Higher genus Gromov-Witten invariants of the Grassmannian, and the Pfaffian Calabi-Yau threefolds　小西由紀子

§0 Introduction

(A-model) (B-model)

CY3
$X \subset Gr(2,7)$ $\xleftrightarrow[x=0]{\text{Mirror}}$ · 1 para CY fold a family
$h^{1,1}(X)=1$

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ B = \mathbb{P}^1 \setminus \{0, \infty \cup 3点\} \\ \cup \\ x \end{array}$$

$\uparrow\uparrow$ Maximal Unipotent monodromy pts

CY3
$X' \subset Pfaff(7)$ $\xleftrightarrow[x=\infty]{\text{Mirror}}$
$h^{1,1}(X')=1$

- $D^b(X) \cong D^b(X')$ ← Borisov-Caldararu 2006

- Mirror thm for genus 0 GW invariants

  X : Batyrev ... '98
  Bertram & Kim '99

  X' : Rødland '98
  Tjotta 2001

- higher genus calculation
  Hosono-K 2007
  ↑
  BCOV's holomorphic anomaly equation

Plan

§1 Mirror Symmetry
§2 BCOV's holomorphic anomaly equation
§3 X & X'

No. Date . .

§1 Mirror Symmetry

$$\begin{array}{ccc} \mathcal{X} & \supset & X_w \\ \downarrow & & \downarrow \\ W & \ni & w \end{array} \qquad \begin{array}{ccc} \mathcal{Y} & \supset & Y_z \\ \downarrow & & \downarrow \\ B & \ni & z \end{array}$$

families of smooth CY 3folds

$\mathcal{X}$ と $\mathcal{Y}$ が Mirror pair であるとは、

(1) $h^{1,1}(X_w) = h^{2,1}(Y_z)$
$h^{2,1}(X_w) = h^{1,1}(Y_z)$

CY 3fold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(2) X_w の "A-model Yukawa coupling" $\overset{☆}{=}$ $\mathcal{Y}$ の "B-model Yukawa coupling"

A-model Yukawa coupling ↑ triple intersection number + Genus 0 GW invariants of X_w

B-model Yukawa coupling ↑ Variation of Hodge str

簡単の為 以下 $h^{1,1}(X_w) = 1 \rightsquigarrow \dim_{\mathbb{C}} B = 1$

" A-model Yukawa coupling" ← w に依らない $X = X_w$

$H \in H^2(X, \mathbb{Z})$ generator

$$C_{ttt} = \int_X H^3 + \left(q\frac{d}{dq}\right)^3 \sum_{d\geq 1} N_d(X)\, q^d \quad \in \mathbb{Q}[\![q]\!]$$

$N_d(X)$: GW inv. of X

"B-model Yukawa coupling"

$\Omega(z)$ holomorphic 3-form

(nowhere vanishing on Y_z)

$$C_{zzz} := \int_{Y_z} \Omega(z) \wedge \nabla_z^3 \Omega(z)$$

$\uparrow$ Gauss–Manin 接続

☆ $\mathcal{D}$: PF operator (diff op in z)

i.e. $\mathcal{D} \int_{\gamma \in H_3(Y_z, \mathbb{Z})} \Omega(z) = 0$

$\exists$ maximal unipotent monodromy pt

say $z=0$ (Exponents $=0$) $\in \overline{B} \setminus B$, $\overline{B} = \mathbb{P}^1$

$\mathcal{D} * = 0$ の solutions

$w_0 = 1 + \cdots$

$w_1 = w_0 \log z + \cdots$

w_2

w_3

$t := \frac{w_1}{w_0}$ ... Mirror map

☆ $$\underbrace{C_{ttt}}_{q\text{の series}} = \underbrace{C_{zzz} \left(\frac{dz}{dt}\right)^3 w_0^{-2}}_{z\text{の series}}$$

$t = t(z) = \log(z) + \cdots$

& $q = e^t$

$z = q + \cdots$

No. Date . .

<u>Example</u> Quintic & Its Mirror

X : quintic hypersurface $\subset \mathbb{P}^4$

$h^{1,1}(X) = 1$

$h^{2,1}(X) = 101$

$\chi = -200$

$\int_X H^3 = 5$

・ $Y \leftarrow$ orbifold construction (Green – Plesser '90)

$$Y_\psi = \overbrace{\{x_1^5 + \cdots + x_5^5 - \psi x_1 \cdots x_5 = 0\}}^{\text{orbifold construction}} / G \quad \subset \mathbb{P}^4/G$$

$$G \cong (\mathbb{Z}_5)^5/\mathbb{Z}_5$$

・true parameter $z = \dfrac{1}{5^5 \psi^5}$

$$\begin{array}{c} Y \\ \downarrow \\ B = \mathbb{P}^1 \setminus \{0, \infty, 1/5^5\} \end{array}$$

$\mathcal{D} = \theta_z^4 - 5z(5\theta_z+1)\cdots(5\theta_z+4) \qquad \theta_z = z\dfrac{d}{dz}$

・ $C_{zzz} = \dfrac{5}{z^3(1-5^5 z)}$

$$\left\{ \begin{array}{ccc} 0 & \infty & 1/5^5 \\ \hline 0 & 1/5 & 0 \\ 0 & 2/5 & 1 \\ 0 & 3/5 & 1 \\ 0 & 4/5 & 2 \end{array} \right\}$$

$\uparrow$

MUM

$\mathcal{D} = 0$ の 原点まわりの解

$w_0 = 1 + 120z + \cdots$

$w_1 = \log z \, w_0 + \cdots$

$t = \dfrac{w_1}{w_0} = \log z + 770z + \cdots$

$\Leftrightarrow \quad z = q - 770q^2 + \cdots$

$$C_{ttt} = 5 + \underbrace{2875}_{N_{1,0}(X)} q + \cdots$$

Mirror thm ⟵ Givental '98

<u>other known cases</u>

- CICY in toric varieties
- CICY in Grassmannian
- $X' \subset \mathrm{Pfaff}(7)$

§2 BCOV's holomorphic anomaly equation

'93 Bershadsky – Cecotti – Ooguri – Vafa

A – model B – model

$$\mathcal{F}^{(g)}(X) = \sum_{d \geq 0} N_{g,d}(X)\, q^d \qquad \underset{\bar{z} \to 0}{q = e^t}$$

$\mathcal{F}^{(g)}(z, \bar{z})$

BCOV's holomorphic anom eq を解いて考える.

しばらく Y → B側を考える.

<u>Notations</u>

- $K(z,\bar{z}) = -\log\left(\sqrt{-1}\,\underbrace{(\Omega(z), \overline{\Omega(z)})}_{\int_{Y_z} \Omega(z) \wedge \overline{\Omega(z)}}\right)$
- $G(z,\bar{z}) = \partial_z \partial_{\bar{z}} K$ ... Weil – Petersson metric

Special geom rel

$$\partial_{\bar{z}} \Gamma^z_{zz} = 2 G_{z\bar{z}} - C_{zzz} \overline{C_{zzz}}\, e^{2K} / G_{z\bar{z}}^2$$

$\mathcal{L}$: the line bundle on B

Fibre at z = $H^{3,0}(Y_z)$

$\uparrow$

$D_z = \partial_z + \partial_z K$ ··· covariant derivative

Let $\mathcal{F}^{(g)}$ be the "B-model top string amplitude"

- C^∞ - section of $\mathcal{L}^{2-2g}$
- monodromy inv

$\mathcal{F}_n^{(g)}$ を次のように def

$$\mathcal{F}_3^{(0)} := C_{zzz}$$
$$\mathcal{F}_0^{(g)} := \mathcal{F}^{(g)}$$
$$\mathcal{F}_n^{(g)} := D_z \mathcal{F}_{n-1}^{(g)}$$

Physic thm ('93 BCOV)

(1) $g = 1$

$\chi(X) = -\chi(Y_z)$

$$\partial_{\bar z} \mathcal{F}_1^{(1)} = \frac{1}{2} C_{zzz} \overbrace{\overline{C_{zzz}}\, e^{2K} / G_{z\bar z}^2}^{\tilde C} - \left(\frac{\chi}{24} - 1\right) G_{z\bar z}$$

$$\Longrightarrow \mathcal{F}_1^{(1)} = \frac{1}{2} \partial_z \log \left[e^{(3 + h^{1,1}(X) - \frac{\chi}{12})K} / G_{z\bar z} \times \left| f^{(1)}(z) \right|^2 \right]$$

$g \geq 2$

$$\partial_{\bar z} \mathcal{F}^{(g)} = \frac{\tilde C}{2} \left[\mathcal{F}_2^{(g-1)} + \sum_{r=1}^{g-1} \mathcal{F}_1^{(g-r)} \mathcal{F}_1^{(r)} \right]$$

(2) Mirror correspondence

Assume $z = 0$: MUM pt

ω_0, ω_1

t ··· Mirror map

$$\boxed{\mathcal{F}^{(g)}(X) = \lim_{z \to 0} \mathcal{F}^{(g)} \omega_0^{2-2g}}$$

$\|$

$$\sum_{d \geq 0} N_{g,d}(X)\, q^d$$

- Huang – Klemm – Quackenbush '06
  - × New boundary condition at "conifold pt" e.g. $z = 1/5^5$ for quintic
  - × solved $F^{(g)}$ up to $g \le 51$ for quintic

- Walcher '06 '07 ← Quintic
  - { Mirror Sym
  - { Extended holo anom eq
  - for open string

§ Grassmanian CY X & Pfaffian CY X'

A-model　　　　　　　　B-model

CY3

$X \subset Gr(2,7)$ ⟷　　　　$\mathcal{Y}$

　　　　　　　　　　　　$\downarrow$

　　　　　　　　　　　　$B = \mathbb{P}^1 \setminus \{0, \infty, d_1, d_2, d_3\}$

$X' \subset Pfaff(7)$ ⟷

$\underline{X}$

$$X = \underbrace{H \cap \cdots \cap H}_{7コ} \cap \underbrace{Gr(2,7)}_{10次元} \subset \mathbb{P}(\Lambda^2 \mathbb{C}^7) = \mathbb{P}^{20}$$

3次元

- Mirror family ('98 Batyrev et al)
  - $Gr(2,7)$ の toric degeneration を考え
  - Batyrev – Borisov による toric CYCI の
  - ミラー構成を応用

$\underline{X'}$

$$\mathbb{P}(\{\rho \in M_7(\mathbb{C}) \mid \rho + {}^t\rho = 0\}) = \mathbb{P}(\Lambda^2 \mathbb{C}^7) = \mathbb{P}^{20}$$

$\cup$

$Pfaff(7) = \{[\rho] \mid \mathrm{rank}\,\rho \le 4\} = \{[\rho] \mid \sqrt{\text{diagonal minor}}\text{'s} = 0\}$

17次元

$X' = \mathbb{P}^6 \cap Pfaff(7) \subset \mathbb{P}^{20}$

- Mirror family ← orbifold construction ('98 Rødland)

	$h^{1,1}$	$h^{2,1}$	H^3	$c_2.H$
X	1	50	42	84
X'	1	50	14	56

$(D^b(X) \simeq D^b(X'))$

- Mirror family について

PF op

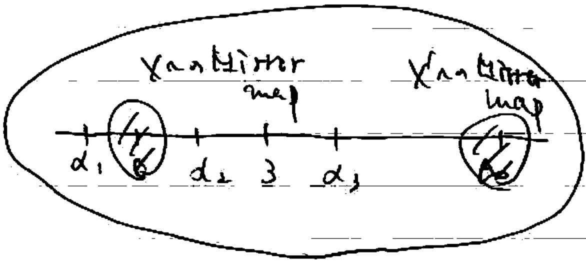
$$\mathcal{D} = \theta_x^4 + x(\theta_x \text{ の 4次式}) + \cdots + x^4(\quad〃\quad) + x^5(\theta_x+1)^4$$

$x \in B \qquad \theta_x = x\dfrac{d}{dx}$

$\Delta = 1 - 57x - 289x^2 + x^3 = 0$

$$\left\{\begin{array}{cccccc} 0 & \infty & \alpha_1 & \alpha_2 & \alpha_3 & 3 \\ \hline 0 & 1 & 0 & & & 0 \\ 0 & 1 & 1 & & & 1 \\ 0 & 1 & 1 & 〃 & 〃 & 3 \\ 0 & 1 & 2 & & & 4 \end{array}\right\}$$

no monodromy

$B =$ 

<u>Thm</u> Mirror thm $(g=0)$ が成立

X $(x=0)$ … Bertram et al, Kim '99

X' $(x=\infty)$ … Tjøtta (2001)

- $1 \le g \le 5$ で $F^{(g)}$ を計算

(Hosono-K 2007)