Date 07. 8. 7 No.

行列因子とミラー対称性 高橋篤史(大阪大学)

k : 体 $\bar{k} = k$ char $k = 0$.

$S = k[x_1, x_2, x_3]$ $\deg x_i = a_i \in \mathbb{Z}_{>0}$ (AS regular)

$f \in S$ $\deg(f) = h \in \mathbb{Z}_{>0}$

f : isolated sing (at 0)

$R := S/(f)$

gr R : cat of f.g. R-modules

$D^{gr}_{sg}(R) := D^b \mathrm{gr} R / \mathrm{perf} R$ perf R : cat of b. cpx of gr. proj R-modul

・目標 "よい" f に対して $D^{gr}_{sg}(R)$ の full strongly exceptional collection $(E_1, \dots, E_n)$ をみつけたい.

$\iff$ $A := \mathrm{End}(\bigoplus E_i)$ f.d. basic alg $A/\mathrm{rad}A \simeq k^{\oplus n}$

$A \simeq k\Gamma/\langle \rho \rangle$ Γ unique

① $D^{gr}_{sg}(R)$ の性質を調べる

・CM module (/R) の cat

・matrix factorization の cat

・Krull. Schmidt

・Serre functor をもつ

・enhanced tri. cat

⋮

② 主定理 ・ADE sing

・Arnold's 14 exceptional sing

③ weighted proj line (Orlov's thm)

④ Mirror Symmetry

⑤ ....

S, f as above

f : isolated sing $\iff$ $\mu := \dim_k S/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) < \infty$

$\varepsilon := \sum_{i=1}^{3} a_i - h$

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(1) $\mathrm{gr}R \longrightarrow \mathrm{gr}R$

$M \longmapsto M(1)$ $\qquad M(1)_i := M_{i+1}$

R : Gorenstein $\quad K_R \xrightarrow{\sim} R(-\varepsilon)$

$D_{sg}^{gr}(R)$: too difficult.

· CM module

$M \in \mathrm{gr}R$ is Cohen-Macauley

$\overset{\mathrm{def}}{\Longleftrightarrow}$ $\mathrm{Ext}_R^i(R/m, M) = 0 \quad i < 2 = \dim R.$ $\qquad m$: sing pt に対応する極大イデアル

$CM^{gr}(R)$ cat of. gr. CM-module.

$\cap$

$\mathrm{gr}R$

<u>Fact</u> (Auslander) $\quad CM^{gr}(R)$ is a Frobenius category

Frobenius cat $\overset{\mathrm{def}}{\Longleftrightarrow}$ · an exact cat with enough injectives and enough projectives

· $\{inj\} = \{proj\}$

Rem $\quad \{inj\} = \{proj\} = \{$free R module$\}$

<u>Fact</u> (Happel)

$\underline{CM}^{gr}(R)$ を次で def.

$ob(\underline{CM}^{gr}(R)) = ob(CM^{gr}(R))$

$\underline{\mathrm{Hom}}(M, N) := \mathrm{Hom}_{\mathrm{gr}R}(M, N) / P(M, N)$

$P(M, N) = \{ g \in \mathrm{Hom}_{\mathrm{gr}R}(M, N) \mid$
$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr] & \circlearrowleft & N \\ & \exists P \arrow[ur] & \end{tikzcd}$ $\}$

projective とする

$\underline{CM}^{gr}(R)$ は tri. cat となる.

<u>Thm</u> (Buchweitz. '85 Orlov '05)

$D_{sg}^{gr}(R) \underset{\mathrm{tri}}{\xrightarrow{\sim}} \underline{CM}^{gr}(R)$

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Fact (f, f^*) strange dual pair $\Rightarrow$ f と f^* は dual

Rem f と f^* が dual $\Longleftrightarrow$ f と f^* が K. Saito の意味で dual
(Milnor monodromy の char. poly を用いて定義)

Homological Mirror Symmetry Conjecture
(f, f^*) dual pair
($D^{gr}_{sg}(R_f)$ は well-defined)
$\exists\, Fuk^{\to}(f^*)$ が def でき、(*) $D^{gr}_{sg}(R_f) \xrightarrow{\sim} D^b Fuk^{\to}(f^*)$ になる。

$Fuk^{\to}(f^*)$: obj : a distinguished basis of vanishing cycles in $f^*(21)$
hom : Floer homology

HMS の $K_0(T)$ level での話.
$(*) \Rightarrow (K_0(T), \chi + {}^t\chi) \underset{\text{lattice}}{\xrightarrow{\sim}} (H_2(f^{*-1}(1), \mathbb{Z}), -I)$

Work in progress

(f, f^*) dual pair $\Rightarrow$ f は次の 5 つの type に分類できる.
I : $x^p + y^q + z^r$ $(p, q) = (q, r) = (r, p) = 1$
II : $x^p + y^q + yz^r$,,
III : $x^p + zy^q + yz^r$,,
IV : $x^p + xy^q + yz^r$,,
V : $zx^p + xy^q + yz^r$,,

Rem (f, f^*) dual pair $\Rightarrow$ $g = 0$, $r = 3$.
とくに $\varepsilon = -1$ なら 14 コの例外型

Thm f : Type I or II or III とする.
Then T は s.e.c. をもつ.

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· $\mathcal{T}$: Krull-schmidt ($\underline{CM}^{gr}(R)$ で known)

Fact (Auslander, Reiton)
$\exists$ S : Serre functor on $\mathcal{T}$ (on $\underline{CM}^{gr}(R)$).
$$\mathrm{Hom}_{\mathcal{T}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(N, S(M))^* \quad \text{for } M, N \in \mathcal{T}$$

$$S \simeq T \circ (-\varepsilon) = T^{\dim R - 1} \circ (-\varepsilon)$$

Fact $T^2 = (h)$ ($h = \deg(f)$)
$\Rightarrow$ $S^h = T^{h-2\varepsilon}$ ($\mathcal{T}$: fractional CY of dim $1 - 2\frac{\varepsilon}{h}$)

Orlov's thm · $D^{gr}_{sg}(R) = D^b grR / perf R$
$qgrR := grR / tor R$

① $\varepsilon > 0$ $D^b qgrR \cong \langle R, R(1), \ldots, R(\varepsilon - 1), D^{gr}_{sg}(R) \rangle$
② $\varepsilon = 0$ $D^b qgrR \xrightarrow{\sim} D^{gr}_{sg}(R)$
③ $\varepsilon < 0$ $D^{gr}_{sg}(R) \xrightarrow{\sim} \langle R/m, R/m(1), \ldots, R/m(-\varepsilon+1), D^b qgrR \rangle$

ex. $R = k[x_1, x_2]$; $\varepsilon = 2$ $\Rightarrow$ ① $D^b qgrR \simeq \langle R, R(1) \rangle$
$$\begin{array}{ccc} D^b qgrR & \simeq & \langle R, R(1)\rangle \\ \wr| & & \| \\ D^b \mathrm{coh}\,\mathbb{P}^1 & & \langle \mathcal{O}, \mathcal{O}(1) \rangle \end{array}$$

ex. ② $\varepsilon = 0$ $f = x_1^3 + x_2^3 + x_3^3 + x_1 x_2 x_3$
$$D^b qgrR \xrightarrow{\sim} D^{gr}_{sg}(R) \simeq D^b \mathrm{coh}\, E$$
$$E := \{f = 0\} \subset \mathbb{P}^2$$

full strongly exceptional collection を def する.

$E \in \mathcal{T}$ が exceptional $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Hom}(E, T^i E) = \begin{cases} k & i = 0 \\ 0 & i \neq 0 \end{cases}$

$(E_1, \ldots, E_n)$ exceptional objects の順序つきの組まりが
strongly exceptional collection
$\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Hom}(E_i, T^a E_j) \neq 0$ only if $a = 0$ and $i < j$

$(E_1, \dots, E_n)$: strongly exceptional collection is full

$\overset{\Longleftrightarrow}{\text{def}}$ $\langle E_1, \dots, E_n \rangle$ ($E_1, \dots, E_n$ を含む最小の $\mathcal{T}$ の full sub tri. cat.)

$\simeq \mathcal{T}$

<u>Fact</u> $(E_1, \dots, E_n)$ full s.e.c.

$\Rightarrow$ $A := \mathrm{End}_{\mathcal{T}}(\oplus E_{n'})$ basic f.d. alg/k ($\Rightarrow$ $\Gamma, \langle \rho \rangle$)

$\Rightarrow$ $\mathcal{T} \simeq D^b \mathrm{mod} A$ は一般に不明

もし $\mathcal{T}$: enhanced tri. cat なら

$\mathcal{T} \simeq D^b \mathrm{mod} A$ (Bondal Kapranov)

<u>Fact</u> $R = S/(f)$ $\mathcal{T} = \mathrm{HMF}^{gr}_S(f)$

$\Rightarrow$ $\mathcal{T}$: enhanced tri. cat.

($\overset{\Longleftrightarrow}{\text{def}}$ $^{\exists}\mathcal{A}$: a dg-cat s.t. $H^0(\mathrm{Tw}\mathcal{A}) \simeq \mathcal{T}$)

<u>Main Theorems</u>

<u>Thm</u> (Kajiwara, K. Saito —)

f : ADE sing

then $\mathcal{T} \simeq D^b \mathrm{mod}\, k\Gamma$ Γ : 対応する型の Dynkin quiver

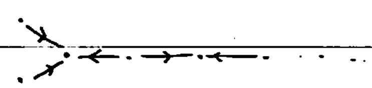
(relation なし)

<u>Rem</u> 次のような orientation が「この構成に関して自然」

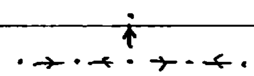
(Bridgeland's stab. cond を保たせる).

A_ℓ :

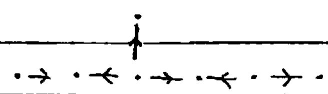
D_ℓ :



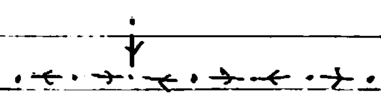
E_6



E_7



E_8



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Thm (KST)

f: 14 exceptional sing ($\Leftarrow \varepsilon = -1$)

$x_1^2 + x_2^3 + x_3^7$ $(2,3,7)$

$x^5 + xy^3 + z^2$ $(2,3,8)$

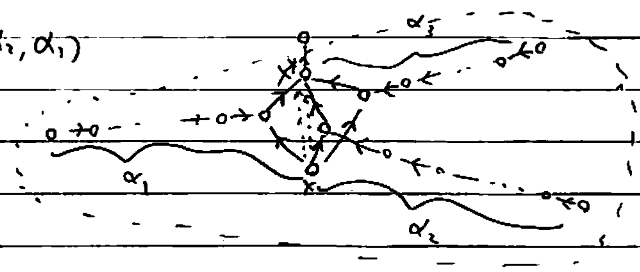
$x^3 + xy^5 + z^2$ $(2,4,5)$ 14つ

$\vdots$ $\vdots$

A: Dolgachev #

Then $\mathcal{T} \xrightarrow{\sim} D^b \operatorname{mod} k\Gamma_A/\langle \rho \rangle$

Γ_A $A = (\alpha_1, \alpha_2, \alpha_3)$



$\langle \rho \rangle$: relation · ρ_1, ρ_2 : two generic relations $x \to x^*$

さらに $(K_0(\mathcal{T}), \chi + {}^t\chi)$ $\chi(M,N) = \sum_n (-1)^n \dim_k \operatorname{Hom}(M, T^n N)$

$\underset{\text{lattice}}{\xrightarrow{\sim}} (H_2(f^{*-1}(1), \mathbb{Z}), -I)$

f^*: f の strange dual

I: intersection form

(Homological Mirror Symmetry)

Rem $\varepsilon > 0 \Rightarrow f$: ADE sing

$\varepsilon = 0 \Rightarrow$ · simple elliptic sing

$\varepsilon < 0 \Rightarrow 14 + 8 + 9$ 種類 の sing

Sketch of proof.

① 「局所」で m.f を計算する

$R/\mathfrak{m} \in \mathcal{T}$ $T^n R/\mathfrak{m}(a)$ の hom を計算

② s.e.c. を見つける

· Serre duality を使う.

③ 見つけた s.e.c. は full か？

$M \in CM^{gr}(R)$ $\quad \mathrm{Ext}^i(R/m, M) = 0 \quad {}^\forall i \neq \dim R \Rightarrow M : \text{free}$

と

<u>lem</u> $(E_1, \dots, E_n)$ (s).e.c $\subset \mathcal{T}$

$\langle E_1, \dots, E_n \rangle \ni R/m(a) \quad a \in \mathbb{Z}$

$\Longrightarrow (E_1, \dots, E_n) : \text{full}.$ を使う. //

<u>Rem</u> $\varepsilon = -1 \quad \mathcal{T} = \langle R/m, D^b \mathrm{qgr} R \rangle$ (by Orlov's thm)

$\Gamma_{A,9}$ $\quad //$

$D^b \mathrm{qgr} R \xrightarrow{\sim} D^b \mathrm{coh}\, \mathcal{C} \qquad \mathcal{C} = [\mathrm{Spec}(R) \setminus \{0\} / k^*]$

quotient stack

$C := \mathrm{Proj}\, R$

<u>Fact</u> C : smooth proj curve. $\quad g := g(C)$ genus

<u>Assumption</u> f としては $g = 0$ となるものに限って考える.

<u>Def.</u> $A'_f := \{ a_i \mid h/a_i \notin \mathbb{Z} \} \coprod_{1 \leq i < j \leq 3} \{ \gcd(a_i, a_j)^{m(a_i, a_j; h) - 1} \}$

$m(a_i, a_j; h) = \sharp \{ (u, v) \in \mathbb{Z}^2_{\geq 0} \mid a_i u + a_j v = h \}$

$A'_f = \{1, \dots, 1\} \sqcup A_f \qquad A_f \not\ni 1.$

(g, A_f) : signature $(g = 0)$

$\varepsilon = -1 \quad n = 3 \quad A_f = \{ \alpha_1, \dots, \alpha_r \}$ Dolgachev number

<u>Thm</u> (KST, in preparation)

$D^b \mathrm{coh}\, \mathcal{C}$ は full strongly exceptional collection を持つ.

<u>Cor</u> $\mathcal{T} = D^{gr}_{Sg}(R) \simeq \langle R/m, \dots, R/m(-\varepsilon+1), D^b \mathrm{qgr} R \rangle$ は full exceptional collection を持つ.

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Proof $R_{A_f} := \mathbb{C}[X_1, \dots, X_r]/I$

$$I = \langle X_i^{\alpha_i} - X_2^{\alpha_2} + \lambda_i X_1^{\alpha_1} : i = 1, \dots, r \rangle$$

$$\lambda_i \neq \lambda_j \ (i \neq j) \qquad \lambda_i \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

$$L := \bigoplus \mathbb{Z}\vec{X}_i \,/\, (\alpha_i \vec{X}_i - \alpha_j \vec{X}_j,\ i < j)$$ $\vec{X}_i = \deg(X_i)$. generator

claim · $R \hookrightarrow R_{A_f}$ sub alg (fを正確に固定する)

$$\Rightarrow \operatorname{gr} R/\operatorname{tor} R \xrightarrow{\sim} \operatorname{gr}^L R_{A_f}/\operatorname{tor}^L R_{A_f}$$

· $D^b \operatorname{gr}^L R_{A_f}/\operatorname{tor}^L R_{A_f}$ は s.e.c. をもつ (Geigle-Lenzing) //

Mirror Symmetry

Algebra $\longleftrightarrow$ Geometry

$f \in k[x_1, x_2, x_3]$ $\quad G \subset GL(3, k)$ finite, diagonal

$\{(f, G)\}$ を考える.

$$(A_\ell, \mathbb{Z}/(\ell+1)\mathbb{Z}) \xleftrightarrow{\text{mirror}} (A_\ell, \{1\})$$

$$X = \{x_1^5 + \cdots + x_5^5 = 0\} \subset \mathbb{P}^4$$

$$X^* := \widehat{X/(\mathbb{Z}/5\mathbb{Z})^3}$$

top. mirror test

$\chi(f, G)(t, \bar{t})$ orbifold Poincaré polynomial

(Vafa's formula)

$(f, G) \longleftrightarrow (f^*, G^*)$ top. mirror

$$\overset{}{\underset{\text{def}}{\iff}} \chi(f, G)(t, \bar{t}) = (-1)^{\cdot}\, \bar{t}^{\cdot}\, \chi(f^*, G^*)(t, \bar{t}^{-1})$$

fとf^*が dual $\underset{\text{def}}{\iff}$ $(f, \{1\})$ と $(f^*, \mathbb{Z}/h^*\mathbb{Z})$ が top. mirror

Fact $0 \to M \to P_n \to P_{n-1} \to \cdots \to P_0 \to N \to 0$ exact

P_i : gr. proj $\quad n \geq \dim R \Rightarrow M \in CM^{gr}(R)$.

(R : Gorenstein , AS-Gorenstein)

$R = S/(f)$ hypersurface sing

$M \in CM^{gr}(R)$ $\quad 0 \to F_1(-h) \xrightarrow{f_1} F_0 \to M \to 0$ free resolution in gr S

$\Rightarrow {}^\exists f_0 : F_0 \to F_1$ S-hom deg 0

s.t. $f_1 \circ f_0 = f \cdot id_{F_0}$

$f_0 \circ f_1 = f \cdot id_{F_1}$

Eisenbud $\quad (F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1)$ が matrix factorization of f

$\underset{def}{\Leftrightarrow}$ F_0, F_1 : free S-module

$f_0 : F_0 \to F_1$

$f_1 : F_1 \to F_0(h)$

s.t. $\begin{cases} f_1 \circ f_0 = f \cdot id_{F_0} \\ f_0 \circ f_1 = f \cdot id_{F_1} \end{cases}$

Rem $\quad \overline{F} = (F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1) \longmapsto \operatorname{coker}(f_1(-h)) \in CM^{gr}(R)$

$\overline{F}, \overline{F}'$ の間の hom $\Phi = (\phi_0, \phi_1)$ は

$$\begin{tikzcd} F_0 \arrow[r, shift left, "f_0"] \arrow[d, "\phi_0"'] \arrow[dr, phantom, "\circlearrowleft"] & F_1 \arrow[l, shift left, "f_1"] \arrow[d, "\phi_1"] \\ F_0' \arrow[r, shift left, "f_0'"] & F_1' \arrow[l, shift left, "f_1'"] \end{tikzcd}$$

なるもの.

同様に通常の方法で $\Phi \sim 0$ (null homotopic) を def できる.

Def. Category $HMF^{gr}_S(f)$ を次で def する.

object : { matrix factorization of f }

hom : { m.f. の hom }/{ null homotopic }

Thm (Buchweitz, Orlov, ...)

$\mathcal{T} := D^{gr}_{Sg}(R) \underset{tri}{\xrightarrow{\sim}} \underline{CM}^{gr}(R) \underset{tri}{\xrightarrow{\sim}} HMF^{gr}_S(f)$

· T : translation functor of $\mathcal{T}$

Fact · $\mathcal{T}$: locally finite $\Leftrightarrow \sum_i \dim_k \operatorname{Hom}_{\mathcal{T}}(M, T^i N) < \infty \quad M, N \in \mathcal{T}$
(R : isolated sing を仮定)

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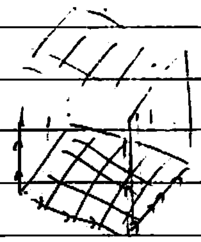
$(\Gamma, \langle \rho \rangle)$ I : $A_p \otimes A_q \otimes A_r$



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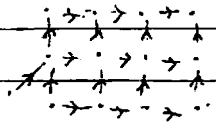
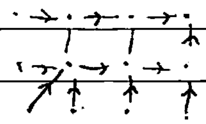
 $\otimes A_p$

III

 $\otimes A_p$