DERIVED CATEGORIES IN REPRESENTATION THEORY

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We survey recent methods of derived categories in the representation theory of algebras.

1. Triangulated Categories and Brown Representability

Definition 1.1. *A triangulated category $\mathcal{C}$ is an additive category together with (1) an autofunctor $T : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ (i.e. there is T^{-1} such that $T \circ T^{-1} = T^{-1} \circ T = \mathbf{1}_{\mathcal{C}}$) called the translation, and*
(2) a collection $\mathcal{T}$ of sextuples (X, Y, Z, u, v, w):

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

called (distinguished) triangles. These data are subject to the following four axioms:

(TR1) *(1) Every sextuple (X, Y, Z, u, v, w) which is isomorphic to a triangle is a triangle.*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \wr\downarrow f & & \wr\downarrow g & & \wr\downarrow h & & \wr\downarrow T(f) \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X') \end{array} \quad \textit{triangle}$$

(2) Every morphism $u : X \to Y$ is embedded in a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

(3) For any $X \in \mathcal{C}$,

$$X \xrightarrow{1} X \to 0 \to T(X)$$

is a triangle

(TR2) *A sextuple*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is a triangle.

(TR3) *For any triangles $(X,\ Y,\ Z,\ u,\ v,\ w)$, (X',Y',Z',u',v',w') and a commutative diagram*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X')
\end{array}$$

there exists $h : Z \to Z'$ which makes a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X')
\end{array}$$

(TR4) *(Octahedral axiom) For any two consecutive morphisms $u : X \to Y$ and $v : Y \to Z$, if we embed u, vu and v in triangles (X,Y,Z',u,i,i'), (X,Z,Y',vu,k,k') and (Y,Z,X',v,j,j'), respectively, then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commute*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & T(X) \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & T(X) \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle T(u)} \\
 & & X' & = & X' & \xrightarrow{j'} & T(Y) \\
 & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle T(i)j'} & & \\
 & & T(Y) & \xrightarrow{T(i)} & T(Z') & &
\end{array}$$

and the third column is a triangle.

Sometimes, we write $X[i]$ for $T^i(X)$.

Definition 1.2 (∂-functor)**.** *Let $\mathcal{C}$, $\mathcal{C}'$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called ∂-functor (sometimes exact functor) provided that there is a functorial isomorphism $\alpha : FT_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{C}'}F$ such that*

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\alpha_X F(w)} T_{\mathcal{C}'}(F(X))$$

is a triangle in $\mathcal{C}'$ whenever

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_{\mathcal{C}}(X)$$

is a triangle in $\mathcal{C}$*. Moreover, if a* ∂*-functor* F *is an equivalence, then* F *is called a triangulated equivalence . In this case, we denote by* $\mathcal{C} \overset{\triangle}{\cong} \mathcal{C}'$*.*

For $(F,\alpha),(G,\beta) : \mathcal{C} \to \mathcal{C}'$ ∂*-functors, a functorial morphism* $\phi : F \to G$ *is called a* ∂*-functorial morphism if*

$$(T_{\mathcal{C}'}\phi)\circ\alpha = \beta\circ\phi T_{\mathcal{C}} \qquad \begin{array}{ccc} FT_{\mathcal{C}} & \xrightarrow{\alpha} & T_{\mathcal{C}'}F \\ {\scriptstyle \phi T_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle T_{\mathcal{C}'}\phi} \\ GT_{\mathcal{C}} & \xrightarrow{\beta} & T_{\mathcal{C}'}G \end{array}$$

We denote by $\partial(\mathcal{C},\mathcal{C}')$ *the collection of all* ∂*-functors from* $\mathcal{C}$ *to* $\mathcal{C}'$*, and denote by* $\partial\,\mathrm{Mor}(F,G)$ *the collection of* ∂*-functorial morphisms from* F *to* G*.*

Proposition 1.3. *Let* $F : \mathcal{C} \to \mathcal{C}'$ *be a* ∂*-functor between triangulated categories. If* $G : \mathcal{C}' \to \mathcal{C}$ *is a right (or left) adjoint of* F*, then* G *is also a* ∂*-functor.*

Definition 1.4. *A contravariant (resp., covariant) additive functor* $H : \mathcal{C} \to \mathcal{A}$ *from a triangulated category* $\mathcal{C}$ *to an abelian category* $\mathcal{A}$ *is called a homological functor (resp., cohomological functor), if for any triangle* (X,Y,Z,u,v,w) *in* $\mathcal{C}$ *the sequence*

$$H(T(X)) \to H(Z) \to H(Y) \to H(X)$$
$$(\text{resp., } H(X) \to H(Y) \to H(Z) \to H(T(X))\ )$$

is exact. Taking $H(T^i(X)) = H^i(X)$*, we have the long exact sequence:*

$$\cdots \to H^{i+1}(X) \to H^i(Z) \to H^i(Y) \to H^i(X) \to \cdots$$

Proposition 1.5. *The following hold.*

(1) *If* (X,Y,Z,u,v,w) *is a triangle, then* $vu = 0$*,* $wv = 0$ *and* $T(u)w = 0$*.*
(2) *For any* $X \in \mathcal{C}$*,* $\mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C} \to \mathfrak{Ab}$ *(resp.,* $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathfrak{Ab}$*) is a homological functor (resp., cohomological functor).*
(3) *For any homomorphism of triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X') \end{array}$$

if two of f*,* g *and* h *are isomorphisms, then the rest is also an isomorphism.*

Definition 1.6 (Compact Object). *Let $\mathcal{C}$ be a triangulated category. An object $C \in \mathcal{C}$ is called a compact object in $\mathcal{C}$ if the canonical morphism*

$$\coprod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(C, X_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(C, \coprod_{i \in I} X_i)$$

is an isomorphism for any set $\{X_i\}_{i \in I}$ of objects (if $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$).

A triangulated category $\mathcal{C}$ is compactly generated if $\mathcal{C}$ contains arbitrary coproducts, and if there is a set $\mathcal{S}$ of compact objects such that

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{S}, X) = 0 \Rightarrow X = 0$$

For a compactly generated triangulated category $\mathcal{C}$, a set $\mathcal{S}$ of compact objects is called a generating set if

(1) $\mathrm{Hom}_{\mathcal{C}}(\mathcal{S}, X) = 0 \Rightarrow X = 0$,
(2) $T(\mathcal{S}) = \mathcal{S}$.

Definition 1.7 (Homotopy Limit). *Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts (resp., products). For a sequence $\{X_i \to X_{i+1}\}_{i \in \mathbb{N}}$ (resp., $\{X_{i+1} \to X_i\}_{i \in \mathbb{N}}$) of morphisms in $\mathcal{C}$, the homotopy colimit (resp., homotopy limit) of the sequence is the third (resp., second) term of the triangle*

$$\coprod_i X_i \xrightarrow{1-\ shift} \coprod_i X_i \to \underrightarrow{\mathsf{hocolim}}\ X_i \to T(\coprod_i X_i)$$

$$(resp.,\ T^{-1}(\prod_i X_i) \to \underleftarrow{\mathsf{holim}}\ X_i \to \prod_i X_i \xrightarrow{1-\ shift} \prod_i X_i)$$

where the above shift morphism is the coproduct (resp., product) of $X_i \xrightarrow{f_i} X_{i+1}$ (resp., $X_{i+1} \xrightarrow{f_i} X_i$) ($i \in \mathbb{N}$).

Theorem 1.8 (Brown Representability Theorem [Ne], [Ke]). *Let $\mathcal{C}$ be a compactly generated triangulated category which contains arbitrary coproducts. If a homological functor $H : \mathcal{C} \to \mathfrak{Ab}$ sends coproducts to products, then it is representable, that is , there is an object $X \in \mathcal{C}$ such that $H \cong \mathrm{Hom}_{\mathcal{C}}(-, X)$.*

Sketch of Proof. Here we set $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$. Let $\mathcal{S}$ be a generating set of $\mathcal{C}$. There exist a coproduct X_1 of objects of $\mathcal{S}$ and a morphism $h_{X_1} \to H$ such that

$$\mathrm{Hom}_{\mathcal{C}}(C, X_1) \twoheadrightarrow H(C)$$

is surjective for any $C \in \mathcal{S}$. For a functor

$$K_1 = \mathrm{Ker}(h_{X_1} \to H)$$

there exists a coproduct Z_2 of objects in $\mathcal{S}$ and a morphism $h_{Z_2} \to K_1$ such that

$$\mathrm{Hom}_{\mathcal{C}}(C, Z_2) \twoheadrightarrow K_1(C)$$

is surjective for any $C \in \mathcal{S}$. Then we have a triangle:

$$Z_2 \to X_1 \to X_2 \to Z_2[1]$$

Since H is a homological functor, we have a commutative diagram

$$\begin{array}{ccccc} H(X_2) & \longrightarrow & H(X_1) & \longrightarrow & H(Z_2) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \mathrm{Mor}(h_{X_2}, H) & \longrightarrow & \mathrm{Mor}(h_{X_1}, H) & \longrightarrow & \mathrm{Mor}(h_{Z_2}, H) \end{array}$$

Then there is a morphism $h_{X_2} \to H$ satisfying a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{K}_1 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(-, X_1) & \longrightarrow & H \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathrm{K}_2 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(-, X_2) & \longrightarrow & H \end{array}$$

and we have a morphism of exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{K}_1(C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C, X_1) & \longrightarrow & H(C) & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{K}_2(C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C, X_2) & \longrightarrow & H(C) & \longrightarrow & 0 \end{array}$$

for any $C \in \mathcal{S}$. By inductive step, we have a triangle

$$\coprod_i X_i \xrightarrow{1-\text{ shift}} \coprod_i X_i \to \underrightarrow{\mathsf{hocolim}}\ X_i \to T\coprod_i X_i$$

and we have an exact sequence

$$\begin{array}{ccccc} H(\underrightarrow{\mathsf{hocolim}}\ X_i) & \longrightarrow & \prod_i H(X_i) & \longrightarrow & \prod_i H(X_i) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \mathrm{Mor}(h_{\underrightarrow{\mathsf{hocolim}}\ X_i}, H) & \longrightarrow & \prod_i \mathrm{Mor}(h_{X_i}, H) & \longrightarrow & \prod_i \mathrm{Mor}(h_{X_i}, H) \end{array}$$

Therefore there is a morphism $\mathrm{Hom}_{\mathcal{C}}(-, \underrightarrow{\mathsf{hocolim}}\ X_i) \to H$ such that

$$\mathrm{Hom}_{\mathcal{C}}(C, \underrightarrow{\mathsf{hocolim}} X_i) \cong H(C)$$

for any $C \in \mathcal{S}$. Considering the case $H = \mathrm{Hom}_{\mathcal{C}}(-, M)$, it is easy to see that $\underrightarrow{\mathsf{hocolim}} X_i \cong M$. Moreover, this result implies that

$$\mathrm{Hom}_{\mathcal{C}}(-, \underrightarrow{\mathsf{hocolim}} X_i) \cong H$$

□

Remark 1.9 (Yoneda's Lemma)**.** *For a category $\mathcal{C}$, the following hold.*

(1) *For $X \in \mathcal{C}$ and a contravariant functor $F : \mathcal{C} \to \mathfrak{Set}$, we have the bijection $FX \to \mathrm{Mor}(h_X, F)$.*
(2) *For $X, Y \in \mathcal{C}$, we have the bijection $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}(h_X, h_Y)$.*

Corollary 1.10 (Adjoint Functor Theorem [Ne])**.** *Let $\mathcal{C}$ be a compactly generated triangulated category which contains arbitrary coproducts. If a ∂-functor $F : \mathcal{C} \to \mathcal{D}$ commutes with arbitrary coproducts, then there exists a ∂-functor $G : \mathcal{D} \to \mathcal{C}$ which is a right adjoint of F.*

Proof. For any $Y \in \mathcal{D}$, the functor

$$\mathrm{Hom}_{\mathcal{D}}(F(-), Y) : \mathcal{C} \to \mathfrak{Ab}$$

is a homological functor. By Brown representability theorem there is an object $GY \in \mathcal{C}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(F(-), Y) \cong \mathrm{Hom}_{\mathcal{C}}(-, GY)$$

□

Definition 1.11 (Quotient Category)**.** *Let S be a multiplicative system in a triangulated category $\mathcal{C}$ which satisfies the following conditions:*

(FR0) *For a morphism s in $\mathcal{C}$, if there exist f, g such that $sf, gs \in \mathsf{S}$, then $s \in \mathsf{S}$.*
(FR1) *(1) $1_X \in \mathsf{S}$ for every $X \in \mathcal{C}$.*
(2) For $s, t \in \mathsf{S}$, if st is defined, then $st \in \mathsf{S}$.

(FR2) *(1) Every diagram in $\mathcal{C}$*

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle f}\downarrow & & \\ X' & & \end{array}$$

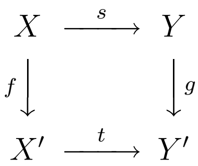
with $s \in \mathsf{S}$, can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{t} & Y' \end{array}$$

with $s, t \in \mathsf{S}$.
(2) Every diagram in $\mathcal{C}$

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{t} & Y' \end{array}$$

with $t \in \mathsf{S}$, can be completed to a commutative square



with $s, t \in \mathsf{S}$.

(FR3) *For $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the following are equivalent.*
(1) There exists $s \in \mathsf{S}$ such that $sf = sg$.
(2) There exists $t \in \mathsf{S}$ such that $ft = gt$.

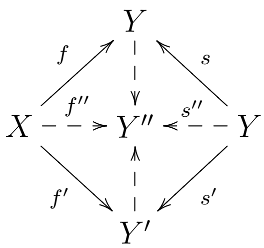
(FR4) *For a morphism u in $\mathcal{C}$, $u \in \mathsf{S}$ if and only if $Tu \in \mathsf{S}$.*

(FR5) *For triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and morphisms $f : X \to X'$, $g : Y \to Y'$ in S with $gu = u'f$, there exists $h : Z \to Z'$ in S such that (f, g, h) is a homomorphism of triangles.*

We define the quotient category $\mathsf{S}^{-1}\mathcal{C}$ of $\mathcal{C}$, as follows:

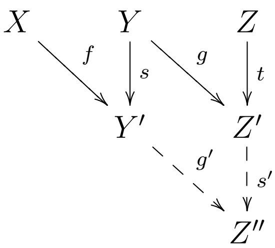
(1) $\mathrm{Ob}(\mathsf{S}^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.

(2) *For $X, Y \in \mathrm{Ob}(\mathcal{C})$, let $V(X, Y) = \{(s, Y', f) | s : Y \to Y' \in \mathsf{S}, f : X \to Y\}$. In $V(X, Y)$, we define $(s, Y', f) \sim (s', Y'', f')$ if there is (s'', Y''', f') such that all triangles are commutative in the following diagram:*



Then we define a morphism from X to Y by an equivalence class $s^{-1}f$ of (s, Y', f).

(3) *For $s^{-1}f : X \to Y, t^{-1}g : Y \to Z$, by (FR2) there are $s' : Z' \to Z'' \in \mathsf{S}$ and $g' : Y' \to Z''$ such that $s' \circ g = g' \circ s$. Then we define $(t^{-1}g) \circ (s^{-1}f) = (s' \circ t)^{-1} g \circ f$.*



Moreover, we define the quotient functor $Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ *by*

(1) $Q(X) = X$ *for* $X \in \mathcal{C}$.
(2) $Q(f) = 1_Y^{-1} f$ *for a morphism* $f : X \to Y$ *in* $\mathcal{C}$.

Remark 1.12. *Can we define (2) in the above?*

Definition 1.13 (Épaisse Subcategory)**.** *Let* $\mathcal{C}$ *be a triangulated category. An additive full subcategory* $\mathcal{U}$ *of* $\mathcal{C}$ *is called a full triangulated subcategory if* $X \to Y$ *is a morphism in* $\mathcal{U}$*, then there is a triangle* $X \to Y \to Z \to TX$ *with* $Z \in \mathcal{U}$.

A full triangulated subcategory $\mathcal{U}$ *is called an épaisse subcategory if it is closed under direct summands. In this case, let* $\mathsf{S}(\mathcal{U})$ *be the collection of morphisms* s *such that* $X \xrightarrow{s} Y \to Z \to X[1]$ *is a triangle with* $Z \in \mathcal{U}$*. Then* $\mathsf{S}(\mathcal{U})$ *is a multiplicative system satisfying (FR0) - (FR5). We write* $\mathcal{C}/\mathcal{U} = \mathsf{S}(\mathcal{U})^{-1}\mathcal{C}$.

In the case that $\mathcal{C}$ *contains arbitrary coproducts, a full triangulated subcategory* $\mathcal{U}$ *is called a localizing subcategory if it is closed under coproducts.*

Proposition 1.14 ([BN])**.** *Let* $\mathcal{C}$ *be a triangulated category which contains arbitrary coproducts. Then any localizing subcategory is an épaisse subcategory.*

Proposition 1.15. *Let* $\mathcal{C}$ *be a triangulated category. For a multiplicative system* S *satisfying the conditions (FR0) - (FR5), let* $\mathcal{U}(\mathsf{S})$ *be the full triangulated subcategory consisting of objects* Z *which is in a triangle* $X \xrightarrow{s} Y \to Z \to X[1]$ *with* $s \in \mathsf{S}$*. Then the following hold.*

(1) $\mathsf{S}(\mathcal{U})$ *and* $\mathcal{U}(\mathsf{S})$ *induce a 1 - 1 correspondence between the collection of multiplicative systems* S *satisfying the conditions (FR0) - (FR5) and the collection of épaisse subcategories* $\mathcal{U}$.
(2) *For an épaisse subcategory* $\mathcal{U}$, $\mathcal{C}/\mathcal{U}$ *is a triangulated category whose triangles are defined to be isomorphic to triangles of* $\mathcal{C}$.
(3) *Assume* $\mathcal{C}$ *contains arbitrary coproducts. For a localizing subcategory* $\mathcal{U}$, $\mathcal{C}/\mathcal{U}$ *also contains arbitrary coproducts.*

Definition 1.16 (stable t-structure)**.** *For full subcategories* $\mathcal{U}$ *and* $\mathcal{V}$ *of a triangulated category* $\mathcal{C}$, $(\mathcal{U}, \mathcal{V})$ *is called a stable* t*-structure in* $\mathcal{C}$ *provided that*

(1) $\mathcal{U}$ *and* $\mathcal{V}$ *are stable for translations.*
(2) $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$.
(3) *For every* $X \in \mathcal{C}$*, there exists a triangle*

$$U \to X \to V \to T(U)$$

with $U \in \mathcal{U}$ *and* $V \in \mathcal{V}$.

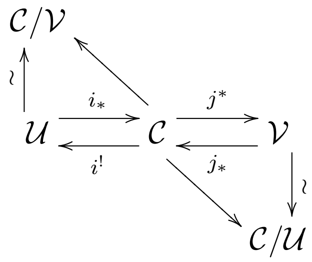
Proposition 1.17 ([BBD], c.f. [Mi])**.** *Let $\mathcal{C}$ be a triangulated category, $(\mathcal{U}, \mathcal{V})$ a stable t-structure in $\mathcal{C}$, and $i_* : \mathcal{U} \to \mathcal{C}, j_* : \mathcal{V} \to \mathcal{C}$ the canonical embeddings. Then the following hold.*

(1) *$\mathcal{U}$ and $\mathcal{V}$ is épaisse subcategories of $\mathcal{C}$.*
(2) *i_* (resp., j_*) has a right adjoint $i^!$ (resp., a left adjoint j^*).*
(3) *The adjunction arrows induce a triangle*

$$i_*i^!X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} j_*j^*X \to i_*i^!X[1]$$

for any $X \in \mathcal{C}$.

(4) *$\mathcal{C}/\mathcal{U}$ (resp., $\mathcal{C}/\mathcal{V}$) exists , and it is triangulated equivalent to $\mathcal{V}$ (resp., $\mathcal{U}$).*



2. Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ which is closed under isomorphisms.

Definition 2.1 (Complex)**.** *A (cochain) complex is a collection $X^{\boldsymbol{\cdot}} = (X^n, d^n_X : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d^{n+1}_X d^n_X = 0$. A complex $X^{\boldsymbol{\cdot}} = (X^n, d^n_X : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ is called bounded below (resp., bounded above, bounded) if $X^n = 0$ for $n \ll 0$ (resp., $n \gg 0$, $n \ll 0$ and $n \gg 0$).*

A complex $X^{\boldsymbol{\cdot}} = (X^n, d^n_X)$ is called a stalk complex if there exists an integer n_0 such that $X^i = O$ if $i \neq n_0$. We identify objects of $\mathcal{B}$ with a stalk complexes of degree 0.

A morphism $f : X^{\cdot} \to Y^{\cdot}$ of complexes is a collection of morphisms $f^n : X^n \to Y^n$ which makes a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X^n & \xrightarrow{d^n_X} & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & Y^n & \xrightarrow{d^n_Y} & Y^{n+1} & \longrightarrow & \cdots
\end{array}$$

We denote by $\mathsf{C}(\mathcal{B})$ (resp., $\mathsf{C}^+(\mathcal{B})$, $\mathsf{C}^-(\mathcal{B})$, $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An autofunctor $T : \mathsf{C}(\mathcal{B}) \to \mathsf{C}(\mathcal{B})$ is called *translation if* $(T(X^{\bullet}))^n = X^{n+1}$ *and* $(Td_X)^n = -d_X^{n+1}$ *for any complex* $X^{\bullet} = (X^n, d_X^n)$.

In $\mathsf{C}(\mathcal{A})$*, a morphism* $u : X^{\cdot} \to Y^{\cdot}$ *is called a quasi-isomorphism if* $\mathrm{H}^n(u)$ *is an isomorphism for any* n.

In this section, "$*$" means "nothing", "$+$", "$-$" or "b".

Definition 2.2. *For* $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$*, the mapping cone of* u *is a complex* $\mathrm{M}^{\bullet}(u)$ *with*

$$\mathrm{M}^n(u) = X^{n+1} \oplus Y^n,$$
$$d^n_{\mathrm{M}^{\bullet}(u)} = \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_X^n \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}.$$

Definition 2.3 (Homotopy Relation)**.** *Two morphisms* $f, g \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$ *are said to be homotopic (denote by* $f \underset{h}{\simeq} g$*) if there is a collection of morphisms* $h = (h^n)$*,* $h^n : X^n \to Y^{n+1}$ *such that*

$$f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$.

Definition 2.4 (Homotopy Category)**.** *The homotopy category* $\mathsf{K}^*(\mathcal{B})$ *of* $\mathcal{B}$ *is defined by*

(1) $\mathrm{Ob}(\mathsf{K}^*(\mathcal{B})) = \mathrm{Ob}(\mathsf{C}^*(\mathcal{B}))$,
(2) $\mathrm{Hom}_{\mathsf{K}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{\mathsf{C}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) / \underset{h}{\simeq}$ *for* $X^{\cdot}, Y^{\cdot} \in \mathrm{Ob}(\mathsf{K}^*(\mathcal{B}))$.

Proposition 2.5. *A category* $\mathsf{K}^*(\mathcal{B})$ *is a triangulated category whose triangles are defined to be isomorphic to*

$$X^{\cdot} \xrightarrow{u} Y^{\cdot} \to \mathrm{M}^{\cdot}(u) \to T(X^{\cdot})$$

for any $u : X^{\cdot} \to Y^{\cdot}$ *in* $\mathsf{K}^*(\mathcal{B})$.

Definition 2.6 (Derived Category)**.** *The derived category* $\mathsf{D}^*(\mathcal{A})$ *of an abelian category* $\mathcal{A}$ *is* $\mathsf{K}^*(\mathcal{A}) / \mathsf{K}^{*,\phi}(\mathcal{A})$*, where* $\mathsf{K}^{*,\phi}(\mathcal{A})$ *is the full subcategory of* $\mathsf{K}^*(\mathcal{A})$ *consisting of null complexes, that is, complexes whose all homologies are* 0.

Proposition 2.7. *The following hold.*

(1) $\mathsf{D}^*(\mathcal{A})$ *is a triangulated category, and the canonical functor* $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ *is a* ∂*-functor.*
(2) *The* i*-th cohomology of complexes is a cohomological functor in the sense of Definition 1.4.*

Definition 2.8. *A complex $X^{\cdot}$ of $\mathsf{K}(\mathcal{B})$ is called* K-*injective (resp.,* K-*projective) if*

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(N^{\cdot}, X^{\cdot}) = 0$$
$$(\text{ resp., } \mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(X^{\cdot}, N^{\cdot}) = 0 \text{ })$$

for any null complex $N^{\cdot}$.

Example 2.9. *Let A be a ring, $\mathsf{Mod}\,A$ the category of right A-modules, and $\mathsf{Inj}\,A$ (resp., $\mathsf{Proj}\,A$) the category of injective (resp., projective) right A-modules. Then any complex $I^{\cdot} \in \mathsf{K}^{+}(\mathsf{Inj}\,A)$ (resp., $P^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,A)$) is a* K-*injective (resp.,* K-*projective) complex in $\mathsf{K}(\mathsf{Mod}\,A)$.*

Example 2.10. *Let k be a field, $A = k[x]/(x^2)$, and*

$$X^{\cdot} : \cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots .$$

Then $X^{\cdot}$ is a null complex of finitely generated projective-injective A-modules. But it is neither K-*projective nor* K-*injective, because $0 \neq 1 \in \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(X^{\cdot}, X^{\cdot})$.*

Theorem 2.11 ([Sp], [Ne], [LAM], [Fr])**.** *Let $\mathsf{K}^{inj}(\mathsf{Mod}\,A)$ (resp., $\mathsf{K}^{proj}(\mathsf{Mod}\,A)$) be the category of* K-*injective (resp.,* K-*projective) complexes, then the following hold.*

1. *$(\mathsf{K}^{proj}(\mathsf{Mod}\,A), \mathsf{K}^{\phi}(\mathsf{Mod}\,A))$ is a stable t-structure in $\mathsf{K}(\mathsf{Mod}\,A)$, and hence $\mathsf{D}(\mathsf{Mod}\,A)$ exists and is triangulated equivalent to $\mathsf{K}^{proj}(\mathsf{Mod}\,A)$.*
2. *$(\mathsf{K}^{\phi}(\mathsf{Mod}\,A), \mathsf{K}^{inj}(\mathsf{Mod}\,A))$ is a stable t-structure in $\mathsf{K}(\mathsf{Mod}\,A)$, and hence $\mathsf{D}(\mathsf{Mod}\,A)$ is triangulated equivalent to $\mathsf{K}^{inj}(\mathsf{Mod}\,A)$.*
3. *For a Grothendieck category $\mathcal{A}$, $(\mathsf{K}^{\phi}(\mathcal{A}), \mathsf{K}^{inj}(\mathcal{A}))$ is a stable t-structure in $\mathsf{K}(\mathcal{A})$, and hence $\mathsf{D}(\mathcal{A})$ exists and is triangulated equivalent to $\mathsf{K}^{inj}(\mathcal{A})$.*

Proof. (1) For a complex $X^{\bullet} = (X^i, d^i)$, we define the following truncation:

$$\sigma_{\leq n} X^{\bullet} : \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}\, d^n \to 0 \to \cdots$$

For any n, there is a complex $P_n^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,A)$ which has a quasi-isomorphism $P_n^{\cdot} \to \sigma_{\leq n} X^{\bullet}$. Then we have the following quasi-isomorphisms (qis)

$$X^{\cdot} \cong \varinjlim \sigma_{\leq n} X^{\cdot} \overset{\text{qis}}{\longleftarrow} \underrightarrow{\mathsf{hocolim}}\, \sigma_{\leq n} X^{\cdot} \overset{\text{qis}}{\longleftarrow} \underrightarrow{\mathsf{hocolim}}\, P_n^{\cdot}$$

It is easy to see that $\underrightarrow{\mathsf{hocolim}}\, P_n$ is K-projective.

(2) Similarly.

(3) Because there is a ring A such that $\mathcal{A}$ is a localization of $\mathsf{Mod}\,A$ (Gabriel-Popescu Theorem). See [LAM] or [Fr]. $\square$

Remark 2.12 (Grothendieck Category). *An abelian category $\mathcal{C}$ is called a Grothendieck category if*

(1) *$\mathcal{C}$ has coproducts of objects indexed by arbitrary sets,*
(2) *the filtered colimit of exact sequences is exact,*
(3) *$\mathcal{C}$ has a generator U.*

In this case, we have Gabriel-Popescu Theorem:
Let $R = \mathrm{End}_{\mathcal{C}}(U)$, then there are functors $F : \mathsf{Mod}\, R \to \mathcal{C}$, $G := \mathrm{Hom}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathsf{Mod}\, R$ such that

(1) *F is an exact functor,*
(2) *G is a right adjoint of F,*
(3) *$FG \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}}$.*

Remark 2.13. *If $P^{\cdot}$ is K-projective complex (e.g. bounded above complex of projective A-modules), then we have*

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(P^{\cdot}, X^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(P^{\cdot}, X^{\cdot})$$

for any complex $X^{\cdot}$. Similarly, for a K-injective complex $I^{\cdot}$ (e.g. bounded below complex of injective A-modules), then we have

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\cdot}, I^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(X^{\cdot}, I^{\cdot})$$

for any complex $X^{\cdot}$.

Proposition 2.14. *If $0 \to X^{\bullet} \xrightarrow{u} Y^{\bullet} \xrightarrow{v} Z^{\bullet} \to 0$ is a exact sequence in $\mathsf{C}(\mathcal{A})$, then it can be embedded in a triangle in $\mathsf{D}(\mathcal{A})$*

$$QX^{\bullet} \xrightarrow{Qu} QY^{\bullet} \xrightarrow{Qv} QZ^{\bullet} \xrightarrow{w} TQ(X^{\bullet}).$$

Definition 2.15 (Right Derived Functor). *For a ∂-functor $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$, the right derived functor of F is a ∂-functor*

$$\boldsymbol{R}^* F : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$$

together with a functorial morphism of ∂-functors

$$\xi \in \partial \,\mathrm{Mor}(Q_{\mathcal{A}'} \circ F, \boldsymbol{R}^* F \circ Q^*_{\mathcal{A}})$$

with the following property:
For $G \in \partial(\mathsf{D}^*(\mathcal{A}), \mathsf{D}(\mathcal{A}'))$ and $\zeta \in \partial \,\mathrm{Mor}(Q_{\mathcal{A}'} \circ F, G \circ Q^*_{\mathcal{A}})$, there exists a unique morphism $\eta \in \partial \,\mathrm{Mor}(\boldsymbol{R}^* F, G)$ such that

$$\zeta = (\eta Q^*_{\mathcal{A}})\xi.$$

In other words, we can simply write the above using functor categories. For triangulated categories $\mathcal{C}, \mathcal{C}'$, the ∂-functor category $\partial(\mathcal{C}, \mathcal{C}')$ is the category (?) consisting of ∂-functors from $\mathcal{C}$ to $\mathcal{C}'$ as objects and ∂-functorial morphisms as morphisms. Then we have

$$\partial \,\mathrm{Mor}(Q_{\mathcal{A}'} \circ F, -Q^*_{\mathcal{A}}) \cong \partial \,\mathrm{Mor}(\boldsymbol{R}^* F, -)$$

as functors from $\partial(\mathsf{D}^*(\mathcal{A}), \mathsf{D}(\mathcal{A}'))$ to $\mathfrak{Set}$.

Proposition 2.16. *Let $\mathcal{A}, \mathcal{A}'$ be abelian categories, $F : \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$ a ∂-functor. If $\mathcal{A}$ is a Grothendieck category, then we have the right derived functor $\boldsymbol{R}F : \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$ such that $F(X^\cdot) \cong \boldsymbol{R}F(X^\cdot)$ for any K-injective complex $X^\cdot$.*

Definition 2.17 ($\mathrm{Hom}^\bullet_A, \dot{\otimes}_A$). *Let $X^\cdot, Y^\cdot$ be complexes in $\mathsf{C}(\mathsf{Mod}\, A)$, $Z^\cdot$ a complex in $\mathsf{C}(\mathsf{Mod}\, A^{op})$. We define the complex $\mathrm{Hom}^\bullet_A(X^\cdot, Y^\cdot)$ in $\mathsf{C}(\mathfrak{Ab})$ by*

$$\mathrm{Hom}^n_A(X^\cdot, Y^\cdot) = \prod_{j-i=n} \mathrm{Hom}_A(X^i, Y^j)$$

$$d^n_{\mathrm{Hom}^\bullet(X,Y)}(f) = d_X \circ f - (-1)^n f \circ d_Y \quad \text{for } f \in \mathrm{Hom}^n_A(X^\cdot, Y^\cdot)$$

And we define the complex $X^\cdot \dot{\otimes}_A Z^\cdot$ in $\mathsf{C}(\mathfrak{Ab})$ by

$$X^\cdot \overset{n}{\otimes}_A Z^\cdot = \coprod_{i+j=n} X^i \otimes_A Z^j$$

$$d^n_{X \otimes Y} = d_X \otimes 1 + (-1)^n 1 \otimes d_Z$$

Proposition 2.18. *Let A are a ring. Then we have a right derived functor*

$$\boldsymbol{R}\,\mathrm{Hom}^\bullet_A : \mathsf{D}(\mathsf{Mod}\, A)^{op} \times \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathfrak{Ab})$$

and a left derived functor

$$\dot{\otimes}^{\boldsymbol{L}}_A : \mathsf{D}(\mathsf{Mod}\, A) \times \mathsf{D}(\mathsf{Mod}\, A^{op}) \to \mathsf{D}(\mathfrak{Ab})$$

Definition 2.19 (Perfect Complex). *Let A be a ring. A complex $X^\cdot \in \mathsf{D}(\mathsf{Mod}\, A)$ is called a perfect complex if $X^\cdot$ is quasi-isomorphic to a bounded complex of finitely generated projective A-modules.*

Let X be a scheme, $\mathsf{D}(X)$ the derived category of sheaves of $\mathcal{O}_X$-modules. We denote by $\mathsf{D}_{qc}(X)$ the full subcategory of $\mathsf{D}(X)$ consisting of complexes whose cohomologies are quasi-coherent sheaves. A complex $X^\cdot \in \mathsf{D}_{qc}(X)$ is called a perfect complex if $X^\cdot$ is locally quasi-isomorphic to a bounded complex of vector bundles.

We denote by $\mathsf{D}_{pf}(\mathcal{A})$ the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ consisting of perfect complexes.

Proposition 2.20 ([Rd1], [Ne]). *For a ring A, the following hold.*

(1) *A complex $X^\cdot \in \mathsf{D}(\mathsf{Mod}\, A)$ is perfect if and only if it is a compact object in $\mathsf{D}(\mathsf{Mod}\, A)$.*
(2) *$\mathsf{D}(\mathsf{Mod}\, A)$ is compactly generated.*

Theorem 2.21 ([BV]). *Let X be a quasi-compact quasi-separated scheme, then the following hold.*

(1) *A complex $X^{\cdot} \in \mathsf{D}_{qc}(X)$ is perfect if and only if it is a compact object in $\mathsf{D}_{qc}(X)$.*
(2) *$\mathsf{D}_{qc}(X)$ is compactly generated.*

Theorem 2.22 ([BN]). *Let X be a quasi-compact separated scheme, then the canonical functor $\mathsf{D}(\mathrm{Qcoh}\, X) \to \mathsf{D}_{qc}(X)$ is a triangulated equivalence, where $\mathrm{Qcoh}\, X$ is the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules.*

Corollary 2.23 ([BV]). *Let X be smooth over a field, then we have*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, X) \overset{\triangle}{\cong} \mathsf{D}_{pf}(X).$$

where $\mathsf{coh}\, X$ is the category of coherent sheaves of $\mathcal{O}_X$-modules.

For a ring A, we denote by $\mathsf{proj}\, A$ the category of finitely generated projective A-modules.

Theorem 2.24 ([Rd1], [Rd2]). *Let A, B be algebras over a field k. The following are equivalent.*

(1) $\mathsf{D}(\mathsf{Mod}\, A) \overset{\triangle}{\cong} \mathsf{D}(\mathsf{Mod}\, B)$.
(2) $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \overset{\triangle}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$.
(3) *There is a perfect complex $T^{\bullet} \in \mathsf{D}(\mathsf{Mod}\, A)$ such that*
(a) $B \cong \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(T^{\bullet})$,
(b) $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(T^{\bullet}, T^{\bullet}[i]) = 0$ for $i \neq 0$,
(c) $\{T^{\cdot}[i] | i \in \mathbb{Z}\}$ is a generating set in $\mathsf{D}(\mathsf{Mod}\, A)$.
(4) *There is a complex $X^{\cdot}$ of B-A-bimodules such that*

$$\boldsymbol{R}\mathrm{Hom}^{\bullet}_A(X^{\cdot}, -) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

is an equivalence.

In this case, $T^{\cdot}$ is called a tilting complex for A, $X^{\cdot}$ is called two-sided tilting complex, and $\boldsymbol{R}\mathrm{Hom}^{\bullet}_A(X^{\cdot}, -)$ is called a standard equivalence.

Theorem 2.25 ([BO]). *Let X be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If X' is a smooth algebraic variety such that $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, X) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, X')$, then X' is isomorphic to X.*

Theorem 2.26 ([Be]). *Let $\mathbf{P} = \mathbf{P}^n_k$ be the n-dimensional projective space over a field k, and let $\mathcal{T}_1 = \bigoplus_{i=0}^n \mathcal{O}(i)$, $\mathcal{T}_2 = \bigoplus_{i=0}^n \Omega(-i)$, and $B_1 = \mathrm{End}_{\mathbf{P}}(\mathcal{T}_1)$, $B_2 = \mathrm{End}_{\mathbf{P}}(\mathcal{T}_2)$. Then B_i are finite dimensional k-algebra of finite global dimension, and*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, \mathbf{P}) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B_1) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B_2)$$

where $\mathsf{mod}\, B_i$ *is the category of finitely generated* B_i*-modules.*

Definition 2.27. *Let* A *be an algebra over a field* k*. The* derived Picard group *of* A *(relative to* k*) is*

$$\mathrm{DPic}_k(A) := \frac{\{\textit{tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})\}}{\textit{isomorphism}}$$

with identity element A*, product* $(T_1, T_2) \mapsto T_1 \otimes^{\mathrm{L}}_A T_2$ *and inverse* $T \mapsto T^\vee := \mathrm{R}\,\mathrm{Hom}_A(T, A)$*. Given any* k*-linear triangulated category* $\mathcal{C}$ *we let*

$$\mathrm{Out}^{\triangle}_k(\mathcal{C}) := \frac{\{k\textit{-linear triangulated self-equivalences of } \mathcal{C}\}}{\textit{isomorphism}}. \tag{2.1}$$

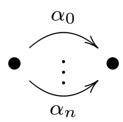
Theorem 2.28 ([MY])**.** *Let* k *be an algebraically closed field, and* A *a finite dimensional hereditary* k*-algebra. Then we have*

$$\mathrm{DPic}_k(A) = \mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)) = \mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$$

M. Kontsevich and A. Rosenberg introduced the notion of non-commutative projective spaces $\mathbf{NP}^n$, and showed that

$$\mathsf{D}^{\mathrm{b}}(\mathrm{Qcoh}\,\mathbf{NP}^n) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, kQ_n)$$
$$\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbf{NP}^n) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, kQ_n)$$

where Q_n is the quiver



Corollary 2.29 ([MY])**.** *For a non-commutative projective spaces* $\mathbf{NP}^n$*, we have*

$$\begin{aligned}\mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{Qcoh}\,\mathbf{NP}^n)) &\cong \mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbf{NP}^n)) \\ &\cong \mathbb{Z} \times \big(\mathbb{Z} \ltimes \mathrm{PGL}_{n+1}(k)\big)\end{aligned}$$

For $\mathbf{P}^1$*, we have* $\mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbf{P}^1)) \cong \mathbb{Z} \times \mathbb{Z} \times \mathrm{PGL}_2(k)$*.*

Theorem 2.30 ([BO])**.** *Let* X *be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then* $\mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\, X))$ *is generated by the automorphisms of variety, the twists by invertible sheaves and the translations, and hence* $\mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\, X)) \cong (\mathrm{Aut}_k X \ltimes \mathrm{Pic}\, X) \times \mathbb{Z}$*.*

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