Versal ガロア被覆について 坂内真三（都立大学）

$/\mathbb{C}$

§ Introduction

Def. Galois cover

X, Y : norm. proj. alg. varieties

$\pi : X \to Y$: surj. finite morphism

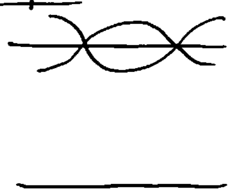
$\mathbb{C}(X)/\mathbb{C}(Y)$: Galois extension

Rem. $\pi : X \to Y$ $\quad \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$

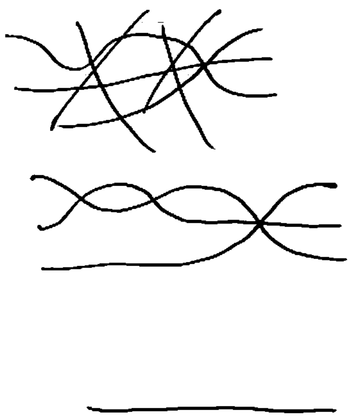
$G \curvearrowright X \longrightarrow X/G \cong Y$

$\underbrace{G \curvearrowright X'}_{\text{G-variety と呼ぶ}} \longrightarrow X'/G$ — G-cover.

Example



$\mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^3$



not Galois cover

$z \mapsto z^2$, $\mathbb{P}^1 \to \mathbb{P}^1$

Basic Problem.

Construct Galois covers

"Top-down" construction

"Bottom-up" — $\left.\begin{array}{l} Y : \text{norm. proj. alg. variety} \\ G : \text{finite group} \end{array}\right\}$

$\Rightarrow \pi : X \to Y$ G-cover

Inverse Galois problem

$\mathbb{Q}$ 上に、ガロア群 G をもつ ガロア拡大を与えよ。

Answer (I)

$\mathbb{C}(Y)$ $\quad K/\mathbb{C}(Y)$: Galois $\quad \mathrm{Gal}(K/\mathbb{C}(Y)) \cong G$

$\tilde{Y}$: K-normalization $\quad \pi : \tilde{Y} \longrightarrow Y$

Answer (II) (Namba) "pull-back" construction

$$\begin{tikzcd} X \arrow[r, dashed, "\exists \mu"] \arrow[d, "\exists \pi"'] & \boxed{X'} \arrow[d, "\pi"] \\ Y \arrow[r, dashed] & Y' \end{tikzcd}$$

G-cover

: Given とする。

この写像の存在？ → π に条件

この上に G-cover をつくる。

§. Versal Galois cover

Def. $\varpi : X \longrightarrow Y$ Versal G-cover

$\overset{}{\underset{\text{def.}}{\Longleftrightarrow}}$ $\forall \pi : X' \longrightarrow Y'$ G-cover

$\exists \mu : X' \dashrightarrow X$: G-equivariant rational map

s.t. $\mu(X') \not\subset \mathrm{Fix}(X, G) := \{ x \in X \mid G_x \neq \{1\} \}$.

Thm (Namba)

G : finite group. $\quad G \,(\nearrow \mathfrak{S}_n) \curvearrowright (\mathbb{P}^1)^{|G|} \longrightarrow (\mathbb{P}^1)^{|G|}/G$

は versal G-cover.

Example
$M \cong \mathbb{Z}^n$

$M \curvearrowleft G \rightsquigarrow \mathbb{C}[M] \curvearrowleft G$

$G \curvearrowright \mathrm{Spec}\,\mathbb{C}[M] \cong (\mathbb{C}^*)^n$ monomial action

$\exists\, X(\Delta)$: proj. toric variety

$G \curvearrowright X(\Delta) \longrightarrow X(\Delta)/G$ versal G-cover

Thm
$\rho : G \hookrightarrow GL(n, \mathbb{C}) \longrightarrow PGL(n, \mathbb{C})$

$G \curvearrowright \mathbb{P}^{n-1}$ faithful action

$\Rightarrow \tilde{\omega} : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}/G$ versal

cf. $\rho \oplus 1$: $G \to GL(n+1, \mathbb{C}) \to PGL(n+1, \mathbb{C})$, $G \curvearrowright \mathbb{P}^n \to \mathbb{P}^n/G$

§ Essential dimension

G : finite group

W, X : faithful G-variety

Def.
$\alpha : X \dashrightarrow W$: dominant G-equivariant rational map

$\overset{\Longleftrightarrow}{\text{def.}}$ α : compression $\boxed{\alpha(X) \not\subset \mathrm{Fix}(W, G)}$

Def. (Beuler - Reichstein)
$ed_G(X) := \min\{\dim W \mid {}^\exists \alpha : X \dashrightarrow W \text{ compression}\}$

Def. Gの essential dimension,

$\rho : G \hookrightarrow GL(n, \mathbb{C})$ $ed_{\mathbb{C}}(G) := ed_G(\mathbb{C}^n)$ (ρの取り方に依らない.)

<u>Fact</u>

$ed_{\mathbb{C}}(G) = 1$

$G = \mathbb{Z}/n\mathbb{Z},\ \underbrace{D_{2n}}_{}$ order $2n$ $(n:\text{odd})$

$G \curvearrowright \mathbb{P}^1 \longrightarrow \mathbb{P}^1/G$

<u>Fact</u>

$ed_{\mathbb{C}}(G) = \min\{\dim X \mid \tilde{\omega}: X \to Y \text{ versal } G\text{-cover}\}$

<u>$ed_{\mathbb{C}}(G) = 2$ の場合</u> $\Rightarrow G \subset Cr_2(\mathbb{C}) := \mathrm{Aut}_{\mathbb{C}}\,\mathbb{C}(x,y)$

$= Br(\mathbb{P}^2)$

<u>Thm (Tokunaga)</u>

$ed_{\mathbb{C}}(G) = 2 \Rightarrow \exists$ rational G-surface X

$\tilde{\omega}: X \longrightarrow X/G$ is versal

<u>Fact</u>

$\tilde{\omega}: X \to Y$ versal G-cover $\Rightarrow X$: rational.

$G \subset Cr_2(\mathbb{C})$ $\quad \mathbb{C}(x,y) \curvearrowleft G$

$G \subset \mathrm{Aut}(X)$

J. Blanc Abelian case math. AG/0610368

Dolgachev - Iskovskikh General case math. AG/0610595

$ed_{\mathbb{C}}(G) = 2$

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\mathbb{P}^2 \longrightarrow \mathbb{P}^2/(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$: versal

$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$

$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \curvearrowright$

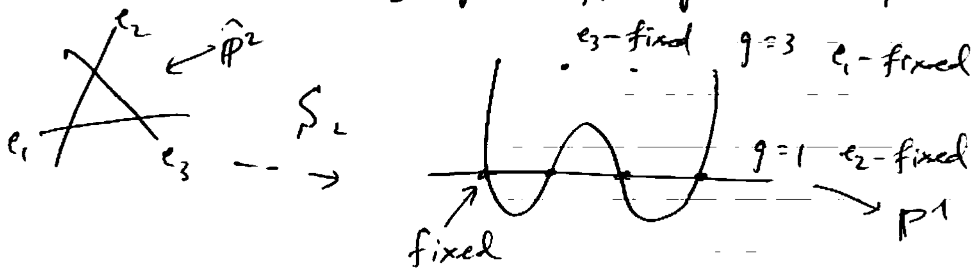
$$\begin{array}{ccc} \mathbb{P}^1\times\mathbb{P}^1 & \xrightarrow{pr.} & \mathbb{P}^1 \curvearrowleft \\ \downarrow & & \\ \mathbb{P}^1\times\mathbb{P}^1/\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & & \end{array}$$

← versal でない.

$\mathbb{P}(2,1,1,1)$

$(w;x;y;z) \longmapsto (\pm w; x:y:\pm z)$

$w^2 = z^4 + f_2(x,y)z^2 + f_4(x,y)$ $\quad f_i$: deg. i homog. poly.



fixed point の "Going-up" / "Going-down" $\quad$ Kollar Szabo

H: finite $\quad Z, Z'$: faithful H-varieties

$p: Z \dashrightarrow Z'$

$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{P}^2 \dashrightarrow S_2$

$p: \mathbb{P}^2 \dashrightarrow S_2$ $\quad$ $\deg p$ odd は存在しない.

$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$-equiv.

dominant