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Deformation quantization in algebraic geometry I

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References

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§1 Background

$(X, \mathcal{O}_X)$ smooth alg var $/\mathbb{C}$ or

C^∞-manifold $\mathcal{O}_X$ sheaf of $\mathbb{C}$-valued C^∞-functions

Want to deform $\mathcal{O}_X$ into sheaf of non-commutative algebra

$\rightsquigarrow$ deformation of $\mathrm{Coh}(X)$

Alg var $\ni X \longmapsto \mathrm{Coh}\,X \in$ abelian category

$\uparrow$ abelian category of coherent sheaf on X

Noncommutative algebraic geometry

= "geometry of abelian categories"

$\supset$ deformation theory ?

No. Date

We have to take account of non commutative deformation of $\mathcal{O}_X$

- 1st deformation $\longleftrightarrow$ $\mathbb{C}[h]/h^2$ - alg str on $\mathcal{O}_X[h]/h^2$

$r \in \Gamma(X, \overset{2}{\wedge} T_X)$

$\longrightarrow \langle , \rangle_r : \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$ bi-diff op

$r = \sum r_1 \wedge r_2$ locally $\quad \langle f, g \rangle_r = \frac{1}{2} \sum (r_1(f) r_2(g) - r_1(g) r_2(f))$

$\longrightarrow \mathbb{C}[h]/h^2$ str on $\mathcal{O}_X[h]/h^2$ given by

$$(f_1 + \hbar f_2)(g_1 + \hbar g_2) = f_1 g_1 + (\langle f_1, g_1 \rangle_r + f_1 g_2 + f_2 g_1) \hbar$$

Def

$*$ - product on $\mathcal{O}_X[[h]]$ is a $\mathbb{C}[[\hbar]]$ - alg str on $\mathcal{O}_X[[h]]$ s.t. $\forall f, g \in \mathcal{O}_X$

$$f * g = fg + \beta_1(f, g) h + \beta_2(f, g) h^2$$

$\beta_i(-, -) : \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$ bi-diff op

$*_1, *_2$ product on $\mathcal{O}_X[[h]]$ are gauge equivalent

$\Leftrightarrow \exists \phi (\mathcal{O}_X[[h]], *_2) \xrightarrow{\sim} (\mathcal{O}_X[[h]], *_1)$

$\mathbb{C}[[h]]$ alg isom

$\phi(f) = f + \phi_1(f) t + \phi_2(f) t^2 + \cdots \quad \phi_i$: diff

$f \in \mathcal{O}_X^{op}$

$\left(\begin{array}{l} \phi : \mathcal{O}_X \to \mathcal{O}_X \text{ diff} \\ \Leftrightarrow \text{locally} \\ \quad \phi = \sum a_{i_1 \cdots i_k} \frac{d}{d x_{i_1}} \cdots \frac{d}{d x_{i_k}} \end{array} \right)$

Question

Given $r \in \Gamma(X, \overset{2}{\wedge} T_X)$, $\exists *$ product on $\mathbb{C}[[\hbar]]$ s.t. $\beta_1(-, -) = \langle -, - \rangle_r$?

Associativity modulo $\hbar^2 \Leftrightarrow \langle -, - \rangle_r$ satisfies Jacobi id

i.e. $\langle \langle f, g \rangle_r, h \rangle_r + \langle \langle g, h \rangle_r, f \rangle_r + \langle \langle h, f \rangle_r, g \rangle_r = 0$

$\forall f, g, h \in \mathcal{O}_X$

Def $\pi \in \Gamma(X, \wedge^2 T_X)$ is a Poisson str

$\underset{\text{def}}{\Leftrightarrow}$ $\{\ ,\ \}_\pi$ satisfies Jacobi identity

Thm (Kontsevich)

X C^∞-manifold

$\Rightarrow$ ${}^\forall \pi \in \Gamma(X, \wedge^2 T_X)$ Poisson structure

Question is true

More precisely

$$\{\text{Formal Poisson str } \pi \in \Gamma(X, \wedge^2 T_X)[[\hbar]]\} \Big/ \mathrm{Diff}_0(X)_{\to \mathrm{id}} \overset{1:1}{\longleftrightarrow} \{\ast\text{-product on } \mathcal{O}_X[[\hbar]]\} \Big/ \text{gauge eq}$$

$$\mathrm{Diff}_0(X)_{\to \mathrm{id}} \subset \{f : X \xrightarrow{\sim} X \text{ diffeo}\}$$

π : Poisson $\Rightarrow$ $\hbar\pi$ formal Poisson str

§2 L_∞-algebras and DGLA
(Differential Graded Lie Algebra)

V graded vector space $/\mathbb{C}$

$\rightsquigarrow$ $\mathrm{Sym}^n V$, $\wedge^n V$ graded vector sp

$x, y \in V$ homogeneous elements of deg $|x|, |y|$

$x \otimes y \rightsquigarrow (-1)^{|x||y|}\, y \otimes x$

defines S_n action on $V^{\otimes n}$

$$V^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n V \hookrightarrow V^{\otimes n}$$
$$v_1 \otimes \cdots \otimes v_n \longmapsto \sum_{\sigma \in S_n} \frac{1}{n!}\, \sigma(v_1 \otimes \cdots \otimes v_n)$$

$$V^{\otimes n} \longrightarrow \wedge^n V \hookrightarrow V^{\otimes n}$$
$$v_1 \otimes \cdots \otimes v_n \longmapsto \sum_{\sigma \in S_n} \frac{1}{n!}\, \mathrm{sgn}(\sigma)\, \sigma(v_1 \otimes \cdots \otimes v_n)$$

$$V[1]^k = V^{k+1}$$
$$\mathrm{Sym}^n(V[1]) \simeq (\wedge^n V)[n]$$
$$v_1 \cdots v_n \longmapsto \pm v_1 \wedge \cdots \wedge v_n$$

No. Date

$$S(V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(V)$$

$\exists$ co-algebra str

$\Delta : S(V) \longrightarrow S(V)^{\otimes 2}$ co multiplication (graded alg mor)

$\Delta(v) = v\otimes 1 + 1\otimes v \quad v \in V$

$\Delta(v_1 v_2) = (v_1\otimes 1 + 1\otimes v_1)(v_2\otimes 1 + 1\otimes v_2)$

Def (g, Q) is a L_∞-algebra

$\overset{def}{\Leftrightarrow}$ (i) g is a graded vector sp

(ii) $Q \quad S(g[1]) \to S(g[1])[1]$ co derivation

i.e.

$$\begin{tikzcd} S(g[1]) \arrow[r, "Q"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & S(g[1])[1] \arrow[d, "\Delta"] \\ S(g[1])^{\otimes 2} \arrow[r, "Q\otimes 1 + 1\otimes Q"'] & S(g[1])^{\otimes 2}[1] \end{tikzcd} \quad , \quad Q^2 = 0$$

Ex (Moyal product)

$X = \mathbb{R}^d \quad \pi = \sum \pi^{ij} \partial_i \wedge \partial_j \qquad \pi^{ij} \in \mathbb{R} \quad \pi^{ij} = -\pi^{ji}$

$(x_1, \cdots, x_d)$

coord. system $\quad \pi$ is Poisson str on $\mathbb{R}^d$

$\partial_i = \frac{\partial}{\partial x_i}$

$\longrightarrow \qquad f * g = fg + \sum \pi^{ij} \partial_i(f)\partial_j(g) + \sum \pi^{ij}\pi^{kl} \partial_i\partial_k(f)\partial_j\partial_l(g) + \cdots$

$\ast$-product on

$\mathcal{O}_X[[h]] \qquad = \sum \frac{h^n}{n!} \sum_{\substack{i_1 \cdots i_n \\ j_1 \cdots j_n}} \pi^{i_1 j_1} \cdots \pi^{i_n j_n} \partial_{i_1} \cdots \partial_{i_n}(f) \, \partial_{j_1} \cdots \partial_{j_n}(g)$

Q is determined by

$$\mathrm{Sym}^n(\mathfrak{g}[1]) \hookrightarrow S(\mathfrak{g}[1]) \xrightarrow{Q} S(\mathfrak{g}[1])[1] \underset{\text{projection}}{\longrightarrow} \mathfrak{g}[2]$$

is $(\bigwedge^n \mathfrak{g})[n]$

$$Q \leftrightarrow \{\Theta_i\}_{i \geq 1}, \quad \Theta_j : \textstyle\bigwedge^j \mathfrak{g} \to \mathfrak{g}[2-j]$$

$\Theta_1 : \mathfrak{g} \to \mathfrak{g}[1]$ $\qquad$ $\Theta_1^2 = 0$, $(\mathfrak{g}, \Theta_1)$ is a complex

$\Theta_2 : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ $\qquad$ Θ_1 satisfies Leibniz rule w.r.t. Θ_2

$\Theta_3 : \bigwedge^3 \mathfrak{g} \to \mathfrak{g}[-1]$ $\qquad$ Θ_2 satisfies Jacobi id upto contribution of Θ_1

Def $(\mathfrak{g}, Q)$ L_∞-alg is DGLA

$\underset{\text{def}}{\Leftrightarrow}$ $\Theta_3 = \Theta_4 = \cdots = 0$

write $d = \Theta_1$, $[\ ,\ ] = \Theta_2$

Ex (i) $\mathfrak{g}$: Lie algebra

$\Rightarrow$ $\mathfrak{g}$ is DGLA, concentrated in deg 0

$d = 0$, $[\ ,\ ]$ bracket of $\mathfrak{g}$

(ii) X : complete mfd

$$\mathfrak{g}^\bullet = \Gamma(X, \underbrace{T_X \otimes \Omega^{(0,\bullet)}(X)}_{T_X\text{-valued }(0,\bullet)\text{-forms}})$$

$\mathfrak{g}^\bullet$ is DGLA

d : induced by $\bar\partial : \Omega^{(0,\bullet)}(X) \to \Omega^{(0,\bullet+1)}(X)$

$[\ ,\ ]$ " $\quad [\ ,\ ] : T_X \times T_X \longrightarrow T_X$

(iii) A associative alg $/\mathbb{C}$

$\mathfrak{g}^p = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p+1}, A)$

$d : \mathfrak{g}^p \longrightarrow \mathfrak{g}^{p+1}$

$$\begin{aligned} d\Phi(a_0 \otimes \cdots \otimes a_{p+1}) = {} & a_0 \Phi(a_1 \otimes \cdots \otimes a_{p+1}) \\ & - \sum_{i=0}^{p} (-1)^i \Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}) \\ & + (-1)^p \Phi(a_0 \otimes \cdots \otimes a_p) a_{p+1} \end{aligned}$$

No. Date

$(g^\bullet, d)$ Hochschild complex

$\Phi_1 \in g^{p_1}$, $\Phi_2 \in g^{p_2}$

$[\Phi_1, \Phi_2] = \Phi_1 \circ \Phi_2 - (-1)^{p_1 p_2} \Phi_2 \circ \Phi_1 \qquad \Phi_1 \circ \Phi_2 \in g^{p_1+p_2}$

$$\Phi_1 \circ \Phi_2 (a_0 \otimes \cdots \otimes a_{p_1+p_2})$$
$$= \sum_i (-1)^{i p_2} \Phi_1 (a_0 \otimes \cdots \otimes a_{i-1} \otimes \Phi_2 (a_i \otimes \cdots \otimes a_{i+p_2}) \otimes a_{i+p_2+1} \otimes \cdots \otimes a_{p_1+p_2})$$

$[\ ,\ ]$ Gerstein Bager bracket

◦ Deformation functors

$g^\bullet$ DGLA

$$MC(g) = \{ \omega \in g^1 \mid \underbrace{d\omega + \tfrac{1}{2}[\omega, \omega] = 0}_{\text{Maurer–Cartan equation}} \}$$

$r \in g^0$, $\omega \in g^1$, $\dot{r} = dr + ad(r)(\omega) \qquad ad(r) = [r, \cdot]$

$\rightsquigarrow$ integrated to give an action of $\exp(g^\circ)$ on $MC(g^\bullet)$

$\omega \in MC(g^\bullet)$

$$\exp(r)(\omega) = \exp(ad(r))(\omega) + \frac{1 - \exp ad(r)}{ad(r)} dr \qquad (*)$$

(we assume $(*)$ converges
eg. $g^\bullet$ is nilpotent)

$\omega_1, \omega_2 \in MC(g^\bullet)$ are gauge equivalent

$\overset{\text{def}}{\Leftrightarrow} \omega_2 = \exp(r)(\omega_1) \quad \exists r \in g^0$

Def Define $\mathrm{Def}_g : (\mathrm{Artin\,loc}/\mathbb{C}) \to \mathrm{Set}$

$$(R, m) \longmapsto MC(g^\bullet \otimes_{\mathbb{C}} m) / \text{gauge eq}$$

Ex (i) X complex mfd

$g^\bullet = \Gamma(X, T_X \otimes_{\mathcal{O}(X)} \Omega^{(0,\bullet)}(X))$

$\mathrm{Def}_g(R) = \{ \mathcal{X} \to \operatorname{spec} R$, flat$/R$, $\mathcal{X} \times_{\operatorname{spec} R} \operatorname{spec} R/m \xrightarrow{\sim} X \} / \text{isom}$

(ii) A associative alg

$\mathfrak{g}^p = \mathrm{Hom}(A^{\otimes(p+1)}, A)$ $\quad$ $\mathfrak{g}$ = Hochschild complex

$\mathrm{Def}_{\mathfrak{g}}(R) \simeq \{$ associative product to an alg $A \otimes R$, ring isom $(A\otimes R)\otimes R/m \simeq A \} /$ isom

$\omega \in \mathfrak{g}^1 \otimes m = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes 2}, A) \otimes_{\mathbb{C}} m$

$f, g \in A$, $f * g = fg + \omega(f, g)$ $\quad$ $*$ is associative
$\Leftrightarrow \omega$ satisfies MC equation

· L_∞-morphism

$(\mathfrak{g}, Q), (\mathfrak{g}', Q')$ DGLA

Def

L_∞-morphism from $(\mathfrak{g}, Q)$ to $(\mathfrak{g}', Q')$ is codg coalg mor of $\mathcal{F}: S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$

i.e.

$$\begin{tikzcd} S(\mathfrak{g}[1]) \arrow[r, "\mathcal{F}"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & S(\mathfrak{g}'[1]) \arrow[d, "\Delta"] \\ S(\mathfrak{g}[1])^{\otimes 2} \arrow[r, "\mathcal{F}\otimes\mathcal{F}"'] & S(\mathfrak{g}'[1])^{\otimes 2} \end{tikzcd} \qquad \begin{tikzcd} S(\mathfrak{g}[1]) \arrow[r, "\mathcal{F}"] \arrow[d, "Q"'] \arrow[dr, phantom, "\circlearrowright"] & S(\mathfrak{g}'[1]) \arrow[d, "Q'"] \\ S(\mathfrak{g}[1])[1] \arrow[r, "\mathcal{F}"'] & S(\mathfrak{g}'[1])[1] \end{tikzcd}$$

$\mathcal{F}$ is determined by

$$\begin{tikzcd} \mathrm{Sym}^n(\mathfrak{g}[1]) \arrow[r, hook] \arrow[d, equal] & S(\mathfrak{g}[1]) \arrow[r, "\mathcal{F}"] & S(\mathfrak{g}'[1]) \arrow[r, "\text{projection}"] & \mathfrak{g}'[1] \\ (\wedge^n \mathfrak{g})[n] & & & \end{tikzcd}$$

$\mathcal{F}_n : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{g}[1-n]$

$\mathcal{F}_1 : \mathfrak{g} \to \mathfrak{g}'$ $\quad$ morphism of complexes

$\mathcal{F}_2 : \wedge^2 \mathfrak{g} \to \mathfrak{g}'[-1]$ $\quad$ $\mathcal{F}_1$ preserves brackets up to the contribution of $\mathcal{F}_2$

$\mathcal{F} : (\mathfrak{g}, Q) \to (\mathfrak{g}, Q')$ is called quasi isomorphism

$\overset{\text{def}}{\Leftrightarrow}$ $H^\cdot(\mathcal{F}) : H^\cdot(\mathfrak{g}) \overset{\sim}{\to} H^\cdot(\mathfrak{g}')$

Prop $\psi: (g, Q) \to (g', Q')$
L_∞-alg q isom
$\Rightarrow$ ψ induces an isom
$$\psi_* : \mathrm{Def}\, g \xrightarrow{\sim} \mathrm{Def}\, g'$$
$$\psi_* \omega = \sum \frac{1}{j!} \psi_j(\underbrace{\omega \wedge \cdots \wedge \omega}_{j}) \in \mathrm{Def}\, g'(R)$$
$\omega \in \mathrm{Def}\, g(R)$ $\qquad \psi_j : \wedge^j g \to g'[1-j]$

§ 3 Formality thm in C^∞-case

$(X, \mathcal{O}_X)$ sm alg var or C^∞-mfd
$D_X[1]$, $T_X[1]$ sheaf of DGLA

Definition of $D_X[1]$
$$\underbrace{D_X[1]^p}_{\text{consist of multi diff operator}} \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X^{\otimes (p+1)}, \mathcal{O}_X)$$

Definition of $T_X[1]$
$$T_X[1]^p = \wedge^{p+1} T_X$$
$d : T_X[1]^p \longrightarrow T_X[1]^{p+1}$ zero map
$\xi_0 \wedge \cdots \wedge \xi_p \in T_X[1]^p$ $\quad p, q \geq 0 \quad \xi_i \in T_X,\ \eta_j \in T_X$
$\eta_0 \wedge \cdots \wedge \eta_q \in T_X[1]^q$
$$[\xi_0 \wedge \cdots \wedge \xi_p, \eta_0 \wedge \cdots \wedge \eta_q] = \sum (-1)^{p+i+j} [\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \hat{i} \cdots \wedge \xi_p \wedge \eta_0 \wedge \cdots \hat{j} \cdots \wedge \eta_q \in T_X[1]^{p+q}$$

$\xi_1 \wedge \cdots \wedge \xi_p \in T_X[1]^p$
$\eta \in \mathcal{O}_X = T_X[1]^{-1}$
$$[\xi_0 \wedge \cdots \wedge \xi_p, \eta] = \sum (-1)^i \xi_i(\eta)\, \xi_0 \wedge \cdots \hat{i} \cdots \wedge \xi_p$$

Schouten Nijenhuis bracket
$T_X[1]$ sheaf of DGLA
Fact $\exists\, \mathcal{U} : T_X[1] \longrightarrow D_X[1]$ q.i.s.
$$\mathcal{U}(\xi_0 \wedge \cdots \wedge \xi_p) = \left(f_0 \otimes \cdots \otimes f_p \longmapsto \sum \frac{1}{(p+1)!} \mathrm{sgn}(\sigma)\, \xi_0(f_{\sigma(0)}) \cdots \xi_p(f_{\sigma(p)}) \right)$$
HKR isom (Hochschild, Kostant, Rosenberg)

Thm (Kontsevich's formality thm)

X C^∞-manifold

$\Rightarrow$ $\exists$ L_∞-q.is

$\{U_i\}_{i\geq 1} : \Gamma(X, T_X[1]) \to \Gamma(X, D_X[1])$

s.t. $U_1 = \Gamma(X, U)$

$\begin{cases} \mathfrak{g}_T = \Gamma(X, T_X[1]) \\ \mathfrak{g}_D = \Gamma(X, D_X[1]) \end{cases}$

In particular

$\mathrm{Def}_{\mathfrak{g}_T} \xrightarrow{\sim} \mathrm{Def}_{\mathfrak{g}_D}$

$(\mathrm{Def}_{\mathfrak{g}_T}(\mathbb{C}[[t]]) \xrightarrow{\sim} \mathrm{Def}_{\mathfrak{g}_D}(\mathbb{C}[[t]]))$

$\mathrm{Def}_{\mathfrak{g}_T}(\mathbb{C}[[t]])$

$= \{\text{Formal Poisson str}\}/\text{gauge}$

$\pi \in \Gamma(X, \wedge^2 T_X)[[h]]$ is a formal Poisson str

$\Leftrightarrow$ $\pi = \pi_1 t + \pi_2 t^2 + \cdots$ and $[\pi, \pi] = 0$

$\mathrm{Def}_{\mathfrak{g}_D}(\mathbb{C}[[t]]) = \{*\text{-product on } \mathcal{O}_X[[t]],\ \mathcal{O}_X[[t]] \otimes \mathbb{C} \simeq \mathcal{O}_X\}/\text{gauge eq}$

$\pi \in \Gamma(X, \wedge^2 T_X)$ Poisson $\Rightarrow$ πh is a formal Poisson str

Explicit description of $*$-products

$X = \mathbb{R}^d$ $(x_1, \dots, x_d)$ coord.

$\pi = \sum \pi^{ij} \partial_i \wedge \partial_j$ $\quad \partial_i = \frac{\partial}{\partial x_i}$ $\quad \pi^{ij} \in \mathcal{O}_X$ Poisson

For $n \geq 1$ consider the set of oriented graphs Γ

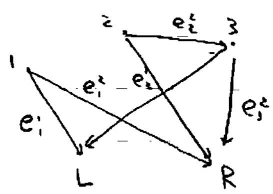
V_Γ = vertex of Γ, E_Γ = edges of Γ

$= \{1, \dots, n, L, R\}$ $\quad \# E_\Gamma = 2n$

Each edge of Γ is labelled by $e_h^1, e_h^2, \cdots$ $(h = 1, \dots, n)$ e_k^i starts from k

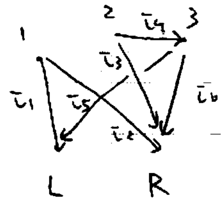
$\nexists$ $\circlearrowright$ at V_Γ

e.g.



$\Gamma \in G_n$, $B_\Gamma : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$

$$B_\Gamma(f, g) = \sum_{I : E_\Gamma \to \{1, \dots, d\}} \prod_k \prod_{e = (k, \cdot)} \partial_{I(e)} \pi^{I(e_k^1) I(e_k^2)} \prod_{e = (*, L)} \partial_{I(e)}(f) \prod_{e = (*, R)} \partial_{I(e)}(g)$$



e.g.

$$B_\Gamma(f, g) = \sum_{i_1 \cdots i_6} \alpha^{i_1 i_2} \alpha^{i_3 i_4} \partial_{i_4} \alpha^{i_5 i_6} \partial_{i_1} \partial_{i_2}(f) \, \partial_{i_3} \partial_{i_5} \partial_{i_6}(g)$$

$\forall \Gamma \in G_n \; \exists w_\Gamma \in \mathbb{R}$

s.t.

$$f * g = \sum_{n \geq 0} h^n \sum_{\Gamma \in G_n} B_\Gamma(f, g) \, w_\Gamma$$ gives a $*$-product

$X = A^d$