Chapter 2
Elements of Convex Analysis

(COSS 2018 Reading Material)

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Minimal technical elements from convex analysis are given in this section. For comprehensive account, the reader is referred to books on convex analysis [1, 2, 3, 5, 6, 7, 8, 9, 10].

2.1 Convex Sets

For two vectors $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in (\mathbb{R} \cup \{-\infty, +\infty\})^n$ we define *closed interval* $[a, b]$ and *open interval* (a, b) as

$$[a,b] = [a,b]_{\mathbb{R}} = \{x \in \mathbb{R}^n \mid a_i \le x_i \le b_i \ (i = 1, 2, \ldots, n)\}, \tag{2.1}$$

$$(a,b) = (a,b)_{\mathbb{R}} = \{x \in \mathbb{R}^n \mid a_i < x_i < b_i \ (i = 1, 2, \ldots, n)\}, \tag{2.2}$$

where, if $a_i = -\infty$, for example, $a_i \le x_i$ is to be understood as $-\infty < x_i$.

A set $S \subseteq \mathbb{R}^n$ is called *convex* if it satisfies the condition

$$x, y \in S, \ \ 0 \le \lambda \le 1 \implies \lambda x + (1 - \lambda) y \in S, \tag{2.3}$$

where an empty set is a convex set. A *convex polyhedron* is a convex set S described by a finite number of linear inequalities as

$$S = \{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} x_j \le b_i \ (i = 1, 2, \ldots, m)\}, \tag{2.4}$$

where $a_{ij} \in \mathbb{R}$ and $b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.

For a finite number of points $x^1, x^2, \ldots, x^m$ in a set S, a point represented as

$$\lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_m x^m \tag{2.5}$$

with nonnegative coefficients λ_i $(1 \leq i \leq m)$ having unit sum $(\sum_{i=1}^{m} \lambda_i = 1)$ is called a *convex combination* of those points. If S is convex, any convex combination of points in S belongs to S, and the converse is also true. Therefore, S is convex if and only if $S = \overline{S}$, where $\overline{S}$ denotes the set of all possible convex combinations of a finite number of points of S.

The intersection of any (finite or infinite) number of convex sets is a convex set. For any set S, the intersection of all the convex sets containing S is the smallest convex set containing S, which is called the *convex hull* of S and denoted as $\mathrm{conv}(S)$. The convex hull of S coincides with the set of all convex combinations of points in S. That is, we have $\mathrm{conv}(S) = \overline{S}$. The convex hull of a set S is not necessarily closed (in the topological sense). The smallest closed convex set containing S is called the *closed convex hull* of S. For a finite set S, the convex hull is always closed.

The *affine hull* of a set S is defined to be the smallest affine set (a translation of a linear space) containing S, and is denoted by $\mathrm{aff}\, S$. The *relative interior* of S, denoted as $\mathrm{ri}\, S$, is the set of points $x \in S$ such that $\{y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon\} \cap \mathrm{aff}\, S$ is contained in S for some $\varepsilon > 0$. In other words, the relative interior of S is the set of the interior points of S with respect to the topology induced from $\mathrm{aff}\, S$.

For two sets S and T, the set

$$S + T = \{x + y \mid x \in S, y \in T\} \tag{2.6}$$

is called the *Minkowski sum* of S and T. If S and T are convex, the Minkowski sum $S + T$ is a convex set.

A set S is a *cone* if it satisfies

$$x \in S, \quad \lambda > 0 \implies \lambda x \in S. \tag{2.7}$$

A cone that is convex is called a *convex cone*. In other words, a set S is a convex cone if and only if it satisfies the condition

$$x, y \in S, \quad \lambda, \mu > 0 \implies \lambda x + \mu y \in S. \tag{2.8}$$

2.2 Convex Functions

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ we define

$$\mathrm{dom}\, f = \mathrm{dom}_{\mathbb{R}} f = \{x \in \mathbb{R}^n \mid -\infty < f(x) < +\infty\}, \tag{2.9}$$

which is called the *effective domain* of f.

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *convex* if it satisfies

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) \qquad (x, y \in \mathbb{R}^n; 0 \leq \lambda \leq 1). \tag{2.10}$$

Note that $-\infty$ is excluded from the possible function values of a convex function, and that the inequality (2.10) is satisfied, by convention, if both sides are equal to $+\infty$. A convex function having a nonempty effective domain is called a *proper convex function*. A function is *strictly convex* if it satisfies (2.10) with strict inequalities, i.e., if

$$\lambda f(x) + (1-\lambda) f(y) > f(\lambda x + (1-\lambda) y) \qquad (x, y \in \mathrm{dom} f; 0 < \lambda < 1). \quad (2.11)$$

A function $g : \mathbb{R}^n \to \underline{\mathbb{R}}$ is *concave* if $-g$ is convex, that is, if

$$\lambda g(x) + (1-\lambda) g(y) \le g(\lambda x + (1-\lambda) y) \qquad (x, y \in \mathbb{R}^n; 0 \le \lambda \le 1). \quad (2.12)$$

The *epigraph* of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, denoted as $\mathrm{epi}\, f$, is the set of points in $\mathbb{R}^n \times \mathbb{R}$ lying above the graph of $\alpha = f(x)$. Namely,

$$\mathrm{epi}\, f = \{(x, \alpha) \in \mathbb{R}^{n+1} \mid \alpha \ge f(x)\}. \quad (2.13)$$

Then we have

$$f \text{ is a convex function} \iff \mathrm{epi}\, f \text{ is a convex set.} \quad (2.14)$$

A function f is said to be *closed convex* if $\mathrm{epi}\, f$ is a closed convex set in $\mathbb{R}^{n+1}$.

The *indicator function* of a set $S \subseteq \mathbb{R}^n$ is a function $\delta_S : \mathbb{R}^n \to \{0, +\infty\}$ defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases} \quad (2.15)$$

Then we have

$$S \text{ is a convex set} \iff \delta_S \text{ is a convex function.} \quad (2.16)$$

For a family of convex functions $\{f_k \mid k \in K\}$, indexed by K, the pointwise maximum of those functions, $f(x) = \sup\{f_k(x) \mid k \in K\}$, is again a convex function, where the index set K here may possibly be infinite. In particular, the maximum of a finite or infinite number of affine functions

$$f(x) = \sup\{\alpha_k + \langle p_k, x\rangle \mid k \in K\} \quad (2.17)$$

is a convex function, where $\alpha_k \in \mathbb{R}$ and $p_k \in \mathbb{R}^n$ for $k \in K$ and

$$\langle p, x\rangle = \sum_{i=1}^{n} p_i x_i \quad (2.18)$$

denotes the *inner product* of $p = (p_1, p_2, \ldots, p_n)$ and $x = (x_1, x_2, \ldots, x_n)$.

A function defined on $\mathbb{R}^n$ is said to be *polyhedral convex* if its epigraph is a convex polyhedron in $\mathbb{R}^{n+1}$. A polyhedral convex function is exactly such a function that can be represented as the maximum of a finite number of affine functions (i.e., (2.17) with finite K) on an effective domain represented as (2.4).

For two functions $f,g:\mathbb{R}^n\to\overline{\mathbb{R}}$, their *sum* is the function $f+g:\mathbb{R}^n\to\overline{\mathbb{R}}$ defined naturally by

$$(f+g)(x)=f(x)+g(x)\qquad(x\in\mathbb{R}^n),\tag{2.19}$$

and their *infimal convolution* is the function $f\,\Box\,g:\mathbb{R}^n\to\mathbb{R}\cup\{-\infty,+\infty\}$ defined by

$$(f\,\Box\,g)(x)=\inf\{f(y)+g(z)\mid x=y+z,\ \ y,z\in\mathbb{R}^n\}\qquad(x\in\mathbb{R}^n).\tag{2.20}$$

The sum of two convex functions is convex, and the infimal convolution of two convex functions is convex if it does not take the value of $-\infty$. If f and g are the indicator functions of sets S and T, then $f+g$ and $f\,\Box\,g$ are the indicator functions of the intersection $S\cap T$ and the Minkowski sum $S+T$, respectively.

For a function f and a vector p, we denote by $f[-p]$ the function defined by

$$f[-p](x)=f(x)-\langle p,x\rangle\qquad(x\in\mathbb{R}^n).\tag{2.21}$$

This is convex for a convex function f.

2.3 Minimization and Subgradients

The most appealing property of a convex function is that local minimality is equivalent to global minimality. A point (or vector) x is said to be a (*global*) *minimizer* of f if the inequality

$$f(x)\le f(y)\tag{2.22}$$

holds for every y. A point x is a *local minimizer* if the inequality (2.22) holds for every y in some neighborhood of x. Obviously, global minimality implies local minimality. The converse is not true in general, but it is true for convex functions.

Theorem 2.1. *For a convex function, local minimality implies global minimality.*

Proof. Let x be a local minimizer of a convex function f. Then we have $f(z)\ge f(x)$ for all z in some neighborhood U of x. For any y, we can choose $\lambda<1$ sufficiently close to 1 such that $z=\lambda x+(1-\lambda)y$ belongs to U. Then it follows from (2.10) that

$$\lambda f(x)+(1-\lambda)f(y)\ge f(\lambda x+(1-\lambda)y)=f(z)\ge f(x).$$

This implies $f(y)\ge f(x)$. □

The set of the minimizers of f is denoted as

$$\operatorname{argmin} f=\operatorname{argmin}_{\mathbb{R}} f=\{x\in\mathbb{R}^n\mid f(x)\le f(y)\ (\forall y\in\mathbb{R}^n)\}.\tag{2.23}$$

This is a convex set if f is convex.

The *subdifferential* of a function f at a point $x\in\operatorname{dom} f$ is defined to be the set

$$\partial f(x)=\{p\in\mathbb{R}^n\mid f(y)-f(x)\ge\langle p,y-x\rangle\ (\forall y\in\mathbb{R}^n)\}.\tag{2.24}$$

Note that $p \in \partial f(x)$ if and only if $x \in \operatorname{argmin} f[-p]$; in particular, $\mathbf{0} \in \partial f(x)$ if and only if $x \in \operatorname{argmin} f$. For a convex function f, the set $\partial f(x)$ is nonempty if x is in the relative interior of $\operatorname{dom} f$. An element of $\partial f(x)$ is called a *subgradient* of f at x. If f is convex and differentiable at x, the subdifferential $\partial f(x)$ consists of a single element, which is the *gradient* of f at x:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right). \tag{2.25}$$

The *directional derivative* of a function f at a point $x \in \operatorname{dom} f$ in a direction $d \in \mathbb{R}^n$ is defined by

$$f'(x;d) = \lim_{\alpha \downarrow 0} \frac{f(x+\alpha d) - f(x)}{\alpha} \tag{2.26}$$

when this limit (finite or infinite) exists, where $\alpha \downarrow 0$ means that α tends to 0 from the positive side ($\alpha > 0$). For a convex function f, the limit exists for all d, and $f'(x;d)$ is a convex function in d. For a polyhedral convex function f, there exists $\varepsilon > 0$, independent of $x \in \operatorname{dom} f$, such that

$$f'(x;d) = f(x+d) - f(x) \qquad (\|d\| \le \varepsilon). \tag{2.27}$$

2.4 Conjugacy

As Fig. 2.1 (a,b) shows, a convex function $f(x)$ can be recovered from tangent lines as the upper envelope of all tangent lines with different slopes. Let α be the vertical intercept of the tangent line with slope p. Since α is dependent on slope p, we denote $\alpha = -f^\bullet(p)$. By considering the minimum distance between the graph of $y = f(x)$ and the line $y = px$, we see that the minimum of $f(x) - px$ over all x is equal to $\alpha = -f^\bullet(p)$; cf., Fig. 2.1(c). That is,

$$f^\bullet(p) = \sup\{px - f(x) \mid x \in \mathbb{R}\} \qquad (p \in \mathbb{R}). \tag{2.28}$$

This function $f^\bullet(p)$ should be equivalent to the original function $f(x)$ in some appropriate sense.

For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $\operatorname{dom} f \neq \emptyset$, the *convex conjugate* (or simply *conjugate*) of f is a function $f^\bullet : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f^\bullet(p) = \sup\{\langle p,x\rangle - f(x) \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n), \tag{2.29}$$

which is indeed a convex function since it is the maximum of (infinitely many) affine functions in p indexed by x. The function $f^\bullet$ is also called the (convex) *Legendre–Fenchel transform* of f, and the mapping $f \mapsto f^\bullet$ is referred to as the (convex) *Legendre–Fenchel transformation*. Similarly, the *concave conjugate* of a function $g : \mathbb{R}^n \to \underline{\mathbb{R}}$ with $\operatorname{dom} g \neq \emptyset$ is a concave function $g^\circ : \mathbb{R}^n \to \underline{\mathbb{R}}$ defined by

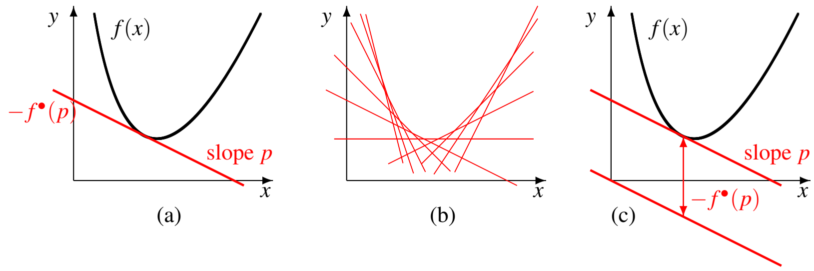


Fig. 2.1 Tangent lines of a convex function

$$g^\circ(p) = \inf\{\langle p,x\rangle - g(x) \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n). \tag{2.30}$$

Note that $g^\circ(p) = -(-g)^\bullet(-p)$.

For a function f, the conjugate function of the conjugate of f, i.e., $(f^\bullet)^\bullet$, is called the *biconjugate* of f and denoted as $f^{\bullet\bullet}$. The biconjugate of f is the largest closed convex function that is dominated pointwise by f.

Theorem 2.2. *The Legendre–Fenchel transform $f^\bullet$ in* (2.29) *is a closed proper convex function for any function f with* $\mathrm{dom}\, f \neq \emptyset$*, and $f^{\bullet\bullet} = f$ for a closed proper convex function f.* ∎

This theorem shows that the Legendre–Fenchel transformation $f \mapsto f^\bullet$ gives a symmetric (or involutive) one-to-one correspondence in the class of all closed proper convex functions.

For a set $S \subseteq \mathbb{R}^n$, the conjugate $\delta_S{}^\bullet$ of its indicator function δ_S is expressed as

$$\delta_S{}^\bullet(p) = \sup\{\langle p,x\rangle \mid x \in S\} \qquad (p \in \mathbb{R}^n), \tag{2.31}$$

which is called the *support function* of S. The biconjugate $\delta_S{}^{\bullet\bullet}$ of the indicator function δ_S of a set S is the indicator function of the closed convex hull of S.

By Theorem 2.2 and the definition (2.24) we obtain the relationship

$$\boxed{\begin{array}{rcl} p \in \partial f(x) & \Longleftrightarrow & x \in \mathrm{argmin}\, f[-p] \\ & & \Updownarrow \\ & & f(x) + f^\bullet(p) = \langle p,x\rangle \\ & & \Updownarrow \\ x \in \partial f^\bullet(p) & \Longleftrightarrow & p \in \mathrm{argmin}\, f^\bullet[-x] \end{array}} \tag{2.32}$$

for a closed proper convex function f and vectors $x, p \in \mathbb{R}^n$. For a closed convex function f and a point x in the relative interior of $\mathrm{dom}\, f$, the support function of the subdifferential $\partial f(x)$ coincides with the directional derivative $f'(x;d)$ as a function in d, i.e.,

$$(\delta_{\partial f(x)})^\bullet(d) = f'(x;d) \qquad (d \in \mathbb{R}^n). \tag{2.33}$$

The addition (2.19) and the infimal convolution (2.20) are conjugate operations with respect to the Legendre–Fenchel transformation. For proper convex functions f and g we have

$$(f \Box g)^{\bullet} = f^{\bullet} + g^{\bullet}, \tag{2.34}$$

$$(f + g)^{\bullet} = f^{\bullet} \Box g^{\bullet}, \tag{2.35}$$

where the latter is true under the assumption that $\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset$.

Example 2.1. The conjugate of a quadratic function $f(x) = \frac{1}{2}x^{\top}Ax$ defined by a positive definite symmetric matrix A can be computed as follows. The maximizer x on the right-hand side of (2.29) is determined from $p = \nabla f(x) = Ax$ as $x = A^{-1}p$. Then

$$f^{\bullet}(p) = p^{\top}x - \frac{1}{2}x^{\top}Ax = \frac{1}{2}p^{\top}A^{-1}p.$$

Since $\nabla f(x) = Ax$ and $\nabla f^{\bullet}(p) = A^{-1}p$, we indeed have the equivalence "$p \in \partial f(x) \iff x \in \partial f^{\bullet}(p)$" in (2.32). ∎

2.5 Duality

The separation theorem for functions asserts that, if a convex function pointwise dominates a concave function, then there exists an affine function that lies between the convex function and the concave function; see Fig. 2.2 (a).

Theorem 2.3 (Separation for convex functions). *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function and $g : \mathbb{R}^n \to \underline{\mathbb{R}}$ a proper concave function, and assume that* (a1) *or* (a2) *below is satisfied:*

(a1) $\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset$,

(a2) *f and g are polyhedral, and* $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$.

If $f(x) \geq g(x)$ ($\forall x \in \mathbb{R}^n$), there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ such that

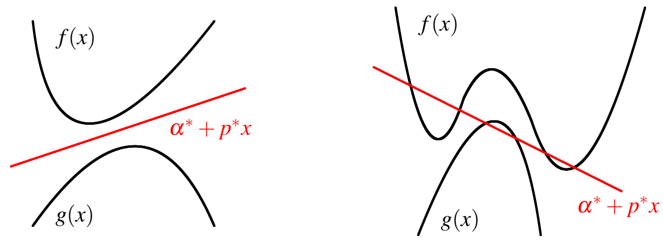
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq g(x) \qquad (\forall x \in \mathbb{R}^n). \tag{2.36}$$

∎

Note that the convexity assumption is critical in Theorem 2.3. In Fig. 2.2 (b), we have $f(x) \geq g(x)$ for all x, but there exists no affine function $\alpha^* + p^*x$ that separates $f(x)$ and $g(x)$.

The *Fenchel duality* is a min-max relation between a pair of convex function f and concave function g and their conjugate functions $f^{\bullet}$ and g°.

Theorem 2.4 (Fenchel duality). *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function and $g : \mathbb{R}^n \to \underline{\mathbb{R}}$ a proper concave function, and assume that at least one of the following four conditions* (a1)∼(b2) *below is satisfied:*



(a) Convex-concave pair (b) Nonconvex-concave pair

Fig. 2.2 Separation theorem

(a1) $\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset$,
(a2) *f and g are polyhedral, and* $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$,
(b1) *f and g are closed*[1], *and* $\mathrm{ri}\,(\mathrm{dom}\, f^{\bullet}) \cap \mathrm{ri}\,(\mathrm{dom}\, g^{\circ}) \neq \emptyset$,
(b2) *f and g are polyhedral, and* $\mathrm{dom}\, f^{\bullet} \cap \mathrm{dom}\, g^{\circ} \neq \emptyset$.
Then it holds that

$$\inf\{f(x) - g(x) \mid x \in \mathbb{R}^n\} = \sup\{g^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbb{R}^n\}. \tag{2.37}$$

Moreover, if this common value is finite, the supremum is attained by some $p \in$ $\mathrm{dom}\, f^{\bullet} \cap \mathrm{dom}\, g^{\circ}$ *under the assumption of* (a1) *or* (a2), *and the infimum is attained by some $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ under the assumption of* (b1) *or* (b2). ∎

If the supremum in (2.37) is attained by $p = p^*$, we have

$$\mathrm{argmin}\,(f - g) = \mathrm{argmin}\, f[-p^*] \cap \mathrm{argmax}\, g[-p^*]. \tag{2.38}$$

Remark 2.1. Theorem 2.4 above is formulated for a pair of convex and concave functions. In some cases it is convenient to reformulate it in terms of two convex functions. For convex functions f and g, the min-max formula (2.37) is rewritten as

$$\inf\{f(x) + g(x) \mid x \in \mathbb{R}^n\} = \sup\{-f^{\bullet}(p) - g^{\bullet}(-p) \mid p \in \mathbb{R}^n\}. \tag{2.39}$$

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[1] By this we mean that f and $-g$ are closed convex functions.

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