}$

$j_{i_0 \cdots i_p} : U_{i_0} \cap \cdots \cap U_{i_p} \hookrightarrow X$: open imm

$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$ čeck resol

$$R\Gamma(X, \mathcal{F}) \simeq \Gamma(X, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))$$

$$R\Gamma(X, T_X^\bullet[1]) \simeq \Gamma(X, \mathrm{Tot}\, \mathcal{C}^\bullet(\mathcal{U}, T_X^\bullet[1]))$$

$$R\Gamma(X, D_X^\bullet[1]) \simeq \Gamma(X, \mathrm{Tot}\, \mathcal{C}^\bullet(\mathcal{U}, D_X^\bullet[1]))$$

double complexes

doesn't have natural DGLA-str

$\mathcal{U} = \{u_{ij}\}_{ij} \in \mathcal{C}^1(\mathfrak{U}, T_X)$ $\qquad u_{ij} \in \Gamma(U_{ij}, T_X|_{U_{ij}})$

$U = \{U_{ij}\} \in$ " $\qquad U_{ij} \in$ "

One may try to define $[u, V]$
by $\{[u_{ij}, U_{jk}]\}_{i,j,k} \in \mathcal{C}^2(\mathfrak{U}, T_X)$

but $[u, U] \neq \pm [U, u]$.

need to construct DGLA.

$R\Gamma^{Lie}(X, T_X^\cdot[1])$, $R\Gamma^{Lie}(X, D_X^\cdot[1])$
q.i.s to $R\Gamma(X, T_X^\cdot[1])$. $R\Gamma(X, D_X^\cdot[1])$

use Thom - Sullivan functor

∘ Thom - Sullivan functor

Δ = category with objects $= \{[0], [1], \cdots, [n], \cdots\}$

$\mathrm{Hom}([i], [j]) = \{\{0, \cdots, i\} \to \{0, \cdots, j\}$ non-decreasing map$\}$

Def
$\mathfrak{g}$ is cosimplicial DGLA
$\leftrightarrow$ $\mathfrak{g}$ is a functor : $\Delta \longrightarrow$ DGLA

$\leftrightarrow$ $\left([0] \rightrightarrows [1] \ \overrightarrow{\underrightarrow{\rightarrow}} \ [2] \ \cdots \right)$

$\mathfrak{g}([0]) \rightleftarrows \mathfrak{g}([1]) \ \cdots$, $\mathfrak{g}([i]) \cdots$ DGLA

EX.

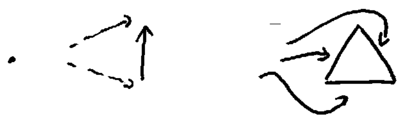
$$\mathcal{C}^0(\mathcal{U}, T_X^\bullet[1]) \;\substack{\rightarrow\\ \rightarrow\\ \leftarrow}\; \mathcal{C}^1(\mathcal{U}, T_X^\bullet[1]) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \leftarrow\\ \leftarrow}\; \mathcal{C}^2(\mathcal{U}, T_X^\bullet[1])$$

sheaf of cosimplicial DGLA.

$$\Delta^i := \mathrm{Spec}\; \mathbb{C}[t_0, \dots, t_i] \,/\, (t_0 + \cdots + t_i - 1)$$

$\{\Delta^i\}_i$ .... cosimplicial scheme

$$\Delta^0 \;\substack{\rightarrow\\ \rightarrow\\ \leftarrow}\; \Delta^1 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \leftarrow\\ \leftarrow}\; \Delta^2 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}$$



Def

g ... cosimplicial DGLA

complex of $\mathbb{C}$-vect. sp

For fixed p, construct $\Omega^p(g)$ as follows :

$$\Omega^p(g) \subset \prod_i \Omega^p(\Delta^i) \otimes_{\mathbb{C}} g([i])$$

$\{u_i\}_i \in \prod_i \Omega^p(\Delta^i) \otimes_{\mathbb{C}} g([i])$ is contained in $\Omega^p(g)$ iff

$$\forall f : [i] \longrightarrow [j] \text{ in } \Delta$$

$$\left(\begin{array}{ll} \Delta^i \to \Delta^j & f^* : \Omega^p(\Delta^j) \to \Omega^p(\Delta^i) \\ g([i]) \xrightarrow[f_*]{} g([j]) & \end{array} \right.$$

$$\Omega^p(\Delta^i)\otimes_{\mathbb C}\mathfrak g([i]) \xrightarrow{1\otimes f_*} \Omega^p(\Delta^i)\otimes_{\mathbb C}\mathfrak g([i]) \xleftarrow{f^*\otimes 1} \Omega^p(\Delta^j)\otimes_{\mathbb C}\mathfrak g([i])$$

$$(1\otimes f_*)u_i = (f^*\otimes 1)u_j$$

$\Omega^p(j)$ is a complex with differential indued by $\{\mathfrak g([i])\}_i$ ⌟

Double complex :

$$0\to\Omega^0(\mathfrak g)\to\Omega^1(\mathfrak g)\to\cdots\to\Omega^p(\mathfrak g)\to\Omega^{p+1}(\mathfrak g)\to\cdots$$

$\Omega^p(\mathfrak g)\to\Omega^{p+1}(\mathfrak g)$ is indued by $\Omega^p(\Delta^i)\to\Omega^{p+1}(\Delta^i)$

Def Thom-Sullivan functor

cosimplicial DGLA $\ni \mathfrak g \longmapsto \mathfrak g^{TS}\in$ DGLA
is defined by $\mathfrak g^{TS} = \mathrm{Tot}(\Omega^\cdot(\mathfrak g))$.

$[\ ,\ ]$ on $\mathfrak g^{TS}$

$u=\{a_i\otimes u_i\}_i$ $a_i\in\Omega^p(\Delta^i)$ $u_i\in\mathfrak g([i])$
$v=\{b_i\otimes v_i\}_i$ $b_i\in\Omega^q(\Delta^i)$ $v_i\in\mathfrak g([i])$

$$[u.v] = \{\pm a_i\wedge b_i\,[u_i,v_i]\}_i \in \prod_i \Omega^{p+q}(\Delta^i)\otimes\mathfrak g([i])$$

$$[u,v] = \pm[v,u]$$

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Def

$$R\Gamma^{Lie}(X, T^{\cdot}[1]) := \Gamma(X, \mathcal{C}^{\cdot}(\mathcal{U}, T_X^{\cdot}[1]))^{TS}$$

$$R\Gamma^{Lie}(X, D_X^{\cdot}[1]) := \Gamma(X, \mathcal{C}^{\cdot}(\mathcal{U}, D_X^{\cdot}[1]))^{TS}$$

$$T_X^{\cdot}[1] \longrightarrow \mathcal{C}^{\cdot}(\mathcal{U}, T_X[1])^{TS}$$

Fact : $R\Gamma^{Lie}(X, T_X^{\cdot}[1])$, $R\Gamma^{Lie}(X, D_X^{\cdot}[1])$ are q.i.s to $R\Gamma(X, T_X^{\cdot}[1])$. $R\Gamma(X, D_X^{\cdot}[1])$

Thm (A. Yekutieli M. van den Bergh)

$$\exists\ L_\infty\text{-q.i.s}\quad \underset{\substack{\parallel \\ \mathfrak{g}_T}}{R\Gamma^{Lie}(X, T_X^{\cdot}[1])} \longrightarrow \underset{\substack{\parallel \\ \mathfrak{g}_D}}{R\Gamma^{Lie}(X, D_X^{\cdot}[1])}$$

Formality thm in alg case

In particular, $\exists$ isom $\mathrm{Def}_{\mathfrak{g}_T} \xrightarrow{\sim} \mathrm{Def}_{\mathfrak{g}_D}$

First order deformation

$$\mathrm{Def}_{\mathfrak{g}_T}(\mathbb{C}[\hbar]/\hbar^2)$$

$$= H^3(R\Gamma(X, T_X^{\cdot}[1])) \qquad T_X^{\cdot}[1] = \mathcal{O}_X[+1] \oplus T_X \oplus \overset{2}{\wedge} T_X[-1] \oplus \cdots$$

$$= \underbrace{H^2(\mathcal{O}_X)}_{\substack{\text{gerby} \\ \text{deformation}}} \oplus \underbrace{H^3(T_X)}_{\substack{\text{Commutative} \\ \text{deformation}}} \oplus \underbrace{H^0(\overset{2}{\wedge} T_X)}_{\substack{\text{non-comm deformation} \\ \text{of } \mathcal{O}_X}}$$

$$\Psi$$

$$\eta = (\alpha, \beta, \gamma)$$

Čech rep.

$\alpha = \{\alpha_{ijk}\}_{i,j,k} \in \Gamma(X, \check{C}^2(\mathcal{U}, \mathcal{O}_X))$ $\alpha_{ijk} \in \mathcal{O}_{ijk}$

$\beta = \{\beta_{ij}\}_{ij} \in \Gamma(X, \check{C}^1(\mathcal{U}, T_X))$ $\beta_{ij} \in T_{U_{ij}}$

γ --- (non-comm) ring str on $\mathcal{O}_X[\hbar]/\hbar^2$:

write $\mathcal{O}_X[\hbar]/\hbar^2{}^{(\gamma)}$

Def Define $\mathbb{C}[\hbar]/\hbar^2$ - linear abelian category $Coh(X, u)$ as follows:

$\tilde{\beta}_{ij} : \mathcal{O}_{U_{ij}}[\hbar]/\hbar^2{}^{(\gamma)} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}[\hbar]/\hbar^2{}^{(\gamma)}$

$a + b\hbar \longmapsto a + (\beta_{ij}(a) + b)\frac{1}{\hbar}$

obj $= \{(\mathcal{F}_i, \phi_i) \mid \mathcal{F}_i \in Coh(\mathcal{O}_{U_i}[\hbar]/\hbar^2)$,
$\phi_i : \mathcal{F}_i \xrightarrow{\sim} \tilde{\beta}_{ij}^* \mathcal{F}_j$ s.t.

$$\mathcal{F}_i \xrightarrow{\phi_i} \tilde{\beta}_{ij}^* \mathcal{F}_j \longrightarrow \tilde{\beta}_{ij}^* \tilde{\beta}_{jk} \mathcal{F}_k \xrightarrow[\tilde{\beta}_{ij}^* \tilde{\beta}_{jk} \phi_k]{} \tilde{\beta}_{ij}^* \tilde{\beta}_{jk}^* \tilde{\beta}_{ki} \mathcal{F}_i = \mathcal{F}_i$$

is equal to $\times(1 - \alpha_{ijk}\hbar)$ }

Mor = obvious

$Coh(X, u)$ ---- first order deformation to $Coh\, X$.

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In general M. Van den Bergh shows.

$\mathrm{Def}_{g_0}(R) = \{$ flat deformations of $\mathrm{Coh}\, X / R \} /$ eq.

defined by using Shilov glueing data.

Poisson str $\rightsquigarrow$ Formal deformation of $\mathrm{Coh}\, X$

§5 Deformations and Fourier - Mukai transforms

X : smooth projective var $/\mathbb{C}$

$D(X) := D^b \mathrm{Coh}\, X$ bounded derived category of coherent sheaves.

obj $= \{ \to 0 \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots \to \mathcal{F}^j \to \cdots \}$

bounded complex.

Kontsevich's philosophy :

It's natural to consider $D(X)$ as "space" in string theory.

has more symmetries due to Fourier-Mukai transforms

Alg var $\ni X \longmapsto D(X) \in$ triangulated category

$\leftarrow \!\!\!\! - \!\!\! \times \!\!\! - -$ does not necessary reconstruct X.

Def

$X, Y \cdots$ smooth proj var

$P \in D(X \times Y)$

$\Rightarrow \Phi^P_{X \to Y} : D(X) \longrightarrow D(Y)$ defined by

$\Phi^P_{X \to Y}(E) \simeq Rp_{Y*}(p_X^*(-) \overset{L}{\otimes} P)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

$\Phi^P_{X \to Y}$ equivales $\Rightarrow$ Fourier – Mukai transform (FM)

Orlovs thm :

$\forall \Phi : D(X) \xrightarrow{\simeq} D(Y)$ is of FM type

EX (i) $A \cdots$ abelian var $\hat{A} = Pic^0(A)$

$\mathcal{P} \in Pic(A \times \hat{A})$ Poincare line bundle

$\Rightarrow \Phi^{\mathcal{P}}_{\hat{A} \to A} : D(\hat{A}) \longrightarrow D(A)$ is Fourier - Mukai transform

(ii) $\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\ & W & \end{tikzcd}$ 3-dim flop $\Rightarrow \Phi^{\mathcal{O}_{X \times_W Y}}_{Y \to X} : D(Y) \longrightarrow D(X)$ is FM transform

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Question:

How do deformation theores of X, Y relate to FM transform $\Phi : D(X) \simeq D(Y)$?

prop $\Phi : D(X) \simeq D(Y)$ FM transform

$\Rightarrow$ Φ indues an isom

$$\bigoplus_{i+j=N} H^i(X, \overset{j}{\wedge} T_X) \xrightarrow[\phi]{\sim} \bigoplus_{i+j=N} H^i(Y, \overset{j}{\wedge} T_Y), \quad \forall N$$

(Remark : ϕ does not necessary preseve direct summands)

· Outline of the proof:

$$HT^N(X) := \bigoplus_{i+j=N} H^i(X, \overset{j}{\wedge} T_X)$$

(HKR-isom) Hochschild cohomology

$$HH^N(X) := \mathrm{Ext}^N_{X\times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$$

$$\begin{array}{ccc} \Delta : X & \longrightarrow & X \times X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & (x, x) \end{array}$$

idea of proof:

$$HH^N(X) \simeq \mathrm{Ext}^N_X(\mathbb{L}\Delta^* \Delta_* \mathcal{O}_X, \mathcal{O}_X)$$

$\exists$ resol

$$\begin{array}{c} \phantom{\longrightarrow \mathcal{O}_X^{\otimes 3} \longrightarrow\ } \mathcal{O}_{X\times X} \\ \phantom{\longrightarrow \mathcal{O}_X^{\otimes 3} \longrightarrow\ } \parallel \\ \longrightarrow \mathcal{O}_X^{\otimes 3} \longrightarrow \mathcal{O}_X^{\otimes 2} \longrightarrow \Delta_* \mathcal{O}_X \longrightarrow 0 \end{array}$$

$$\begin{array}{l} \mathbb{L}\Delta^* \Delta_* \mathcal{O}_X \underset{q.i.s}{\simeq} (\longrightarrow \mathcal{O}_X^{\otimes 3} \longrightarrow \mathcal{O}_X^{\otimes 2} \longrightarrow \mathcal{O}_X \longrightarrow 0) \\ \phantom{\mathbb{L}\Delta^* \Delta_* \mathcal{O}_X \simeq (\longrightarrow\ } \downarrow \hspace{18em} q.i.s \\ \phantom{\mathbb{L}\Delta^* \Delta_* \mathcal{O}_X \simeq (} \longrightarrow \Omega_X \xrightarrow{=} \Omega_X \xrightarrow{0} \mathcal{O}_X \longrightarrow 0 \end{array}$$

$$\begin{array}{ccc} \mathcal{O}_X^{\otimes p+1} & \longrightarrow & \Omega_X^p \\ f_0 \otimes f_1 \otimes \cdots \otimes f_p & \longmapsto & f_0\, df_1 \wedge \cdots \wedge df_p \end{array}$$

$$\mathbb{L}\Delta^* \Delta_* \mathcal{O}_X \simeq \bigoplus_{p \geq 0} \Omega_X^p [p] \quad \in D(X)$$

$$\Rightarrow HT^N(X) \simeq HH^N(X)$$

STEP 2. $HH^N(X) \simeq HH^N(Y)$

idea of the proof $\quad HH^N(X) = \mathrm{Hom}_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[N])$

$$\fallingdotseq \{ id_X \longrightarrow [N],\ \text{natural transforms} \}$$

$$HH^N(Y) \qquad \wr\wr$$

$$\fallingdotseq \{ id_Y \longrightarrow [N],\ \text{"} \}$$

More particularly.

$$\exists \Phi : \begin{array}{ccc} D(X \times X) & \xrightarrow{\sim} & D(Y \times Y) \\ \mathcal{O}_{\Delta_X} & \longmapsto & \mathcal{O}_{\Delta_Y} \end{array}$$

$$\begin{array}{ccccc} \mathrm{Defg}_{T_X}(\mathbb{C}[\hbar]/\hbar^2) & \ni & u & \longrightarrow & \mathrm{Coh}(X, u) \\ \| & & \downarrow & & \\ HT^2(X) & & & & \\ \downarrow \wr\ \phi & & & & \\ HT^2(Y) & & & & \\ \| & & \downarrow & & \\ \mathrm{Defg}_{T_Y}(\mathbb{C}[\hbar]/\hbar^2) & \ni & v = \phi(u) & \longrightarrow & \mathrm{Coh}(Y, v) \end{array}$$

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Thm (T)

$\Phi : D(X) \xrightarrow{\simeq} D(Y)$ $u \in HT^2(X)$ $v = \phi(u) \in HT^2(Y)$

$\Rightarrow \Phi$ extends to an equivalent

$\widetilde{\Phi} : D^b Coh(X, u) \xrightarrow{\simeq} D^b Coh(Y, v)$

EX

(i) A A.V. $\hat{A}$: $Pic^\circ(A)$

$\Phi^{u}_{S \to A} : D(\hat{A}) \xrightarrow{\simeq} D(A)$ $u \in Pic(\hat{A} \times A)$

$\Rightarrow \phi : H^2(\mathcal{O}_{\hat{A}}) \oplus H^1(T_{\hat{A}}) \oplus H^0(\wedge^2 T_{\hat{A}}) \xrightarrow{\simeq} H^2(\mathcal{O}_A) \oplus H^1(T_A) \oplus H^0(\wedge^2 T_A)$

$(0, 0, \gamma) \longmapsto (\alpha, 0, 0)$

(ii) $X \dashrightarrow Y$ 3-dim flop $\Phi^{\mathcal{O}_{X \times_W Y}}_{Y \to X} : D(Y) \longrightarrow D(X)$

$\searrow \swarrow$
W

ϕ takes $H^1(T_Y) \longrightarrow H^1(T_X)$

problem : Extend $\widetilde{\Phi}$ to higher order deformations

(Pointer, Ben-Bassat, Block)

Problem holds true for Ex (i)

Conjecture

$$\Phi : D(X) \xrightarrow{\sim} D(Y)$$

(i) $\exists\, L_\infty$ - q.i.s $\quad \underbrace{R\Gamma^{Lie}(X, T_X^{\cdot}[1])}^{g_{T_X}} \longrightarrow \underbrace{R\Gamma^{Lie}(Y, T_Y^{\cdot}[1])}^{g_{T_Y}}$

In particular $\exists$ isom $\mathrm{Def}_{g_{T_X}} \xrightarrow[\sim]{\phi} \mathrm{Def}_{g_{T_Y}}$

(ii) $u \in \mathrm{Def}_{g_{T_X}}(R) \longrightarrow \mathrm{Coh}(X, u)$ $\quad$ R-flat def of Coh X

$\downarrow$

$v = \phi(u) \in \mathrm{Def}_{g_{T_Y}}(R) \longrightarrow \mathrm{Coh}(Y, v)$ $\quad$ " $\quad$ Coh Y

$\Rightarrow$ Φ extends to an eq.

$$\tilde{\Phi} : D^b\mathrm{Coh}(X, u) \xrightarrow{\sim} D^b\mathrm{Coh}(Y, v)$$