Chapter 2
Elements of Convex Analysis

(COSS 2018 Reading Material)

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Minimal technical elements from convex analysis are given in this section. For comprehensive account, the reader is referred to books on convex analysis [1,2,3,5,6,7,8,9,10].

2.1 Convex Sets

For two vectors \( a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in (\mathbb{R} \cup \{-\infty, +\infty\})^n \) we define closed interval \([a, b] = \{ x \in \mathbb{R}^n | a_i \leq x_i \leq b_i \, (i = 1, \ldots, n) \}\) and open interval \((a, b) = \{ x \in \mathbb{R}^n | a_i < x_i < b_i \, (i = 1, \ldots, n) \}\) as

\[
\begin{align*}
[a, b] &= [a, b]_\mathbb{R} = \{ x \in \mathbb{R}^n | a_i \leq x_i \leq b_i \, (i = 1, \ldots, n) \}, \\
(a, b) &= (a, b)_\mathbb{R} = \{ x \in \mathbb{R}^n | a_i < x_i < b_i \, (i = 1, \ldots, n) \},
\end{align*}
\]

(2.1)

where, if \( a_i = -\infty \), for example, \( a_i \leq x_i \) is to be understood as \(-\infty < x_i\).

A set \( S \subseteq \mathbb{R}^n \) is called convex if it satisfies the condition

\[
x, y \in S, \ 0 \leq \lambda \leq 1 \implies \lambda x + (1 - \lambda) y \in S,
\]

(2.3)

where an empty set is a convex set. A convex polyhedron is a convex set \( S \) described by a finite number of linear inequalities as

\[
S = \{ x \in \mathbb{R}^n | \sum_{j=1}^{n} a_{ij} x_j \leq b_i \, (i = 1, 2, \ldots, m) \},
\]

(2.4)

where \( a_{ij} \in \mathbb{R} \) and \( b_i \in \mathbb{R} \, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n) \).

For a finite number of points \( x^1, x^2, \ldots, x^m \) in a set \( S \), a point represented as

\[
\lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_m x^m
\]

(2.5)

where \( \lambda_i \in \mathbb{R} \, (i = 1, 2, \ldots, m) \).
with nonnegative coefficients $\lambda_i (1 \leq i \leq m)$ having unit sum ($\sum_{i=1}^{m} \lambda_i = 1$) is called a convex combination of those points. If $S$ is convex, any convex combination of points in $S$ belongs to $S$, and the converse is also true. Therefore, $S$ is convex if and only if $S = S$, where $\mathcal{S}$ denotes the set of all possible convex combinations of a finite number of points of $S$.

The intersection of any (finite or infinite) number of convex sets is a convex set. For any set $S$, the intersection of all the convex sets containing $S$ is the smallest convex set containing $S$, which is called the convex hull of $S$ and denoted as $\text{conv}(S)$. The convex hull of $S$ coincides with the set of all convex combinations of points in $S$. That is, we have $\text{conv}(S) = \mathcal{S}$. The convex hull of a set $S$ is not necessarily closed (in the topological sense). The smallest closed convex set containing $S$ is called the closed convex hull of $S$ and denoted as $\overline{\text{conv}}(S)$.

The affine hull of a set $S$ is defined to be the smallest affine set (a translation of a linear space) containing $S$, and is denoted by $\text{aff}S$. The relative interior of $S$, denoted as $\text{ri}S$, is the set of points $x \in S$ such that $\{y \in \mathbb{R}^n | \|y - x\| < \varepsilon\} \cap \text{aff}S$ is contained in $S$ for some $\varepsilon > 0$. In other words, the relative interior of $S$ is the set of the interior points of $S$ with respect to the topology induced from $\text{aff}S$.

For two sets $S$ and $T$, the set
\[
S + T = \{x + y | x \in S, y \in T\}
\]
(2.6)
is called the Minkowski sum of $S$ and $T$. If $S$ and $T$ are convex, the Minkowski sum $S + T$ is a convex set.

A set $S$ is a cone if it satisfies
\[
x \in S, \quad \lambda > 0 \implies \lambda x \in S.
\]
(2.7)
A cone that is convex is called a convex cone. In other words, a set $S$ is a convex cone if and only if it satisfies the condition
\[
x, y \in S, \quad \lambda, \mu > 0 \implies \lambda x + \mu y \in S.
\]
(2.8)

### 2.2 Convex Functions

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ we define
\[
\text{dom} f = \text{dom}_{\mathbb{R}^n} f = \{x \in \mathbb{R}^n | -\infty < f(x) < +\infty\},
\]
(2.9)
which is called the effective domain of $f$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if it satisfies
\[
\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) \quad (x, y \in \mathbb{R}^n; 0 \leq \lambda \leq 1).
\]
(2.10)
Note that $-\infty$ is excluded from the possible function values of a convex function, and that the inequality (2.10) is satisfied, by convention, if both sides are equal to $+\infty$. A convex function having a nonempty effective domain is called a proper convex function. A function is strictly convex if it satisfies (2.10) with strict inequalities, i.e., if

$$
\lambda f(x) + (1-\lambda)f(y) > f(\lambda x + (1-\lambda)y) \quad (x, y \in \text{dom } f; 0 < \lambda < 1). 
$$

(2.11)

A function $g : \mathbb{R}^n \to \mathbb{R}$ is concave if $-g$ is convex, that is, if

$$
\lambda g(x) + (1-\lambda)g(y) \leq g(\lambda x + (1-\lambda)y) \quad (x, y \in \mathbb{R}^n; 0 \leq \lambda \leq 1).
$$

(2.12)

The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$, denoted as $\text{epi } f$, is the set of points in $\mathbb{R}^n \times \mathbb{R}$ lying above the graph of $\alpha = f(x)$. Namely,

$$
\text{epi } f = \{(x, \alpha) \in \mathbb{R}^{n+1} | \alpha \geq f(x)\}.
$$

(2.13)

Then we have

$$
f \text{ is a convex function } \iff \text{epi } f \text{ is a convex set.}
$$

(2.14)

A function $f$ is said to be closed convex if $\text{epi } f$ is a closed convex set in $\mathbb{R}^{n+1}$.

The indicator function of a set $S \subseteq \mathbb{R}^n$ is a function $\delta_S : \mathbb{R}^n \to \{0, +\infty\}$ defined by

$$
\delta_S(x) = \begin{cases} 
0 & (x \in S), \\
+\infty & (x \notin S).
\end{cases}
$$

(2.15)

Then we have

$$
S \text{ is a convex set } \iff \delta_S \text{ is a convex function.}
$$

(2.16)

For a family of convex functions $\{f_k | k \in K\}$, indexed by $K$, the pointwise maximum of those functions, $f(x) = \sup \{f_k(x) | k \in K\}$, is again a convex function, where the index set $K$ here may possibly be infinite. In particular, the maximum of a finite or infinite number of affine functions

$$
f(x) = \sup \{\alpha_k + \langle p_k, x \rangle | k \in K\}
$$

(2.17)

is a convex function, where $\alpha_k \in \mathbb{R}$ and $p_k \in \mathbb{R}^n$ for $k \in K$ and

$$
\langle p, x \rangle = \sum_{i=1}^n p_i x_i
$$

(2.18)

denotes the inner product of $p = (p_1, p_2, \ldots, p_n)$ and $x = (x_1, x_2, \ldots, x_n)$.

A function defined on $\mathbb{R}^n$ is said to be polyhedral convex if its epigraph is a convex polyhedron in $\mathbb{R}^{n+1}$. A polyhedral convex function is exactly such a function that can be represented as the maximum of a finite number of affine functions (i.e., (2.17) with finite $K$) on an effective domain represented as (2.4).
For two functions $f, g : \mathbb{R}^n \to \mathbb{R}$, their sum is the function $f + g : \mathbb{R}^n \to \mathbb{R}$ defined naturally by
\[(f + g)(x) = f(x) + g(x) \quad (x \in \mathbb{R}^n), \quad (2.19)\]
and their infimal convolution is the function $f \boxplus g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by
\[(f \boxplus g)(x) = \inf \{ f(y) + g(z) \mid x = y + z, \; y, z \in \mathbb{R}^n \} \quad (x \in \mathbb{R}^n). \quad (2.20)\]
The sum of two convex functions is convex, and the infimal convolution of two convex functions is convex if it does not take the value of $-\infty$. If $f$ and $g$ are the indicator functions of sets $S$ and $T$, then $f + g$ and $f \boxplus g$ are the indicator functions of the intersection $S \cap T$ and the Minkowski sum $S + T$, respectively.

For a function $f$ and a vector $p$, we denote by $f[p]$ the function defined by
\[f[p](x) = f(x) - \langle p, x \rangle \quad (x \in \mathbb{R}^n). \quad (2.21)\]
This is convex for a convex function $f$.

### 2.3 Minimization and Subgradients

The most appealing property of a convex function is that local minimality is equivalent to global minimality. A point (or vector) $x$ is said to be a (global) minimizer of $f$ if the inequality
\[f(x) \leq f(y) \quad (2.22)\]
holds for every $y$. A point $x$ is a local minimizer if the inequality (2.22) holds for every $y$ in some neighborhood of $x$. Obviously, global minimality implies local minimality. The converse is not true in general, but it is true for convex functions.

**Theorem 2.1.** For a convex function, local minimality implies global minimality.

**Proof.** Let $x$ be a local minimizer of a convex function $f$. Then we have $f(z) \geq f(x)$ for all $z$ in some neighborhood $U$ of $x$. For any $y$, we can choose $\lambda < 1$ sufficiently close to $1$ such that $z = \lambda x + (1 - \lambda) y$ belongs to $U$. Then it follows from (2.10) that
\[
\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) = f(z) \geq f(x).
\]
This implies $f(y) \geq f(x)$. \(\square\)

The set of the minimizers of $f$ is denoted as
\[\text{argmin } f = \text{argmin}_{\mathbb{R}} f = \{ x \in \mathbb{R}^n \mid f(x) \leq f(y) \; (\forall y \in \mathbb{R}^n) \}. \quad (2.23)\]
This is a convex set if $f$ is convex.

The subdifferential of a function $f$ at a point $x \in \text{dom } f$ is defined to be the set
\[\partial f(x) = \{ p \in \mathbb{R}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \; (\forall y \in \mathbb{R}^n) \}. \quad (2.24)\]
2.4 Conjugacy

Note that $p \in \partial f(x)$ if and only if $x \in \text{argmin } f[-p]$: in particular, $0 \in \partial f(x)$ if and only if $x \in \text{argmin } f$. For a convex function $f$, the set $\partial f(x)$ is nonempty if $x$ is in the relative interior of $\text{dom } f$. An element of $\partial f(x)$ is called a subgradient of $f$ at $x$. If $f$ is convex and differentiable at $x$, the subdifferential $\partial f(x)$ consists of a single element, which is the gradient of $f$ at $x$:

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

(2.25)

The directional derivative of a function $f$ at a point $x \in \text{dom } f$ in a direction $d \in \mathbb{R}^n$ is defined by

$$f'(x; d) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

(2.26)

when this limit (finite or infinite) exists, where $\alpha \downarrow 0$ means that $\alpha$ tends to 0 from the positive side ($\alpha > 0$). For a convex function $f$, the limit exists for all $d$, and $f'(x; d)$ is a convex function in $d$. For a polyhedral convex function $f$, there exists $\varepsilon > 0$, independent of $x \in \text{dom } f$, such that

$$f'(x; d) = f(x + d) - f(x) \quad (\|d\| \leq \varepsilon).$$

(2.27)

2.4 Conjugacy

As Fig. 2.1 (a,b) shows, a convex function $f(x)$ can be recovered from tangent lines as the upper envelope of all tangent lines with different slopes. Let $\alpha$ be the vertical intercept of the tangent line with slope $p$. Since $\alpha$ is dependent on slope $p$, we denote $\alpha = -f^*(p)$. By considering the minimum distance between the graph of $y = f(x)$ and the line $y = px$, we see that the minimum of $f(x) - px$ over all $x$ is equal to $\alpha = -f^*(p)$; cf., Fig. 2.1(c). That is,

$$f^*(p) = \sup\{px - f(x) \mid x \in \mathbb{R}\} \quad (p \in \mathbb{R}).$$

(2.28)

This function $f^*(p)$ should be equivalent to the original function $f(x)$ in some appropriate sense.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom } f \neq \emptyset$, the convex conjugate (or simply conjugate) of $f$ is a function $f^* : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f^*(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbb{R}^n\} \quad (p \in \mathbb{R}^n),$$

(2.29)

which is indeed a convex function since it is the maximum of (infinitely many) affine functions in $p$ indexed by $x$. The function $f^*$ is also called the (convex) Legendre–Fenchel transform of $f$, and the mapping $f \mapsto f^*$ is referred to as the (convex) Legendre–Fenchel transformation. Similarly, the concave conjugate of a function $g : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom } g \neq \emptyset$ is a concave function $g^* : \mathbb{R}^n \to \mathbb{R}$ defined by
Fig. 2.1 Tangent lines of a convex function

\[ g^\circ(p) = \inf \{ \langle p, x \rangle - g(x) \mid x \in \mathbb{R}^n \} \quad (p \in \mathbb{R}^n). \]  

Note that \( g^\circ(p) = -(-g)^\ast(-p) \).

For a function \( f \), the conjugate function of the conjugate of \( f \), i.e., \((f^\ast)^\ast\), is called the biconjugate of \( f \) and denoted as \( f^{\ast\ast} \). The biconjugate of \( f \) is the largest closed convex function that is dominated pointwise by \( f \).

**Theorem 2.2.** The Legendre–Fenchel transform \( f^\ast \) in (2.29) is a closed proper convex function for any function \( f \) with \( \text{dom} f \neq \emptyset \), and \( f^{\ast\ast} = f \) for a closed proper convex function \( f \).

This theorem shows that the Legendre–Fenchel transformation \( f \mapsto f^\ast \) gives a symmetric (or involutive) one-to-one correspondence in the class of all closed proper convex functions.

For a set \( S \subseteq \mathbb{R}^n \), the conjugate \( \delta_S^\ast \) of its indicator function \( \delta_S \) is expressed as

\[ \delta_S^\ast(p) = \sup \{ \langle p, x \rangle \mid x \in S \} \quad (p \in \mathbb{R}^n), \]

which is called the support function of \( S \). The biconjugate \( \delta_S^{\ast\ast} \) of the indicator function \( \delta_S \) of a set \( S \) is the indicator function of the closed convex hull of \( S \).

By Theorem 2.2 and the definition (2.24) we obtain the relationship

\[
\begin{align*}
p \in \partial f(x) & \iff x \in \text{argmin} f[-p] \\
f(x) + f^\ast(p) & = \langle p, x \rangle \\
x \in \partial f^*(p) & \iff p \in \text{argmin} f^\ast[-x]
\end{align*}
\]  

for a closed proper convex function \( f \) and vectors \( x, p \in \mathbb{R}^n \). For a closed convex function \( f \) and a point \( x \) in the relative interior of \( \text{dom} f \), the support function of the subdifferential \( \partial f(x) \) coincides with the directional derivative \( f'(x; d) \) as a function in \( d \), i.e.,

\[
(\delta_{\partial f(x)})^\ast(d) = f'(x; d) \quad (d \in \mathbb{R}^n).
\]
The addition (2.19) and the infimal convolution (2.20) are conjugate operations with respect to the Legendre–Fenchel transformation. For proper convex functions \( f \) and \( g \) we have

\[
(f \square g)^* = f^* + g^*,
\]

(2.34)

\[
(f + g)^* = f^* \square g^*,
\]

(2.35)

where the latter is true under the assumption that \( \text{ri} (\text{dom} f) \cap \text{ri} (\text{dom} g) \neq \emptyset \).

**Example 2.1.** The conjugate of a quadratic function \( f(x) = \frac{1}{2} x^\top A x \) defined by a positive definite symmetric matrix \( A \) can be computed as follows. The maximizer \( x \) on the right-hand side of (2.29) is determined from \( p = \nabla f(x) = A x \) as \( x = A^{-1} p \). Then

\[
f^*(p) = p^\top x - \frac{1}{2} x^\top A x = \frac{1}{2} p^\top A^{-1} p.
\]

Since \( \nabla f(x) = A x \) and \( \nabla f^*(p) = A^{-1} p \), we indeed have the equivalence \( \text{”} p \in \partial f(x) \text{”} \iff x \in \partial f^*(p) \) in (2.32).

### 2.5 Duality

The separation theorem for functions asserts that, if a convex function pointwise dominates a concave function, then there exists an affine function that lies between the convex function and the concave function; see Fig. 2.2 (a).

**Theorem 2.3 (Separation for convex functions).** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a proper convex function and \( g : \mathbb{R}^n \to \mathbb{R} \) a proper concave function, and assume that at least one of the following four conditions (a1) or (a2) below is satisfied:

(a1) \( \text{ri} (\text{dom} f) \cap \text{ri} (\text{dom} g) \neq \emptyset \),

(a2) \( f \) and \( g \) are polyhedral, and \( \text{dom} f \cap \text{dom} g \neq \emptyset \).

If \( f(x) \geq g(x) \ (\forall x \in \mathbb{R}^n) \), there exist \( \alpha^* \in \mathbb{R} \) and \( p^* \in \mathbb{R}^n \) such that

\[
f(x) \geq \alpha^* + \langle p^*, x \rangle \geq g(x) \quad (\forall x \in \mathbb{R}^n).
\]

Note that the convexity assumption is critical in Theorem 2.3. In Fig. 2.2 (b), we have \( f(x) \geq g(x) \) for all \( x \), but there exists no affine function \( \alpha^* + p^* x \) that separates \( f(x) \) and \( g(x) \).

The **Fenchel duality** is a min-max relation between a pair of convex function \( f \) and concave function \( g \) and their conjugate functions \( f^* \) and \( g^* \).

**Theorem 2.4 (Fenchel duality).** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a proper convex function and \( g : \mathbb{R}^n \to \mathbb{R} \) a proper concave function, and assume that at least one of the following four conditions (a1)~(b2) below is satisfied:
(a1) $\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset$.
(a2) $f$ and $g$ are polyhedral, and $\text{dom } f \cap \text{dom } g \neq \emptyset$.
(b1) $f$ and $g$ are closed$^1$, and $\text{ri}(\text{dom } f^*) \cap \text{ri}(\text{dom } g^*) \neq \emptyset$.
(b2) $f$ and $g$ are polyhedral, and $\text{dom } f^* \cap \text{dom } g^* \neq \emptyset$.

Then it holds that

$$\inf \{f(x) - g(x) \mid x \in \mathbb{R}^n\} = \sup \{g^*(p) - f^*(p) \mid p \in \mathbb{R}^n\}. \quad (2.37)$$

Moreover, if this common value is finite, the supremum is attained by some $p \in \text{dom } f^* \cap \text{dom } g^*$ under the assumption of (a1) or (a2), and the infimum is attained by some $x \in \text{dom } f \cap \text{dom } g$ under the assumption of (b1) or (b2).

If the supremum in (2.37) is attained by $p = p^*$, we have

$$\arg\min (f - g) = \arg\min f[-p^*] \cap \arg\max g[-p^*]. \quad (2.38)$$

Remark 2.1. Theorem 2.4 above is formulated for a pair of convex and concave functions. In some cases it is convenient to reformulate it in terms of two convex functions. For convex functions $f$ and $g$, the min-max formula (2.37) is rewritten as

$$\inf \{f(x) + g(x) \mid x \in \mathbb{R}^n\} = \sup \{-f^*(p) - g^*(-p) \mid p \in \mathbb{R}^n\}. \quad (2.39)$$

References


$^1$ By this we mean that $f$ and $-g$ are closed convex functions.