$(G \leq GL_2(k))$

$k[[x,y]]^G$

傾斜加群とクラスター傾斜加群

Date 8・8 No. 1

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Given (fin. dim) alg Λ

1. Classify indecomposable Λ-modules

(fin-dim k-alg $k=\bar{k}$)

finite	tame	wild	$k\langle x,y\rangle$ ほとんど全てのalg.
done	進行中	hopeless	

almost split sequence を使って全ての irred. mod. が計算できる

2. Classify $\boxed{\text{tilting}}$ Λ-module

傾斜加群　　hereditary. alg.（一般には ほとんど手つかず）

3. Classify n-cluster tilting modules

McKay 対応　　$X \to \operatorname{Spec} k[[x,y]]^G$

$$D^b(\text{mod}\, k[[x,y]]*G) \simeq D^b(\operatorname{Coh} X)$$

skew. group. ring.

これ上の tilt. mod を分類したい

§1 基本事項
§2 preprojective. algebra
§3 n-cluster. tilt
§4 one-dimension hypersurf 上の cluster. tilt

§1

R: d次元 complete regular local. ring（complete の上に Krull-Schmidt）

Λ: R-order ($\overset{\text{def}}{\Longleftrightarrow}$ R-alg, R上 fin. gen. proj（R-mod として）)

Λ-module M が $\underline{CM}$ $\Leftrightarrow$ R-mod として fin. gen. proj.

<u>rem</u>
- $\operatorname{Hom}_R(-,R)$: $CM\,\Lambda \leftrightarrow CM\,\Lambda^{op}$
- $CM\,\Lambda$ は R に depend
- Λ: commu $\Rightarrow$ フツウと CM と一致

<u>Example</u> $\mathrm{gl.dim}\,\Lambda = d$ のとき. $\mathrm{CM}\,\Lambda = \{\text{fin.gen proj. } \Lambda\text{-mod}\}$

<u>Def</u> Λ: isolated. singularity $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{gl.dim}(\Lambda \otimes_R R_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$
($\forall \mathfrak{p}$: non.maximal. prime)

<u>def</u>. $\underline{\mathrm{CM}}\,\Lambda$ - obj は. $\mathrm{CM}\,\Lambda$
stable.cat. - morphism は. $\underline{\mathrm{Hom}}(X, Y) := \mathrm{Hom}_\Lambda(X, Y) / \{X \to Y \text{ 」} \exists C \text{ proj (resp. inj)}\}$
(resp. $\overline{\mathrm{CM}}\,\Lambda$)
costable.cat.

finite length ($\mathrm{Hom}_R(\Lambda, R)^n$ の直和因子)

<u>rem</u> Λ: isolated. sing $\Leftrightarrow$ $\underline{\mathrm{Hom}}_\Lambda(X, Y) \in \underline{\mathrm{f.l}}\,R$ $(\forall X, Y \in \mathrm{CM}\,\Lambda)$

<u>rem</u> ($\underline{\mathrm{CM}}\,\Lambda$ の indec.obj. の iso.class)
= { indec. non-proj Λ-modules }

<u>Thm</u> (1) $\exists \tau$: $\underline{\mathrm{CM}}\,\Lambda \xrightarrow{\sim} \overline{\mathrm{CM}}\,\Lambda$ Auslander-Reiten translation.
(indec. non-proj) $\overset{1:1}{\longleftrightarrow}$ (indec. non-inj)

(2) $\exists$ functorial. isomorphism $D := \mathrm{Ext}^d_R(-, R)$: f.l.R $\circlearrowleft$.

$$\underline{\mathrm{Hom}}_\Lambda(X, Y) \simeq D\,\mathrm{Ext}^1_\Lambda(Y, \tau X) \qquad (X, Y \in \mathrm{CM}\,\Lambda)$$

(Auslander - Reiten duality)

<u>Application</u> $Y = X$ non-proj indec. $\mathrm{Ext}^1_\Lambda(X, \tau X) \overset{D}{\longleftrightarrow} \underline{\mathrm{End}}_\Lambda(X)$: local. ring.

$\mathrm{soc}_{\mathrm{End}(X)}(\mathrm{Ext}^1_\Lambda(X, \tau X)) \overset{D}{\longleftrightarrow} \underline{\mathrm{End}}_\Lambda(X) / \mathrm{rad}(\underline{\mathrm{End}})$

$$\nabla : 0 \to \underbrace{\tau X \to E \to X}_{\mathrm{CM}\,\Lambda} \to 0 \qquad \nabla \neq 0$$

<u>Thm</u> $\forall X$: non-proj. indec. CM-Λ-mod.

$0 \to \tau X \xrightarrow{b} E \xrightarrow{a} X \to 0$: non-split.

$0 \to X \to E' \to \tau^{-1} X \to 0$
non-inj.

mod = left mod

(1). $\forall f : Y \to X$: split.epi でない.

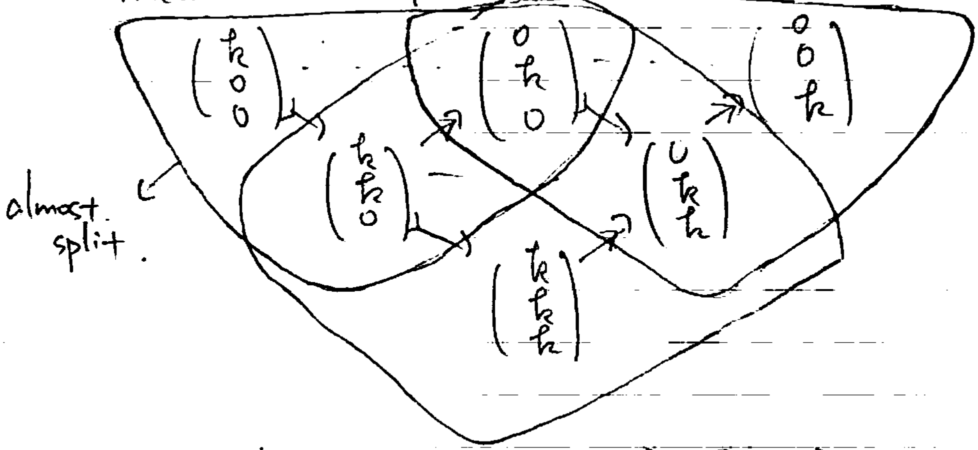
$\exists f' : Y \to E$ s.t. $f = f'a$

(2) dual.

この列のことを almost split seq とよぶ.

<u>Example</u> $\Lambda = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$ (k : 体) ← $\circ \to \circ \to \circ$.

indec. Λ-mod は ~~以下~~ の6つ



このように almost split seq の項を図示するものを AR-quiver という.

<u>Application. 2</u> Λ : R-order ($d = \dim R$), isolated singularity.

Λ : symmetric ($\mathrm{Hom}_R(\Lambda, R) \simeq \Lambda$ as (Λ, Λ)-bimodule)

CM Λ で proj = inj

$\underline{\mathrm{CM}}\,\Lambda = \overline{\mathrm{CM}}\,\Lambda$

[Happel] : $\underline{\mathrm{CM}}\,\Lambda$ は triangulated category.

suspension $[1] = \Omega^{-1}$ $\quad 0 \to X \to \underset{\text{inj. C.M}}{I} \to \underset{\mathrm{CM}\Lambda}{\Omega^{-1} X} \to 0$

[Auslander] $\underset{\Omega^{2-d}}{\tau} : \underline{\mathrm{CM}}\,\Lambda \circlearrowleft$

$\underline{\mathrm{Hom}}_\Lambda(X, Y) \simeq D\mathrm{Ext}^1_\Lambda(Y, \Omega^{2-d} X) \simeq D\underline{\mathrm{Hom}}_\Lambda(Y, X[d-1])$

<u>Cor</u> $\underline{\mathrm{CM}}\,\Lambda$ は $(d-1)$-Calabi-Yau triangulated category.

§1.2 tilting theory

R : commu. noeth. Λ : module-finite R-mod

Def $T \in \operatorname{mod}\Lambda$ が tilt. mod であるとは

(mod ↑ fin. gen.)

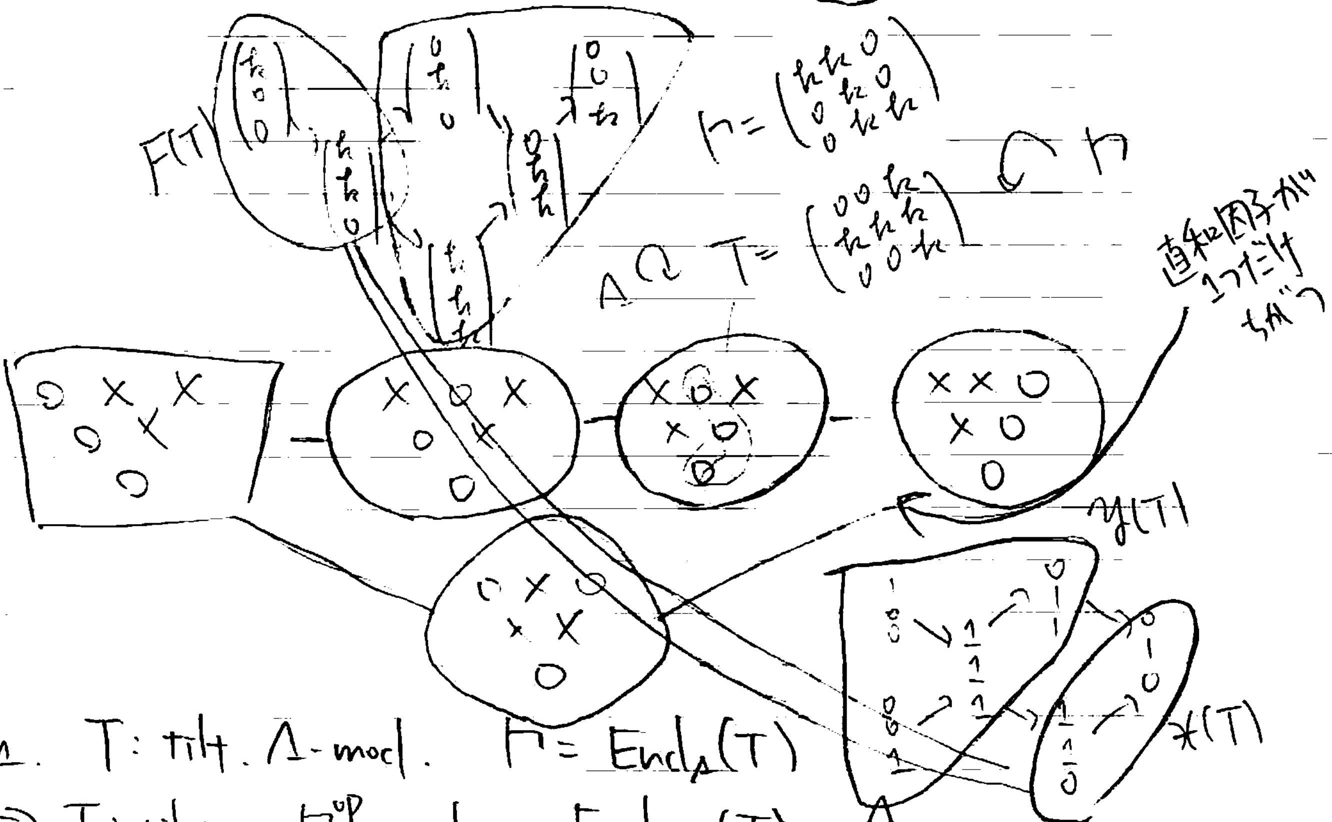
(e.g. $T = \Lambda$ は tilt)

(1). $\operatorname{proj.dim} T \leq 1$

(2). $\operatorname{Ext}^1_\Lambda(T, T) = 0$.

(3). $\exists\, 0 \to \Lambda \to T_0 \to T_1 \to 0$ ($T_1 \in \operatorname{add} T$)

$\operatorname{add} T$: Tの直和因子の直和



Lem. T: tilt. Λ-mod. $\Gamma = \operatorname{End}_\Lambda(T)$

$\Rightarrow T$: tilting. Γ^{op}-mod. $\operatorname{End}_{\Gamma^{op}}(T) = \Lambda$

Thm [Happel] T: tilt. Λ-mod. $\Gamma = \operatorname{End}_\Lambda(T)$

$$\mathbb{R}\operatorname{Hom}_\Lambda(T, -) : D^b(\operatorname{mod}\Lambda) \xrightarrow[\text{triangle equivalence}]{\sim} D^b(\operatorname{mod}\Gamma)$$

$\mathcal{T}(T) := \{X \in \mathrm{mod}\,\Lambda : \mathrm{Ext}^1_\Lambda(T,X) = 0\}$.

$\mathcal{F}(T) := \{ \text{———} : \mathrm{Hom}_\Lambda(T,X) = 0\}$

$\mathcal{X}(T) := \{Y \in \mathrm{mod}\,\Gamma : T \otimes_\Gamma Y = 0\}$

$\mathcal{Y}(T) = \{ \text{———} : \mathrm{Tor}^\Gamma_1(T,Y) = 0\}$

<u>Cor.</u> [Brenner. Butler]

$$\left.\begin{array}{ccc} \mathcal{T}(T) & \underset{T\otimes}{\overset{\mathrm{Hom}(T,-)}{\rightleftarrows}} & \mathcal{Y}(T) \\ \mathcal{F}(T) & \underset{\mathrm{Tor}^\Gamma_1(T,-)}{\overset{\mathrm{Ext}^1_\Lambda(T,-)}{\rightleftarrows}} & \mathcal{X}(T) \end{array}\right\} \text{equiv.}$$

<u>Prop</u> [Riedtmann-Scofield]

(1). $T \oplus X$: basic.tilting X : indecomp.

<u>仮定</u>

$$0 \to X \xrightarrow{a} T' \to Y \to 0, \quad T' \in \mathrm{add}\,T$$

· $\mathrm{Hom}_\Lambda(T',T) \xrightarrow[a]{} \mathrm{Hom}_\Lambda(X,T) \to 0$

·· proj.dim. $Y \leq 1$

$\Rightarrow$ $T \oplus Y$: basic.tilt. Λ-mod.

(2). (1)のdual

このとき. $T \oplus X$ と $T \oplus Y$ は. <u>mutation</u> の関係にある.

<u>Def</u> $T.U$: basic.tilting.module.

$$T \geq U \overset{\text{def}}{\Longleftrightarrow} \mathrm{Ext}^1_\Lambda(T,U) = 0 \quad \left(\Leftrightarrow \mathcal{T}(T) \supseteq \mathcal{T}(U) \right)$$

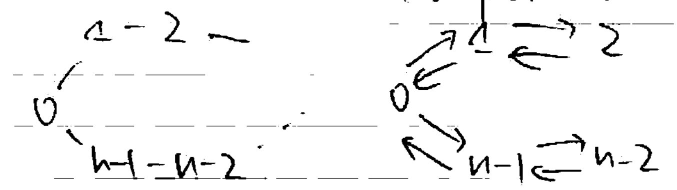
$\geq$ は. partial.order を与える 最大元は Λ

$\mathcal{T}(\Lambda) = \mathrm{mod}\,\Lambda$

k: 体, $k = \bar{k}$, char $k = 0$

§2. Δ: extended Dynkin. Λ: preprojective algebra of Δ
completion

$\Delta = \tilde{A}_{n-1}$



— を $\xrightarrow{a}$ $\xleftarrow{a^*}$ でおきかえる. $\sum_{a:\Delta\text{のedge}} a_1 a_2 = \sum_{a:\Delta\text{のedge}} a_2 a_1$

$k[[x,y]] * G$ $\quad G \subseteq SL_2(k)$: finite subgrp.

$\Lambda \underset{\text{Morita}}{\sim}$

$G = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\rangle$

$\zeta^n = 1$

Δ の vertex $1 \sim n$

$1 = e_1 + e_2 + \cdots + e_n$ (原始直交巾等元)

$I_i := \Lambda(1-e_i)\Lambda \subseteq \Lambda$

<u>Lem</u> I_i: tilting Λ-module, $\mathrm{End}_\Lambda(I_i) = \Lambda$.

—— Λ^{op} —— $\mathrm{End}_{\Lambda^{op}}(I_i) = \Lambda$

<u>Lem</u> T: 2-sided of Λ, tilting Λ-module $\mathrm{End}_\Lambda(T) = \Lambda$

$TI_i \neq T$ を仮定

$\Rightarrow TI_i$ は T の mutation. tilting Λ-module.

$\| \quad (Te_1 \oplus \cdots \oplus Te_i \oplus \cdots \oplus Te_n)$

$(Te_1 \oplus \cdots \oplus TI_ie_i \oplus \cdots \oplus Te_n)$

$\mathrm{End}_\Lambda(TI_i) = \Lambda$

$\langle I_1, \cdots, I_n \rangle$: $I_1 \sim I_n$ の積で表される ideal 全体

<u>Thm</u> [I-. Reiten] ← ext Dynkin をつかう (これ以外は ext Dynkin で使われ)

$\langle I_1, \cdots, I_n \rangle = \{$basic tilting Λ-module$\}$

$= \{$basic tilting Λ^{op}-module$\}$

<u>Prop</u> — $I_i^2 = I_i$

— $I_iI_j = I_jI_i$ $\quad i \not\!\!- j$ in Δ

— $I_iI_jI_i = I_jI_iI_j$ $\quad i \overset{1本}{—} j$ in Δ $\quad =$

$W = \langle S_1, \cdots, S_n \rangle \Big/ \begin{array}{ll} S_i^2 = 1 & \\ S_iS_j = S_jS_i & i \not\!\!- j \\ S_iS_jS_i = S_jS_iS_j & i - j \end{array}$

affine Weyl grp

$$\begin{array}{ccc} & & \{\text{basic tilt. } \Lambda\text{-mod}\} \\ & & \| \\ W & \xrightarrow{1-1} & \langle I_1, \cdots, I_n \rangle \\ \rotatebox{90}{$\in$} & & \\ w = S_{i_1} \cdots S_{i_k} & \longmapsto & I_{i_1} \cdots I_{i_k} = I_w \end{array}$$

(reduced expression)

W の right order (left)

$w, w' \in W$ $\quad w \overset{R}{\underset{L}{\leq}} w' \overset{\text{def}}{\Longleftrightarrow} \exists w''$ $w' = ww''$ $\quad l(w') = l(w) + l(w'')$

l : reduced expression の長さ

<u>Thm</u> (tilt Λ-mod の order) = (right order)^{op}

左加群

(tilt Λ^{op}-mod の order) = (left order)^{op}

§3. <u>n-cluster tilt. mod</u>

Λ : module fin. R-alg. $\quad R$: comm. noeth

$\mathcal{X} \subseteq \operatorname{mod} \Lambda$ の extension closed full subcategory

$M \in \mathcal{X}$ が <u>n-cluster filting</u>

$\overset{\text{def}}{\Leftrightarrow}$ $\operatorname{add} M = \{ X \in \mathcal{X} : \operatorname{Ext}^i(M, X) = 0 \ (i = 1, \cdots, n-1) \}$

$= \{ X \in \mathcal{X} : \operatorname{Ext}^i(X, M) = 0 \ (i = 1, \cdots, n-1) \}$

Thm Λ : fin. dim. alg

(1). $M \in \mathrm{mod}\,\Lambda$ が n-cluster tilting.

$$\Gamma := \mathrm{End}(M)$$

$$\mathrm{gl.dim}\,\Gamma \leq n+1 \leq \mathrm{dom.dim}\,\Gamma$$

min. inj. resol. $0 \to {}_\Gamma\Gamma \to \underbrace{\boxed{I_0 \to \cdots \to I_n}}_{\text{proj.}} \to I_{n+1} \to 0$

S : simple Γ-module.

$$\mathrm{pd}_\Gamma S = n+1 \Rightarrow \mathrm{Ext}^i_\Gamma(S, \Gamma) = \begin{cases} 0 & (i \neq n+1) \\ \text{simple } \Gamma^{op}\text{-module} & (i = n+1) \end{cases}$$

Λ が R-order $\dim R = n+1$ ならば.

CMΛ の n-cluster tilting obj の End は Gorenstein. cond をみたす.

Example $S = k[[x_1, \cdots, x_d]]$ char $k = 0$

$G \leq GL_d(k)$: finite subgrp.

CM $S^G \ni S$ は $(d-1)$-cluster tilting

特に. $d = 2$ add S = CM S^G (1-cluster tilting)

S^G は. representation finite.

§4. $S = k[[x, y]]$ $\Lambda := S/(f)$

$0 \neq f \in (x, y)$ $f = f_1 \cdots f_n$ (f_i : irred.)

Assume Λ is reduced. ($(f_i) \neq (f_j)$ $(i \neq j)$)

— $\mathrm{Hom}_\Lambda(X, Y) \simeq D\mathrm{Ext}^1_\Lambda(Y, \Omega X)$ $(X, Y \in \mathrm{CM}\Lambda)$
$; \Omega^{2-2}$

$\underline{\mathrm{CM}}\Lambda$ は. 0-Calabi. Yau triangulated category.

— MF より. $\Omega^2 = \mathrm{id}$

$\Lambda^n \xrightarrow{Q} \Lambda^n \xrightarrow{P} \Lambda^n \xrightarrow{Q} \Lambda^n \xrightarrow{P} \Lambda^n$

CMΛ : 2-Calabi. Yau
⇝ 2-cluster tilting の mutation

<u>Thm. 1</u> CM Λ に 2-cluster. tilt が存在

$\Leftrightarrow f_i \notin (x.y)^2 \quad (\forall i)$

このとき. $S_i := S/(f_1 \cdots f_i)$ とおくと.

CM $\Lambda \ni \bigoplus_{i=1}^{n} S_i$ が. 2-cluster tilting

$(\Leftarrow)$ MF を使って. $\mathrm{Ext}^1_\Lambda(M.M) = 0$ がいえる.

<u>Lem</u> M: generator $\mathrm{Ext}^1_\Lambda(M.M) = 0$ を仮定

M: 2-cluster. tilt. $\Leftrightarrow$ gl. dim. $\mathrm{End}_\Lambda(M) \leq 3$

(実際に. simple $\mathrm{End}_\Lambda(M)$-mod の proj. res を構成することで.
M: 2-cluster. tilt)

$(\Rightarrow)$ $S' := k[[x.y.z.w]]$

$\Lambda' := S'/(f(x.y) + zw)$

$\underline{\mathrm{CM}}\,\Lambda \simeq \underline{\mathrm{CM}}\,\Lambda'$ Knorrer • periodicity.

CM Λ に 2-cluster. tilt が存在

$\Leftrightarrow$ CM Λ' に ―――

$\Leftrightarrow$ Λ' が. non-commutative. crepant resolution が存在.
[Van. den. Bergh]

M: reflexive Λ'-mod.

$\Gamma := \mathrm{End}_{\Lambda'}(M)$ が. gl. dim $(\Gamma) = 3 = \mathrm{depth}_{\Lambda'}\Gamma$ をみたす.

Van. den. Bergh.
$\Leftrightarrow$ Λ' が. crepant resolution が存在.

Katz
$\Leftrightarrow$ $f = f_1 \cdots f_n \quad (f_i \notin (x.y)^2)$