非可換代数幾何学入門 II 毛利出先生

☆ Classify noncommutative projective surfaces

① q-projective plane ATV 1990
② q-ruled surface IV
③ a surface finite over its center

tools
① Blow up Van den Bergh 2001
② Intersection Theory Smith 2001 V

IV Quantum Ruled Surface

① Bimodule

Def. X, Y : q-schemes

M is X-Y-bimodule $\underset{\text{def}}{\Leftrightarrow}$ $\left.\begin{array}{l} -\otimes_X M : \mathrm{Mod}\,X \to \mathrm{Mod}\,Y \\ \mathcal{H}om_Y(M,-) : \mathrm{Mod}\,Y \to \mathrm{Mod}\,X \end{array}\right\}$ adjoint pair

eg. R, S : rings, $X = \mathrm{Spec}\,R$, $Y = \mathrm{Spec}\,S$

M : R-S-bimodule $\Leftrightarrow$ M : X-Y-bimodule

eg. X = scheme, $M \in \mathrm{Mod}\,X$

$$-\otimes_X M : \mathrm{Mod}\,X \longrightarrow \mathrm{Mod}\,X$$
$$\mathcal{H}om_X(M,-) : \mathrm{Mod}\,X \longrightarrow \mathrm{Mod}\,X$$

eg. $\mathcal{O}_X : \mathrm{Mod}\,X \longrightarrow \mathrm{Mod}\,X$: identity

X, Y, Z : q-schemes

M : X-Y-bimodule
N : Y-Z-bimodule $\Rightarrow$ $M \otimes_Y N$ is X-Z-bimodule

$$-\otimes_X (M \otimes_Y N) = (-\otimes_X M) \otimes_Y N$$
$$\mathcal{H}om_Z(M \otimes_Y N, -) = \mathcal{H}om_Y(M, \mathcal{H}om_Z(N,-))$$

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M = X-Y-bimodule

$\mathrm{Tor}_i^X(-,M) : \mathrm{Mod}\,X \to \mathrm{Mod}\,Y$ is defined by the formula

$$\mathrm{Hom}_Y(\mathrm{Tor}_i^X(-,M), I) \cong \mathrm{Ext}_X^i(-, \mathrm{Hom}_Y(M,I))$$

$$\forall I \in \mathrm{Mod}\,Y : \text{injectives}$$

② Quantum projective space bundle

X : noetherian $\mathscr{Y}$-scheme

Def. A is graded X-algebra $\underset{\text{def}}{\Leftrightarrow}$ $A = \bigoplus_{i\in\mathbb{Z}} A_i$, A_i : X-bimodule

s.t. $A_i \otimes_X A_j \to A_{i+j}$ (multiplication)

$\mathcal{O}_X \to A_0$ (unit)

M : graded A-module $\underset{\text{def}}{\Leftrightarrow}$ $M = \bigoplus_{i\in\mathbb{Z}} M_i$, $M_i \in \mathrm{Mod}\,X$

s.t. $M_i \otimes A_j \to M_{i+j}$

$\mathrm{Proj}\,A = (\mathrm{Tails}\,A, f^*\mathcal{O}_X)$

$f : \mathrm{Proj}\,A \to \mathrm{Mod}\,X$: structure map

$f^* : \mathrm{Mod}\,X \xrightarrow{-\otimes_X A} \mathrm{GrMod}\,A \xrightarrow{\pi} \mathrm{Tails}\,A$ adjoint

$f_* : \mathrm{Tails}\,A \xrightarrow{\omega} \mathrm{GrMod}\,A \xrightarrow{(\ )_0} \mathrm{Mod}\,X$ pair

Def. (—, Smith) X : $\mathscr{Y}$-scheme

A quantum projective space bundle $/X$ is $\mathrm{Proj}\,A$ where A is a noetherian flat regular connected X-algebra.

connected : $A_{<0} = 0$, $A_0 = \mathcal{O}_X$

regular : $\mathrm{Tor}_i^A(-, \mathcal{O}_X) = 0 \quad \forall i \gg 0$

flat : $- \otimes_X A : \mathrm{Mod}\, X \longrightarrow \mathrm{GrMod}\, A$ is exact

noetherian : $- \otimes_X A : \mathrm{Mod}\, X \longrightarrow \mathrm{grmod}\, A$

$K_0(X) = K_0(\mathrm{mod}\, X)$: Grothendieck group of X

Thm. (–, Smith)

X : noetherian smooth projective scheme

$\mathrm{Proj}\, A$: q-projective space bundle $/ X$

$$\Longrightarrow K_0(\mathrm{Proj}\, A) = \frac{K_0(X)[t, t^{-1}]}{\left(\sum_{i \in \mathbb{N}} (-1)^i H_{\mathrm{Tor}_i^A(\mathcal{O}_X, \mathcal{O}_X)}(t)\right)}$$

$M \in \mathrm{grmod}\, A$ Hilbert series of M is

$$H_M(t) = \sum_{i \in \mathbb{Z}} [M_i] t^i \in K_0(X)[[t]][t^{-1}]$$

eg. A : noetherian regular connected k-algebra

$$\Longrightarrow K_0(\mathrm{Proj}\, A) = \frac{\mathbb{Z}[t]}{(H_A(t)^{-1})}$$

eg. $\mathbb{P}^{n-1} = \mathrm{Proj}\, k[x_1, \cdots, x_n]$

$$\Longrightarrow K_0(\mathbb{P}^{n-1}) = \frac{\mathbb{Z}[t]}{((1-t)^n)}$$

<u>Remark</u> $\underset{\mathbb{P}^{n-1}}{\overset{}{X}}$: projective scheme

$X = \mathrm{Proj}\, B(X, D)$ ($B(X,D)$ sometimes regular)

③ Quantum Ruled Surface

X : smooth projective curve

Y is ruled surface $/ X$

$\Leftrightarrow$ $Y = \mathrm{Proj}\, S(\mathcal{E})$ where $\mathcal{E}$ is a locally free sheaf of rank 2

$S(\mathcal{E})$: symmetric algebra on S over $\mathcal{O}_X$

$\quad \shortparallel$

$T(\mathcal{E})/(Q)$ $\qquad T(\mathcal{E}) = \mathcal{O}_X \oplus \mathcal{E} \oplus \mathcal{E}^{\oplus 2} \oplus \cdots$

$Q \subseteq \mathcal{E} \otimes \mathcal{E}$: locally generated by $x \otimes y - y \otimes x$.

R : commutative ring

R-bimodule $= R \otimes R$-module

$X = \mathrm{Spec}\, R$

X-bimodule $= \mathrm{Spec}(R \otimes R)$-module

$$\mathrm{Spec}(R \otimes R) = X \times X$$

<u>Def.</u> (Artin - Van den Bergh)

A coherent $\mathcal{O}_X$-bimodule M is a coherent $\mathcal{O}_{X \times X}$-module s.t. $pr_i|_{\mathrm{Supp}M}$ are finite $i = 1, 2$

$$\begin{array}{c} \mathrm{Supp}\,M \\ \cap\,| \\ pr_i : X \times X \longrightarrow X \\ (p_1, p_2) \longmapsto p_i \end{array}$$

In this case

$$\left.\begin{array}{l} -\otimes_X M = pr_{2*}(pr_1^*(-)\otimes_{X\times X} M) \\ \mathcal{H}om_X(M,-) = pr_{1*}(\mathcal{H}om_X(M, pr_2^!(-))) \end{array}\right\} \text{adjoint pair}$$

Therefore M is a X-bimodule.

$\Delta = \{(p,p) \mid p \in X\} \subseteq X \times X$

$\mathcal{O}_X \longleftrightarrow \mathcal{O}_X$

$\mathcal{E}$: coherent $\mathcal{O}_X$-bimodule

$\mathcal{E}$ is locally free of rank r

$\overset{\text{def}}{\Longleftrightarrow}$ $pr_{i*}\mathcal{E}$ are locally free of rank r

$i = 1, 2$.

$\Longrightarrow$ $\exists$ ${}^*\mathcal{E}$, $\mathcal{E}^*$: locally free $\mathcal{O}_X$-bimodule of rank r

s.t. $-\otimes {}^*\mathcal{E}$ is left adjoint to $-\otimes\mathcal{E}$.
$-\otimes\mathcal{E}^*$ is right adjoint to $-\otimes\mathcal{E}$.

$Q \subseteq \mathcal{E}\otimes\mathcal{E}$ is nondegenerate

$\overset{\text{def}}{\Longleftrightarrow}$ $Q\otimes{}^*\mathcal{E} \longrightarrow \mathcal{E}\otimes\mathcal{E}\otimes{}^*\mathcal{E} \longrightarrow \mathcal{E}\otimes\mathcal{O}_\Delta = \mathcal{E}$ isoms

$\mathcal{E}^*\otimes Q \longrightarrow \mathcal{E}^*\otimes\mathcal{E}\otimes\mathcal{E} \longrightarrow \mathcal{O}_\Delta\otimes\mathcal{E} = \mathcal{E}$.

<u>Def.</u> (Van den Bergh, Patric)

A quantum ruled surface in $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}\, A$ where

$\mathcal{E}$ = locally free $\mathcal{O}_X$-bimodule of rank 2

$Q \subseteq \mathcal{E} \otimes \mathcal{E}$: nondegenerate invertible $\mathcal{O}_X$-bimodule

$$\left(\text{invertible} \quad \begin{array}{l} \exists\, Q' \\ Q \otimes Q' \simeq \mathcal{O}_\Delta \\ Q' \otimes Q \simeq \mathcal{O}_\Delta \end{array} \right)$$

$\mathcal{A} = T(\mathcal{E})/(Q)$

<u>Thm.</u> $K_0(\mathbb{P}(\mathcal{E})) = \dfrac{K_0(X)[t]}{([\mathcal{O}_X] - [pr_{2*}\mathcal{E}]t + [pr_{2*}Q]t^2}$.

$\mathcal{O}_{\mathbb{P}(\mathcal{E})} = f^* \mathcal{O}_X$

$\omega_{\mathbb{P}(\mathcal{E})} = f^*(\omega_X \otimes Q)(-2)$

Then (Serre duality)

$$\mathrm{Ext}^i_{\mathbb{P}(\mathcal{E})}(M, \omega_{\mathbb{P}(\mathcal{E})}) \cong \mathrm{Ext}^{2-i}_{\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, M)^*$$
$$\forall M \in \mathrm{mod}\, \mathbb{P}(\mathcal{E})$$

V Intersection Theory

X : noetherian Q-scheme / k

X is <u>Ext-finite</u> $\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $\dim_k \mathrm{Ext}^i_X(M, N) < \infty$

$$\forall M, N \in \mathrm{mod}\, X$$
$$\forall i$$

$\mathrm{hd}(X) = \sup\{ i \mid \mathrm{Ext}^i_X(M,N) \neq 0,\ \exists M,N \in \mathrm{mod}\, X \}$

X : Ext-finite & $\mathrm{hd}(X) < \infty$.

Euler form $(M,N) := \sum (-1)^i \dim_k \mathrm{Ext}^i_X(M,N)$

can extend

$(\ ,\ ) : K_0(X) \times K_0(X) \to \mathbb{Z}$

$[M], [N] \in K_0(X)$

Intersection multiplicity of $[M]$ and $[N]$ is defined by

$[M] \cdot [N] := (-1)^{\mathrm{codim}[M]} ([M], [N])$

Thm. (Chan) X : smooth projective variety $/k$

$C, D \subseteq X$: subschemes

$\dim C + \dim D \leq \dim X \Rightarrow C \cdot D = [\mathcal{O}_C] \cdot [\mathcal{O}_D]$

$C \cdot D = \sum_{p \in C \cap D} (-1)^i \dim_k \mathrm{Tor}^{\mathcal{O}_{X,p}}_i (\mathcal{O}_{C,p}, \mathcal{O}_{D,p})$

X : Gorenstein scheme

$[M] \cdot [N] = [N] \cdot [M] \quad \forall M, N \in \mathrm{mod}\, X$

$\Leftrightarrow$ Serre vanishing conjecture

$\left\{ \begin{array}{l} \dim C + \dim D < \dim X \\ \quad \Rightarrow C \cdot D = 0 \end{array} \right\}$

<u>Def</u> $D \in \mathrm{WPic}\, X$

$$\mathcal{O}_D = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(X)$$

$C, D \in \mathrm{WPic}\, X$

$$C \cdot D = -(\mathcal{O}_C, \mathcal{O}_D)$$

<u>Def.</u> X is <u>Gorenstein</u> $\underset{\text{def}}{\Leftrightarrow}$ $\exists K \in \mathrm{WPic}\, X$: canonical divisor

s.t. $\mathrm{Ext}^i_X(-, \mathcal{O}_X(K)) \cong \mathrm{Ext}^{d-i}_X(\mathcal{O}_X, -)^*$.

<u>Thm.</u> X = Gorenstein, K : canonical divisor
$D \in \mathrm{WPic}\, X$

(Riemann-Roch)

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D \cdot D - K \cdot D) + 1 + p_a$$

$$p_a = \chi(\mathcal{O}_X) - 1 \quad \left(\underset{\text{Euler characteristic}}{\chi(-) = (\mathcal{O}_X, -)} \right)$$

(Adjunction Formula)

$$2g - 2 = D \cdot D + K \cdot D$$

$$g = 1 - \chi(\mathcal{O}_D).$$

X : smooth projective curve

$\mathbb{P}(\mathcal{E})$: $\mathcal{Q}$ - ruled surface $/ X$

<u>Thm</u> (1) $\mathbb{P}(\mathcal{E})$ is Ext-finite (Nyman)

(2) $\mathrm{hd}(\mathbb{P}(\mathcal{E})) < \infty$

$$f : \overset{\operatorname{Proj} A}{\overset{\|}{\mathbb{P}(\mathcal{E})}} \longrightarrow X \ : \text{structure map}$$

- section of f $\quad H := [\mathcal{O}_{\mathbb{P}(\mathcal{E})}] - [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)] \in K_0(\mathbb{P}(\mathcal{E}))$
- fiber of a closed pt $p \in X$ $\quad f^{-1}p = [f^*\mathcal{O}_X] - [f^*(\mathcal{O}_X(-p))]$
- quasi-canonical divisor $K = [\omega_{\mathbb{P}(\mathcal{E})}] - [\mathcal{O}_{\mathbb{P}(\mathcal{E})}] \in \overset{\cap}{K_0(X)}$

$$\left([\mathcal{O}_{\mathbb{P}(\mathcal{E})}] - [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-K)] \right) \otimes \mathcal{O}_X(K)$$

$$= [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(K)] - [\mathcal{O}_{\mathbb{P}(\mathcal{E})}]$$

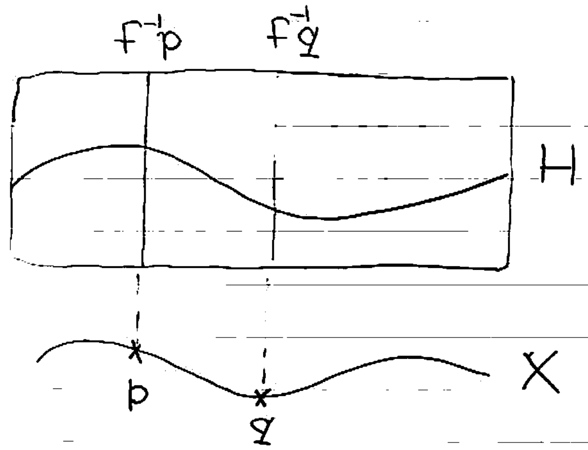
Thm. (− , Smith) $p, q \in X$ closed pts

$$f^{-1}p \cdot f^{-1}q = 0$$

$$f^{-1}p \cdot H = H \cdot f^{-1}p = 1$$

$$H \cdot H = \deg(pr_{2*}\mathcal{E})$$

$$(\operatorname{Pic} \mathbb{P}(\mathcal{E}) = \mathbb{Z} H \oplus \operatorname{Pic} X)$$



Thm. (Adjunction-Formula)

$$D = f^{-1}p \quad \text{or} \quad D = H$$

$$\Rightarrow \ 2g - 2 = D \cdot D + D \cdot K$$

$$g = 1 - \chi(\mathcal{O}_D)$$

Thm. $K \sim -2H + (2g - 2 - e) f^{-1}p$: numerically equivalent

$$g = g(X), \quad e = -H \cdot H$$

Thm. if $\mathcal{E}$ is "nice" (eg. X: Calabi-Yau)

then $K \cdot K = 8(1-g)$, $g = g(X)$

Appendix Classification of Quantum Ruled Surfaces

① Classify locally free bimodule of rank 2 / X

② $\mathcal{E}$, $\mathcal{F}$: locally free bimodule of rank 2, when $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{F})$?

③ Find invariant of $\mathbb{P}(\mathcal{E})$ ($e = -HHX$)

Thm $\mathcal{F} = \mathcal{L} \otimes \mathcal{E} \otimes M$

$\mathcal{L}, M$: invertible $\Rightarrow \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{F})$

M, N : $\mathcal{O}_X$-bimodule

$M \otimes N = pr_{13*}(pr_{12}^* M \underset{X \times X \times X}{\otimes} pr_{23}^* N)$: $\mathcal{O}_X$-bimodule

we can normalize $\mathcal{E}$

$\rightarrow$ $e = -H \cdot H$ invariant ?

$e = \deg(pr_{2*}\mathcal{E})$

If $\mathcal{E}$ is decomposable

$\mathcal{E} = \mathcal{L} \oplus M$ $\mathcal{L}, M$: locally free of rank 1

$\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$, $\mathcal{L}$ is invertible $\mathcal{O}_X$-bimodule

$\{\text{invertible } \mathcal{O}_X\text{-bimodules}\} \longleftrightarrow \underset{(\sigma, \mathcal{L})}{\mathrm{W\,Pic}\, X}$, X : smooth projective curve

$\mathcal{L}_\sigma = p_1{}^*\mathcal{L} \otimes_X \mathcal{O}_\Gamma \quad , \ \Gamma = \{ (p, \sigma(p)) \mid p \in X \}$

$(X, \sigma, \mathcal{L}) \to \mathbb{P}(\mathcal{O}_X \otimes \mathcal{L}_\sigma)$

Question

$\mathbb{P}(\mathcal{O}_X \otimes \mathcal{L}_\sigma) \cong \mathbb{P}(\mathcal{O}_X \otimes M_\tau) \underset{\text{def}}{\Leftrightarrow} (X, \sigma, \mathcal{L}) \ ? \ (X, M, \tau)$

<u>Thm.</u> $A(X, \sigma, \mathcal{L}) \cong A(X', \sigma', \mathcal{L}')$

$\Leftrightarrow \quad \exists \tau : X \xrightarrow{\sim} X'$

① $\tau^* \mathcal{L}' \cong \mathcal{L}$

②
$$\begin{tikzcd} X \arrow[r, "\sim\tau"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d, "\sigma'"] \\ X \arrow[r, "\sim\tau"] & X' \end{tikzcd}$$

<u>Thm.</u> $\mathrm{GrMod}\, A(X, \sigma, \mathcal{L}) \simeq \mathrm{GrMod}\, A(X', \sigma', \mathcal{L}')$

$\Leftrightarrow \quad \exists \tau_n : X \xrightarrow{\sim} X$

s.t. ① $\tau_n^* \mathcal{L}' \cong \mathcal{L}$

②
$$\begin{tikzcd} X \arrow[r, "\tau_n"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d, "\sigma'"] \\ X \arrow[r, "\tau_{n+1}"] & X' \end{tikzcd}$$

$\forall n \in \mathbb{Z}$

$? \Updownarrow \quad A, A' : \text{AS regular}$

$\mathrm{Proj}\, A \cong \mathrm{Proj}\, A'$

$? \Updownarrow \quad |\sigma| = \infty$, $X = \mathbb{P}^2$, $\cancel{X}$, $\emptyset$

$\mathrm{Proj}\, A \overset{\text{b.e.}}{\sim} \mathrm{Proj}\, A'$