毛利 出「非可換代数幾何学入門 I」

Introduction to Noncommutative Algebraic Geometry

I. Over View
II. Classification of Noncomm. Proj. Curves
III. 〃
IV. Quantum Ruled Surface
V. Intersection Theory

I. Over View

k ; field $\quad (\bar{k} = k, \ ch(k) = 0)$

① Motivation

☆ Classify algebras (f.g /k) of low dim.

$R = T(V)/I \qquad (\dim_k V < \infty, \ I \triangleleft T(V))$
$R_n = (k+V)^n$
$\mathrm{GK}\dim R = \limsup_{n\to\infty} \log(\dim_k R_n)/\log n$

Gelfand Kirillov

$R = $ comm. $\Rightarrow \mathrm{GK}\dim R = \mathrm{Kul}\text{-}\dim R$

$\mathrm{GK}\dim R = 0 \Leftrightarrow R$ fin. dim /k

☆ Classfy domains

R = domain. $\quad$ GK dim $R = 0 \Leftrightarrow R = k$

<u>Th</u> <Small - Warfield>

R = domain, $\quad$ GK dim $R = 1$

$\Rightarrow \quad R$ = comm.

☆ Classfy graded domains
(up to iso. of noncomm. proj. schemes)

② Quasi-scheme

<u>Th</u> <Gabriel - Rosenberg>

X = scheme $\Longrightarrow$ Mod X = cat. of quasi-coherent sheaves on X

<u>Def</u> <Rosenberg - Van den Bergh>

A quasi-schem X is a pair $X = (\mathrm{Mod}\, X, \mathcal{O}_X)$

Mod X = Grothendick cat.

$\mathcal{O}_X \in \mathrm{Mod}\, X$

e.g.

X = scheme $\Rightarrow$ $X = (\mathrm{Mod}\, X, \underbrace{\mathcal{O}_X}_{\substack{\| \\ \text{structur sheaf}}})$

Def

$$X \simeq X' \;:\Leftrightarrow\; \begin{array}{ccc} \mathrm{Mod}\, X & \xrightarrow{\sim} & \mathrm{Mod}\, X' \\ \mathcal{O}_X & \longmapsto & \mathcal{O}_{X'} \end{array}$$

e.g.

R = comm $X = \mathrm{Spec}\, R$

$$\Rightarrow \Gamma(X, -) : \begin{array}{ccc} \mathrm{Mod}\, X & \xrightarrow{\sim} & \mathrm{Mod}\, R \\ \mathcal{O}_X & \longmapsto & R \end{array}$$

cat. of R-mod.

Def

- R = ring.

  $\mathrm{Spec}\, R := (\mathrm{Mod}\, R, R)$

- A = graded ring

  $\mathrm{GrMod}\, A$ = cat. of graded right (left) A-module

- $M \in \mathrm{GrMod}\, A$ is torsion

  $:\Leftrightarrow\; M_n = 0 \quad {}^\forall n \gg 0$

- $\mathrm{Tors}\, A$ = {direct limits of torsion modules} $\subset \mathrm{GrMod}\, A$

· Tails A := GrMod A / Tors A ; quatient Cat.

· $M \cong N$ in Tails A $:\Leftrightarrow$ $M_{\geq n} \cong N_{\geq n}$ in GrMod A.

<u>Th</u> <Serre>

$A = k[A_1]$ (A_1; comm. ring), $X = \mathrm{Proj}\ A$

quationt function

$$\mathrm{Mod}\ X \xrightarrow{\bigoplus_{n\in\mathbb{Z}} \Gamma(X, (-)(n))} \mathrm{GrMod}\ A \xrightarrow{\pi} \mathrm{Tails}\ A$$
$$\mathcal{O}_X \longmapsto \pi A$$

is isom.

$$X \simeq (\mathrm{Tails}\ A, \pi A)$$

<u>Def</u> < Artin - Zhang 1994)

A = graded ring

$\Rightarrow \mathrm{Proj}\ A = (\mathrm{Tails}\ A, \pi A)$; noncommutative projective scheme associated to A.

A = graded alg. (f.g. / k, generated in deg 1)
$= k\langle A_1 \rangle$

A_1 = graded domain of GK dim $A = d+1$

$\Rightarrow \mathrm{Proj}\ A$ = noncomm. proj. variety of dim. d

☆ classify noncommutative proj. var. of low dim.

③ Birational Classification

Def

$A =$ noeth. domain , $X = \mathrm{proj}\, A$

The function field of X is defined by

$k(X) = \{ab^{-1} \in Q(A) \mid a, b \in A$; homg. of the same dim.$\}$

$X \overset{bir}{\sim} X' :\Leftrightarrow k(X) \simeq k(X')$

$A =$ graded dom. , $\mathrm{GKdim}\, A = 2$

$X = \mathrm{Proj}\, A$; noncomm. proj curve

$\Rightarrow A =$ noetherian

$K := Z(k(X))$; center of $k\langle X\rangle$

- $\mathrm{tr\text{-}dim}\, K = 0$ $\times$ small-wanfield
- $\mathrm{tr\text{-}dim}\, K = 1$ $\Longrightarrow$ $k(X) \in \mathrm{Br}(K)$

$\bar{k} = k \xRightarrow[\text{Tsen's Thm}]{} k(X) = K \simeq k(C)$

$\exists C$; comm. curve

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補足

$A, A' =$ graded ring

$A \simeq A' \Rightarrow \mathrm{GrMod} \simeq \mathrm{GrMod}\, A'$

$\Rightarrow \mathrm{Proj}\, A \simeq \mathrm{Proj}\, A'$

$\Rightarrow \mathrm{Proj}\, A \overset{\text{bir}}{\sim} \mathrm{Proj}\, A'$

$A, A' =$ noeth. dom.

$A =$ noeth. graded domain, $K = Z(k(X))$

$\mathrm{GK}\dim A = 3$

$X = \mathrm{Proj}\, A$; noncomm proj. surface

- <u>Conjectur</u> ⟨Artin 1997⟩

ATV 1990

- $\deg_k K = 0 \Rightarrow X \overset{\text{bir}}{\sim}$ quantam proj. plain
- $\deg_k K = 1 \Rightarrow X \overset{\text{bir}}{\sim}$ quantam ruled surface [??
- $\deg_k K = 2 \Rightarrow k(X) \in \mathrm{Br}(K)$

II) Classification of Noncomm. Proj. Curves

X = q-scheme

Def

$D \in \mathrm{WPic}\, X$ is weak divisor on X

$:\Leftrightarrow \quad D : \mathrm{Mod}\, X \xrightarrow{\sim} \mathrm{Mod}\, X$; aut equiv.

e.g.

X = scheme , D = div. on X

$- \otimes_X \mathcal{O}_X(D) \; : \mathrm{Mod}\, X \xrightarrow{\sim} \mathrm{Mod}\, X$

Th ⟨Bondal, Orlov⟩

X = smooth proj. var. $(\pm K = \text{ample})$

$\Rightarrow \cdot \mathrm{WPic}\, X \cong \mathrm{Aut}\, X \ltimes \mathrm{Pic}\, X$

$$D = (\sigma, \mathcal{L})$$

$$\mathrm{Mod}\, X \xrightarrow{D} \mathrm{Mod}\, X$$

$$\mathcal{F} \longmapsto \sigma_*(\mathcal{F} \otimes \mathcal{L})$$

$$\mathrm{Mod}\,X \xrightarrow{D} \mathrm{Mod}\,X \xrightarrow{C} \mathrm{Mod}\,X$$

$$\mathcal{F} \longmapsto \mathcal{F}(D) \longmapsto \mathcal{F}(D)(C) =: \mathcal{F}(D+C) \quad \left(\mathcal{F}(C+D)\right)$$

$$\mathrm{Mod}\,X \xrightarrow{D^n} \mathrm{Mod}\,X$$

$$\mathcal{F} \longmapsto \mathcal{F}(nD)$$

$$B(X, D) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(nD))$$

$$\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(mD)) \times \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(nD)) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X((n+m)D))$$

$$(a, b) \longmapsto ab = a(nD)\circ b.$$

$$\begin{array}{ccccc} & & \mathcal{O}_X & \xrightarrow{a} & \mathcal{O}_X(mD) \\ & & \downarrow & & \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(nD) & \xrightarrow{a(nD)} & \mathcal{O}_X(mD)(nD) = \mathcal{O}_X((m+n)D) \end{array}$$

e.g.

A = graded ring

$X = (\mathrm{GrMod}\,A, A)$

$$(1) : \mathrm{GrMod}\,A \xrightarrow{\sim} \mathrm{GrMod}\,A$$

$$M \longmapsto M(1) \quad (M_i(1) := M_{i+1})$$

$$B(X, (1)) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(A, A(n)) \cong A$$

e.g.

X = scheme

$\boxed{X \text{ is proj} \iff {}^\exists D \text{ ample div. on } X}$

$\Rightarrow X \simeq \mathrm{Proj}\, \underbrace{B(X, D)}$

homogenious coordinate ring

X = q-scheme

X = noeth $:\iff$ Mod X = locally neoth

(${}^\exists$set of noeth generators)

$\Rightarrow$ mod X = {noeth objects} $\subset$ Mod X

X is over k $:\iff$ Mod X is k-linear

X is Hom finite $:\iff$ $\dim_k \mathrm{Hom}_X(M, N) < \infty$

${}^\forall M, N \in$ mod X

Def <Artin – Zhang>

X = noeth. q-scheme

$D \in \mathrm{WPic}\, X$ is ample

$:\Leftrightarrow$ ① $\{\mathcal{O}_X(-nD)\}_{n\in\mathbb{N}}$ is a set of generators

② $M \twoheadrightarrow N$ in mod X

$\Rightarrow \mathrm{Hom}(\mathcal{O}_X(-nD), M) \twoheadrightarrow \mathrm{Hom}(\mathcal{O}_X(-nD), N)$ $(n \gg 0)$

Th <Artin – Zhang 1994>

X = noeth Hom-finite q-scheme $/k$

$(\dim_k \mathrm{Ext}^i_A(A/A_{\geq 1}, M) < \infty,\ {}^\forall M \in \text{graded } A)$

$X = \mathrm{Proj}\, A$ $({}^\exists A$ = graded alg satisfying $\chi_1)$

$:\Leftrightarrow$ ${}^\exists D \in \mathrm{WPic}\, X$: ample

$\Rightarrow X = \mathrm{Proj}\, B(X, D)$

X = noeth $:\Leftrightarrow$ ${}^\exists$ set of noeth generators

$\mathcal{O}_X$ = noeth. obj.

e.g.

$X =$ Scheme , $D = (\sigma, \mathcal{L})$; ample ($\sigma = id \Rightarrow B$; comm)

$X \simeq \mathrm{Proj}\, B(X, \sigma, \mathcal{L})$; twisted homg. coodinate ring
$$\underset{B(X,D)}{\shortparallel}$$

$(\underset{\text{schem}}{\overset{}{X}}, \underset{\text{Aut}X}{\sigma}, \underset{\in \mathrm{Pic}X}{\mathcal{L}}) =$ geometric toriple

Th <Artin - Stafford 1995>

$A =$ graded domain $\mathrm{GKdim}\, A = 2$

$\mathrm{Proj}\, A$ noncomm. proj. curve

$\Rightarrow \exists (X, \sigma, \mathcal{L})$; geometric toriple

s.t.

$A_{\geq n} \simeq B(X, \sigma, \mathcal{L})_{\geq n}$ $(n \gg 0)$

$(\sigma, \mathcal{L})$; ample

$\therefore$

$\mathrm{Proj}\, A \simeq \mathrm{Proj}\, B(X, \sigma, \mathcal{L}) \simeq X$

π_A $\quad \pi_B(X, \sigma, \mathcal{L})$ $\quad \mathcal{O}_X$

III) Classification of Quantum Projective Plains

① Artin − Shalter regular alg.

$\mathbb{P}^{n-1} = \mathrm{Proj}\ k[x_1, \cdots, x_n]$

$$A = \text{comm. ring} \Leftrightarrow A = \text{Porynomial algebra}$$
$$\text{comm. ring} = \text{gl dim } A < \infty$$

☆ Classify regular graded algebra

$\text{gl dim } A = 0 \quad \Leftrightarrow \quad A = k$

$\text{gl dim } A = 1 \quad \left(\text{e.g. } A = k[x], \ A = k\langle x_1, \cdots, x_n \rangle = T(V) \right)$

additional condition — GK dim $< \infty$, noeth.

Def <Artin − Schelter>

A is a d dimensional AS-reg. alg.

$:\Leftrightarrow$
- ① $\text{gl dim } A = d < \infty$
- ② $\text{GK dim } A < \infty$
- ③ Gorenstain condition

$$\mathrm{Ext}_A^i(k, A) = \begin{cases} k & i = d \\ 0 & i \neq d \end{cases}$$

※ ③ $\Leftrightarrow$ $0 \to F^d \to \cdots \to F^0 \to k \to 0$

$\Rightarrow$ $0 \to (F^0)^\vee \to \cdots \to (F^d)^\vee \to k \to 0$

$\left((F^i)^\vee := \mathrm{Hom}_A(F^i, A) \right)$

Th

A = 1-dim. AS-reg $\Leftrightarrow$ $A = k[x]$

A = 2-dim AS-reg $\Rightarrow$ $A = k\langle x,y\rangle / \langle \alpha x^2 + \beta xy + \gamma yx + \delta y^2 \rangle$

	$\alpha\delta - \beta\gamma \neq 0$	$\alpha\delta - \beta\gamma = 0$, $\beta \neq \gamma$	$\alpha\delta - \beta\gamma = 0$, $\beta = \gamma$
gl dim	2	2	∞
GKdim	2	2	∞
Gorenstine	O	X	X
noeth	O	X	X
domain	O	X	X
		e.g. $k\langle x,y\rangle / \langle xy \rangle$	e.g. $k\langle x,y\rangle / \langle x^2 \rangle$

☆ Classify 3-dim quadratic AS-reg alg.s

$\Rightarrow$ Proj A quantum proj plane

$$\boxed{\begin{array}{l} A = T(V)/(R) \\ \quad R \subset V \otimes V \quad ; \text{ quadratic alg.} \end{array}}$$

② Geometric alg

X = Prog scheme

$\{$very ample invertivel sheaves$\} \leftrightarrow \{X \hookrightarrow \mathbb{P}(V^*)$

$$\mathcal{L} \longmapsto V = \Gamma(X, \mathcal{L})$$

$$\begin{array}{c} p^* \mathcal{O}_{\mathbb{P}(V^*)}(1) \\ \| \\ \mathcal{O}_X(1) \end{array} \longleftarrow p : X \hookrightarrow \mathbb{P}(V^*)$$

Def

$A = T(V)/(R)$ is geometric

$:\Leftrightarrow$ $\exists (X, \sigma, \mathcal{L})$ = geometric toriple

$\mathcal{L}$; very ample , $V = \Gamma(X, \mathcal{L})$

s.t.

(G1) $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in X\}$

(G2) $R = \{f \in V \otimes V \mid f(p, \sigma(p)) = 0, \forall p \in X\}$

$$f \in R \subset V \otimes V$$

$$\begin{array}{c} f : (V \otimes V)^* \longrightarrow k \\ \wr\| \\ V^* \otimes V^* \end{array}$$

$$\mathcal{V}(R) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p, q) = 0, \forall f \in R\}$$

$f : V^* \times V^* \longrightarrow k$; multi. hom.

e.g.

$A = k\langle x, y\rangle / (f)$ $\quad (f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2)$

$(p, q) = ((a, b), (c, d)) \in \mathcal{V}(f) \subset \mathbb{P}^1 \times \mathbb{P}^1$

$\Leftrightarrow f(p, q) = \alpha ac + \beta ad + \gamma bc + \delta bd = 0$

$\Leftrightarrow (\beta a + \delta b) d = (-\alpha a - \gamma b) c$

$\Leftrightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \beta a + \delta b \\ -\alpha a - \gamma b \end{pmatrix} = \begin{pmatrix} \beta & \delta \\ -\alpha & -\gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$\boxed{\mathrm{Proj}_{\mathcal{I}} B(X, \sigma, \mathcal{L}) \to \mathrm{Proj}_{\mathcal{I}} A(X, \sigma, \mathcal{L})}$

$\Leftrightarrow q = \sigma(p) \quad , \sigma = \begin{pmatrix} \beta & \delta \\ -\alpha & -\gamma \end{pmatrix}$

if A is geometric as above write $A = A(X, \sigma, \mathcal{L}) \Rightarrow B(X, \sigma, \mathcal{L})$

A satisfies $(G_1) \Leftrightarrow A = A(\mathbb{P}^1, \sigma, \mathcal{O}_{\mathbb{P}^1}(1))$

$\Leftrightarrow \sigma \in \mathrm{Aut}\,\mathbb{P}^1 = PGL(k)$

$\Leftrightarrow |\sigma| = \alpha\delta - \beta\gamma \neq 0$

$\Leftrightarrow A = 2$-dim AS-reg. parametrization by $\sigma \in \mathrm{Aut}\,\mathbb{P}^1$

Th <Artin - Tate - Van den Bergh 1990>

A is a 3-dim. quadratic AS-reg. alg.

$\Leftrightarrow A = A(X, \sigma, \mathcal{L})$; geometric

s.t.

① $(\mathbb{P}^2, \sigma, \mathcal{O}_{\mathbb{P}^2}(1))$

② $X \hookrightarrow \mathbb{P}^2$: cubic div. , $\mathcal{L} = \mathcal{O}_X(1)$

$\sigma^*\mathcal{L} \neq \mathcal{L}$ but $(\sigma^2)^*\mathcal{L} \otimes \mathcal{L} \cong \sigma^*\mathcal{L} \otimes \sigma^*\mathcal{L}$

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$X = \mathbb{P}^2$, $\equiv$, $\ne$, $\ast$, $\cancel{A}$, $\cancel{b}$, ϕ, ς, α, ς
elliptic

e.g.

$$A = k\langle x, y, z\rangle / \langle zy - \alpha yz, xz - \beta zx, yx - \gamma xy\rangle$$

$$\alpha, \beta, \gamma \in k \setminus \{0\}$$

$\Rightarrow$ 3-dim. AS-reg.

$$X = \begin{cases} \mathbb{P}^2 & \alpha\beta\gamma = 1 \\ \mathcal{V}(xyz) = \cancel{A} & \alpha\beta\gamma \ne 1 \end{cases}$$

$$\sigma_{x=0}(0, b, c) = (0, \alpha b, c)$$

$$\sigma_{y=0}(a, 0, c) = (a, 0, \beta c)$$

$$\sigma_{z=0}(a, b, 0) = (\gamma a, b, 0)$$

$\alpha\beta\gamma = 1 \Rightarrow \sigma^* \mathcal{L} = \mathcal{L}$

2日目に続く。