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Discrete Convex Analysis II:

Properties of Discrete Convex Functions

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Properties of Discrete Convex Functions

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2. L-convex (L$^\natural$-convex) fn on $\mathbb{Z}^n$
2. M-convex (M$^\natural$-convex) fn on $\mathbb{Z}^n$

3. M-convex fn on jump systems
3. L-convex fn on graphs

P1.

Convex Extension

Convex Extension

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is convex-extensible

$\Leftrightarrow$ $\exists$ convex $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

Theorem:

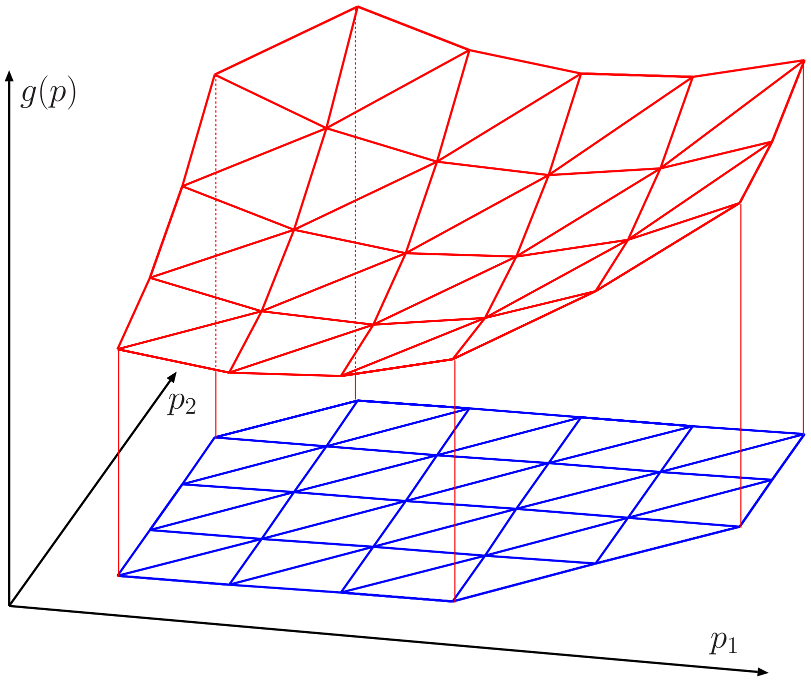
(1) Separable-convex fns are convex-extensible

(2) Integrally-convex fns are convex-extensible (by def)

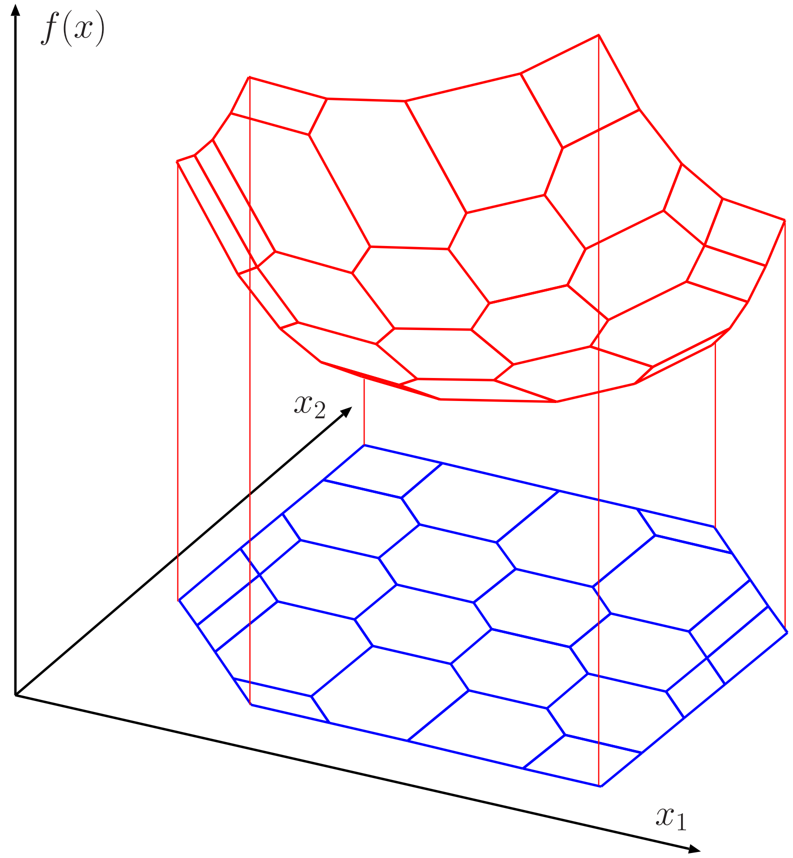
(3) $\mathrm{L}^\natural$-convex fns are convex-extensible (Murota 98)

(4) $\mathrm{M}^\natural$-convex fns are convex-extensible (Murota 96)

Bivariate L♮- and M♮-convex Functions



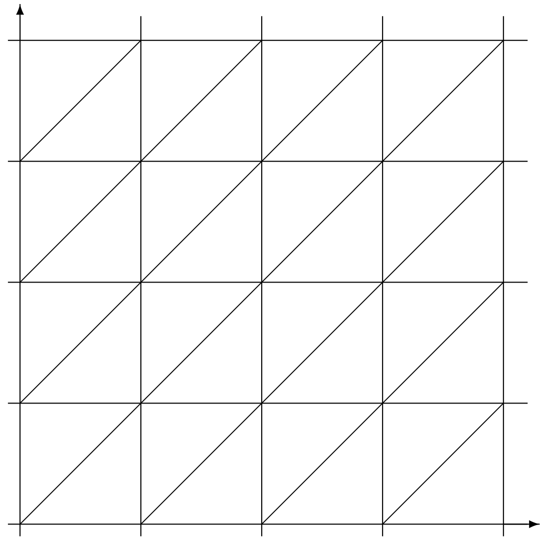
L♮-convex fn



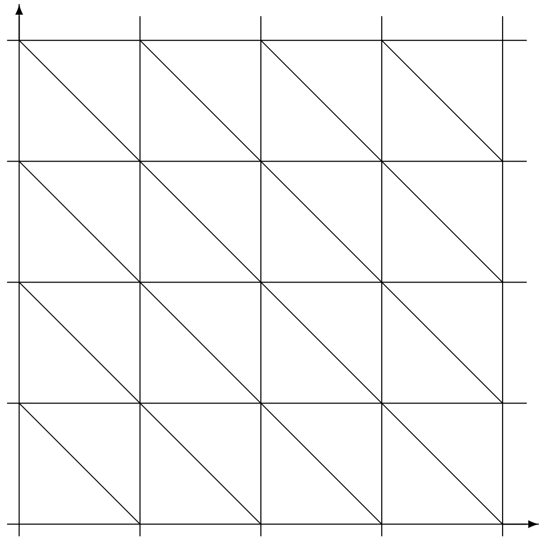
M♮-convex fn

Triangulation for Discrete Convex Functions

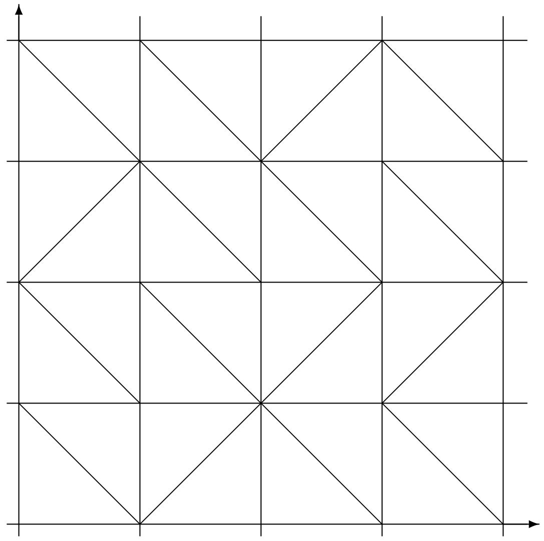
$(n = 2)$



$L^\natural$-convex

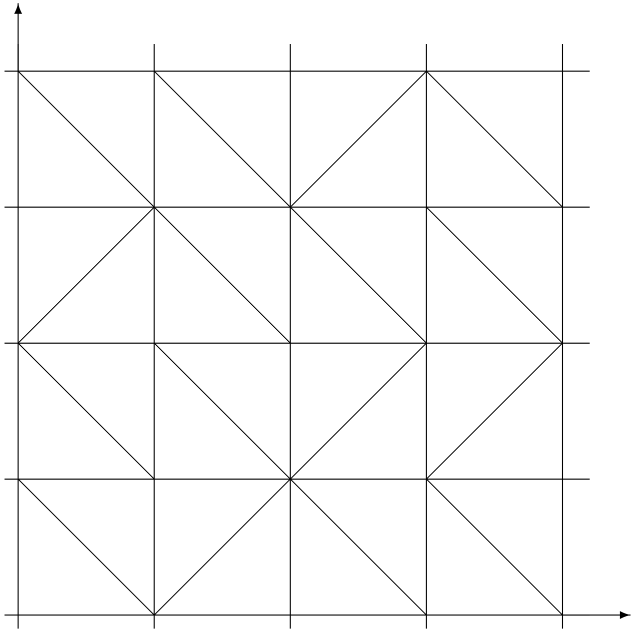


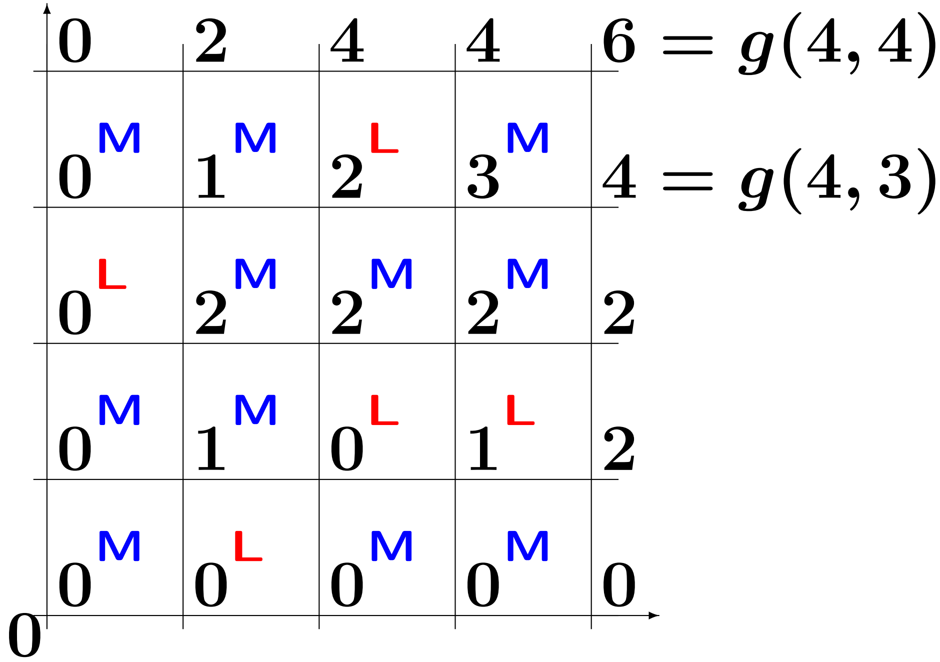
$M^\natural$-convex



Integrally convex

Bivariate Integrally-convex Functions





$$g(x_1, x_2) = \#(\mathsf{M}) - \#(\mathsf{L}) \text{ in } [0, (x_1, x_2)]$$

$$f(x_1, x_2) = A({x_1}^2 + {x_2}^2) + g(x_1, x_2)$$

Convex Extension — Computation

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is convex-extensible

$\Leftrightarrow$ $\exists$ convex $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

Theorem:

(1) Separable-convex

easy to compute (consecutive points)

(2) Integrally-convex

difficult (exp-time) to compute

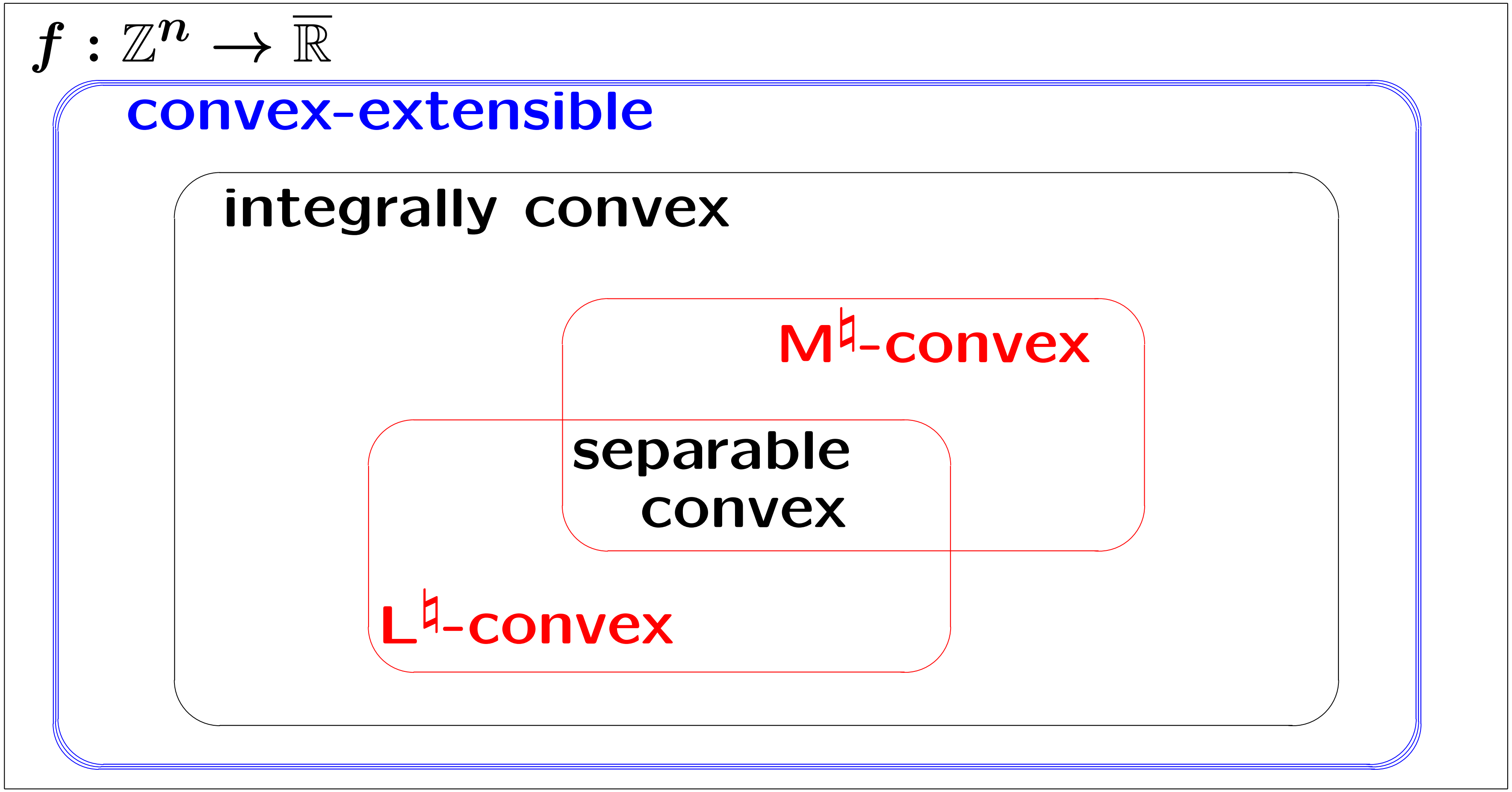
(3) L$^\natural$-convex (Favati–Tardella 90, Murota 98)

easy to compute (Lovász ext.)

(4) M$^\natural$-convex (Shioura 09,15)

poly-time to compute (via conjugacy)

Classes of Discrete Convex Functions



$\mathrm{M}^\natural \cap \mathrm{L}^\natural =$ separable

P2.

Optimality Criterion

(local opt = global opt)

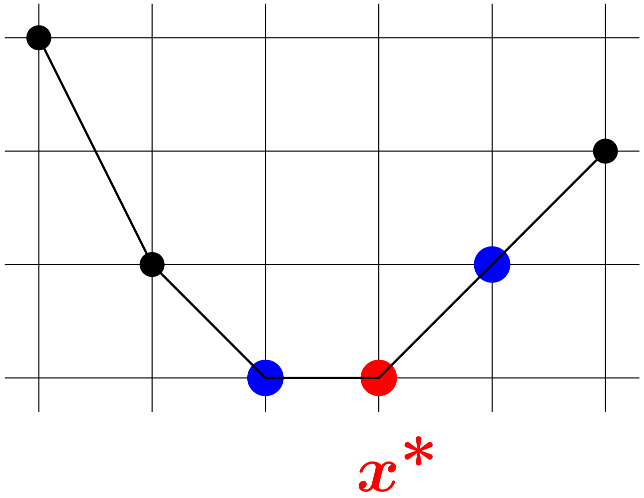
Local vs Global Optimality ($n = 1$)

$f : \mathbb{Z} \to \overline{\mathbb{R}}$

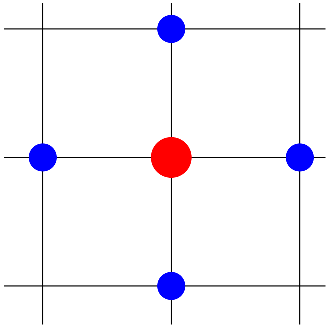
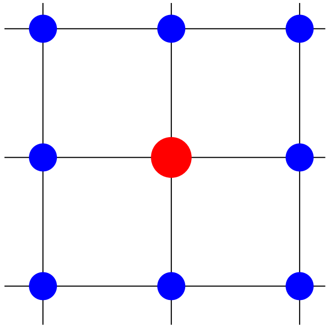
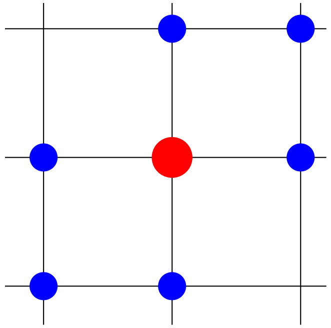
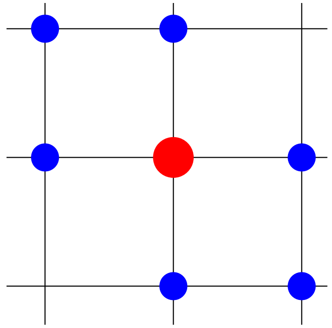
x^*: global opt (min)

$\iff$ x^*: local opt (min)

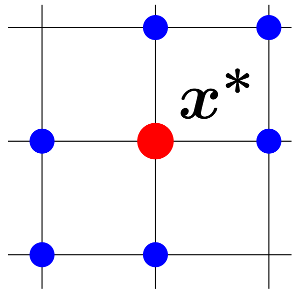
$$f(x^*) \leq \min\{f(x^* - 1),\ f(x^* + 1)\}$$



Neighborhood for Local Optimality

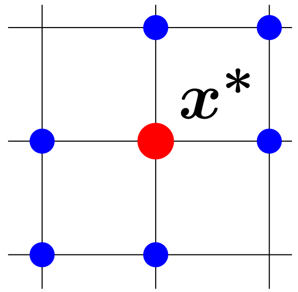
separable convex	integrally convex	L$^\natural$-convex	M$^\natural$-convex
			
$2n+1$	3^n	$2^{n+1}-1$	$n(n+1)+1$
$\{\pm e_1, \ldots, \pm e_n\}$	$\{\chi_X - \chi_Y\}$	$\{\pm \chi_X\}$	$\{e_i - e_j\}$

Local min = Global min



	= ?	#neigh-bors	poly-time algorithm local opt	poly-time algorithm global opt
submodular (set fn)	Y	2^n		
separable-conv	Y	$2n$		
integrally-conv	Y	3^n		
L$^\natural$-conv ($\mathbb{Z}^n$)	Y	2^n		
M$^\natural$-conv ($\mathbb{Z}^n$)	Y	n^2		

Local min = Global min



	= ?	#neigh -bors	poly-time algorithm local opt	global opt
submodular (set fn)	Y	2^n	Y	
separable-conv	Y	$2n$	Y	
integrally-conv	Y	3^n	N	
L$^\natural$-conv ($\mathbb{Z}^n$)	Y	2^n	Y	
M$^\natural$-conv ($\mathbb{Z}^n$)	Y	n^2	Y	

P3.

Operations

Operations

- **scaling:** $af(x) + b$, $\quad f(ax + b)$
- **linear addition:** $f(x) + \langle p, x \rangle$
- **section:** $f(x, 0)$
- **projection (partial minimization):** $\min_y f(x, y)$

- **sum:** $f_1(x) + f_2(x)$
- **convolution:** $(f_1 \square f_2)(x) = \min_y (f_1(y) + f_2(x - y))$

- **transformation by graphs/networks**

Scaling/Linear Addition

	$af(x)$ $(a > 0)$	$f(sx)$ $(s \in \mathbb{Z}_+)$	$f(-x)$	$f(x) + \langle p, x \rangle$
submodular (set fn)	Y	—	Y* $V \setminus X$	Y
separable-conv	Y	Y	Y $\pm x_i$	Y
integrally-conv	Y	N	Y $\pm x_i$	Y
L-conv ($\mathbb{Z}^n$)	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	N	Y	Y

Section/Projection

	section $f(x, 0)$	projection $\min_y f(x, y)$
submodular (set fn)	Y restriction	Y contraction*
separable-conv	Y	Y
integrally-conv	Y	Y
L-conv ($\mathbb{Z}^n$)	N	Y
L$^\natural$-conv	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	N
M$^\natural$-conv	Y	Y

contraction: $\rho_A(X) = \rho(X \cup A) - \rho(A)$

Sum and Convolution

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Theorem: (Murota 98)

$f_1,\ f_2$: L $\Longrightarrow$ $f_1 + f_2$: L

L$^\natural$ $\Longrightarrow$ L$^\natural$

$f_1, f_2, \ldots, f_k$: L / L$^\natural$ $\Longrightarrow$ $f_1 + f_2 + \cdots + f_k$: L / L$^\natural$

- $(f_1 \Box f_2)(x) = \min_y (f_1(y) + f_2(x - y))$

Theorem: (Murota 96)

$f_1,\ f_2$: M $\Longrightarrow$ $f_1 \Box f_2$: M

M$^\natural$ $\Longrightarrow$ M$^\natural$

$f_1, f_2, \ldots, f_k$: M / M$^\natural$ $\Longrightarrow$ $f_1 \Box f_2 \Box \cdots \Box f_k$: M / M$^\natural$

Rem: M $+$ M is not M, L $\Box$ L is not L

Significance of M-convolution Thm

Concave convolution:

$$(U_1 \Box U_2)(x) = \max_y (U_1(y) + U_2(x-y))$$

$U_1, U_2, \ldots, U_k$: gross-substitute (M$^\natural$-concave)

$\Rightarrow$ **aggregated utility** $U_1 \Box U_2 \Box \cdots \Box U_k$ is gross-substitute (M$^\natural$-concave)

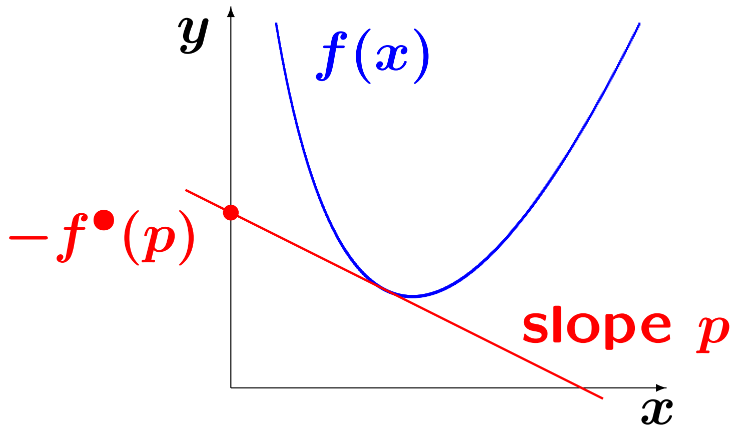
Sum/Convolution

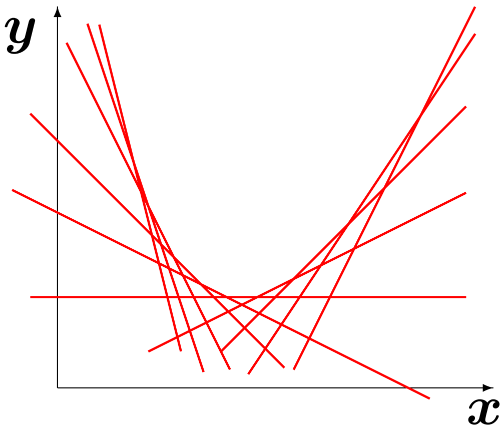
	sum $f_1 + f_2$	convolution $f_1 \,\square\, f_2$
submodular (set fn)	Y	N matroid intersec $\min_{Y \subseteq X} (\rho_1(Y) + \rho_2(X \setminus Y))$
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	N $\rightarrow$ L_2-convex
M-conv ($\mathbb{Z}^n$)	N $\rightarrow$ M_2-conv matr.intersec	Y matroid union

P4.
Conjugacy

(Legendre transform)

Conjugacy: Discrete Legendre Transform





$$f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} \{\langle p, x \rangle - f(x)\}$$

$\Rightarrow$ **If** $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$, **then** $f^{\bullet} : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

(integer-valued)

M-L Conjugacy Theorem

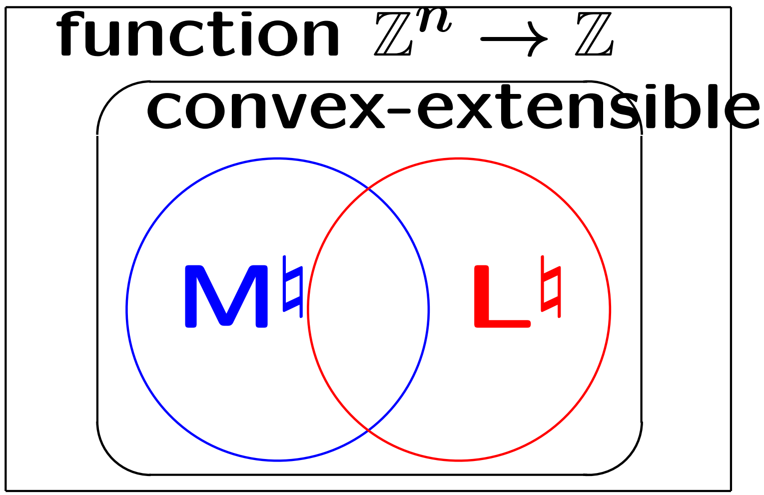
Integer-valued discrete fn $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

Legendre transform: $f^\bullet(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$

(1) M and L are conjugate (Murota 98)

(2) M$^\natural$ and L$^\natural$ are conjugate

$$f \mapsto f^\bullet = g \mapsto g^\bullet = f$$



(3) biconjugacy

$$f^{\bullet\bullet} = f$$

for $f \in$ M$^\natural \cup$ L$^\natural$

Significance of M-L Conjugacy

- Economics (game, auction)
  x: commodity bundle, p: price vector
- Network flow (min-cost flow)
  x: flow, p: tension (potential)
- Electrical network (Iri's book 69)
  x: current, p: voltage (potential)
- Discrete DC programming (Maehara-Murota 15)
- Convexity view on matroids

Conjugacy in Linear Algebra

$$[a_1, \cdots, a_5] = \begin{array}{|c|c|c|c|c|} \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

Bases $\mathcal{B} = \{ \ \{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,4,5\}, \{3,4,5\} \ \}$

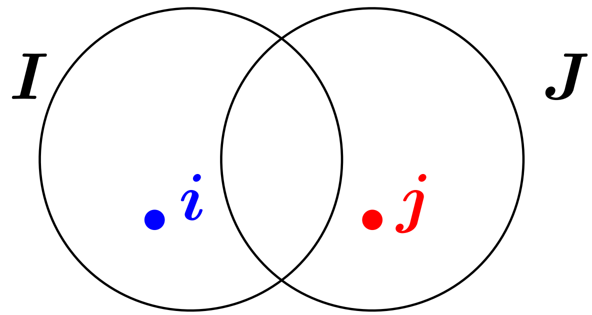
Rank fn $\rho(X) = \mathrm{rank}\,\{a_j \mid j \in X\}$

Equivalence $\mathcal{B} \iff \rho$

$$\rho(X) = \max\{|X \cap J| \mid J \in \mathcal{B}\} \qquad (X \subseteq V)$$

$$\mathcal{B} = \{J \subseteq V \mid \rho(J) = |J| = \rho(V)\}$$

Axioms of Matroid

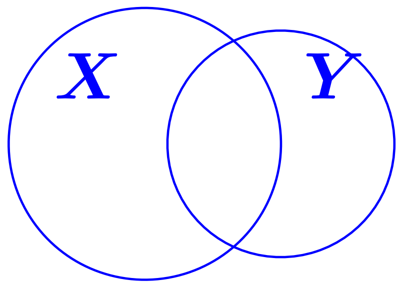


Basis axiom (set family $\mathcal{B}$):

$\forall\ I, J \in \mathcal{B},\ i \in I \setminus J, \quad \exists\ j \in J \setminus I$:

$I - i + j \in \mathcal{B},\ J + i - j \in \mathcal{B}$

Rank axiom (set function ρ):



(R1) $0 \leq \rho(X) \leq |X|$

(R2) $X \subseteq Y \implies \rho(X) \leq \rho(Y)$

(R3) $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$

Equivalence $\mathcal{B} \Leftrightarrow \rho$ (M $\leftrightarrow$ L)

$$\rho(X) = \max\{|X \cap J| \mid J \in \mathcal{B}\} \quad (X \subseteq V)$$

$$\mathcal{B} = \{J \subseteq V \mid \rho(J) = |J| = \rho(V)\}$$

Conjugacy in Matroid

Bases $\mathcal{B} = \{\ \{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,4,5\}, \{3,4,5\}\ \}$

Rank fn $\rho(X) = \mathrm{rank}\,\{a_j \mid j \in X\}$

Equivalence $\mathcal{B} \iff \rho$

$$\rho(X) = \max\{|X \cap J| \mid J \in \mathcal{B}\} \qquad (X \subseteq V)$$

$$\mathcal{B} = \{J \subseteq V \mid \rho(J) = |J| = \rho(V)\}$$

Bases (M$^\natural$-convex)	$\Leftarrow$ conjugate $\Rightarrow$	Rank fn (L$^\natural$-convex)

Dual Character of Matroid Rank

$$\rho(X) = \max\{|I| \mid I : \text{independent},\ I \subseteq X\}$$

is M$^\natural$-concave and L$^\natural$-convex

Edmonds' matroid union formula:

$$\max_{X}\{\rho_1(X)+\rho_2(V\setminus X)\} = \min_{Y}\{\rho_1(Y)+\rho_2(Y)+|V\setminus Y|\}$$

submod maximization (M$^\natural$-concave $\square$ M$^\natural$-concave)

submod minimization (L$^\natural$-convex $+$ L$^\natural$-convex)

Self-Conjugacy: $\rho(X) = |X| - \rho^{\bullet}(\chi_X)$

Conjugacy in Polymatroids

Polyhedron S		**Submodular fn** ρ
$S = \{x \mid x(A) \le \rho(A) \ \ \forall A\}$	$\leftarrow$	
	$\rightarrow$	$\rho(A) = \max_{x \in S} x(A)$

Conjugacy in Polymatroids

Polyhedron S | **Submodular fn ρ**

$$S = \{x \mid x(A) \leq \rho(A) \ \forall A\} \ \leftarrow$$

$$\rightarrow \ \rho(A) = \max_{x \in S} x(A)$$

Indicator fn of S | **Lovász ext. of ρ**

$f(x) \in \{0, +\infty\}$ | $g(p)$

$$\rightarrow: \ g(p) = \max_{x \in S} \langle p, x \rangle = \max_{x} [\langle p, x \rangle - f(x)] = f^{\bullet}(p)$$

$$\leftarrow: \ f(x) = \max_{p} [\langle p, x \rangle - g(p)] = g^{\bullet}(x)$$

Legendre transform

M$^{\natural}$-convex | L$^{\natural}$-convex

History of Discrete Conjugacy

Matroid bases	$\longleftrightarrow$	Matroid rank fn
Whitney 35		Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid	$\longleftrightarrow$	**Submodular fn**
Edmonds 70		Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid	|	Lovász extension
Dress–Wenzel 90	|	Lovász 83
$\Downarrow$	|	$\Downarrow$
	|	Submod. integ. conv. fn
	|	Favati-Tardella 90
		$\Downarrow$
M-convex fn	$\longleftrightarrow$	**L-convex fn**
Murota 96		Murota 98
$\Updownarrow$		$\Updownarrow$
M$^\natural$-convex fn	$\longleftrightarrow$	L$^\natural$-convex fn
Murota -Shioura 99		Fujishige–Murota 00

Integral Subgradient & Biconjugacy

Subdifferential: $\partial f(x)$

$$= \{p \in \mathbb{R}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \quad (\forall y)\}$$

↑ subgradient

Integral subdifferential: $\partial_{\mathbb{Z}} f(x) = \partial f(x) \cap \mathbb{Z}^n$

$$= \{p \in \mathbb{Z}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \quad (\forall y)\}$$

↑ integral subgradient

Prop: $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ (Murota 98)

If $\partial_{\mathbb{Z}} f(x) \neq \emptyset$ **for all** $x \in \mathrm{dom}\, f$, **then** $f^{\bullet\bullet} = f$

$$f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$$

Integ. Subgradient & Biconjugacy: Example

(Discrete life is not easy)

$f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$ $\qquad$ $\partial_{\mathbb{Z}} f(x) \neq \emptyset$? $\qquad$ $f^{\bullet\bullet} = f$?

Example: $D = \{(0,0,0), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\}$

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)/2, & x \in D, \\ +\infty, & \text{o.w.} \end{cases}$$

D is "convex": $\mathrm{conv}(D) \cap \mathbb{Z}^n = D$

$\partial f_{\mathbb{R}}(0) = \{(1/2, 1/2, 1/2)\}$ $\qquad$ $\partial_{\mathbb{Z}} f(0) = \emptyset$

$$f^{\bullet\bullet}(0) = -\inf_{p \in \mathbb{Z}^3} \max\{0, |p_1+p_2-1|, |p_2+p_3-1|, |p_3+p_1-1|\}$$

$f^{\bullet\bullet}(0) = -1 \neq 0 = f(0)$

Integral Subgradient & Biconjugacy

Thm: (Murota 98)

f: integer-valued L$^\natural$- / M$^\natural$- / L$_2^\natural$- / M$_2^\natural$-convex

- Subdifferential $\partial f(x)$ is an integral polyhedron
- Hence integral subgradient p exists
- Hence $f^{\bullet\bullet} = f$

Thm: (Murota - Tamura 18)

f: integer-valued integrally convex

- Subdifferential $\partial f(x)$ is NOT an integral polyhedron
- But integral subgradient p exists
- Hence $f^{\bullet\bullet} = f$

Conjugacy and Biconjugacy

Legendre trans: $f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$

$f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$	$f^{\bullet} : \mathbb{Z}^n \to \overline{\mathbb{Z}}$	$f^{\bullet\bullet} = f$
submodular (set fn)	submodular polyhedron $\{x \in \mathbb{Z}^n \mid x(A) \leq \rho(A)\}$	Y
separable-convex $f(x) = \sum \varphi_i(x_i)$	separable-convex $\varphi_1^{\bullet}(p_1) + \cdots + \varphi_n^{\bullet}(p_n)$	Y
integrally-convex	Not integrally-convex (characterization: open)	Y
L-convex $(\mathbb{Z}^n)$	M-convex	Y
L$^\natural$-convex	M$^\natural$-convex	Y
M-convex $(\mathbb{Z}^n)$	L-convex	Y
M$^\natural$-convex	L$^\natural$-convex	Y

P5.
Duality

(separation theorem)
(Fenchel duality)

Conjugacy/Duality in Matroids

Conjugacy

Exchange axiom $\Leftrightarrow$ Submodularity of rank function

Duality

Matroid intersection theorem (Edmonds)

Discrete separation (Frank)

Fenchel-type duality (Fujishige)

Matroid Intersection Problem

Given two matroids ......

- Find a common indep. set X with max $|X|$
- Find a common base B (if any)

Given two matroids and weight w ......

- Find a common indep. set X with max $w(X)$
- Find a common base B with max $w(B)$

LP formulation with integrality, combinatorial duality, efficient algorithms, many applications

Edmonds' Intersection Theorem

Submodular polyhedron $(\rho(\emptyset) = 0,\ \rho(V) < +\infty)$

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^n \mid x(X) \leq \rho(X)\ (\forall X \subseteq V)\} \qquad (|V| = n)$$

Theorem: (Edmonds 70)

(1) For $\rho_1, \rho_2 : 2^V \to \overline{\mathbb{R}}$: submodular,

$$\max_x \{x(V) \mid x \in \mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2)\} = \min_X \{\rho_1(X) + \rho_2(V \setminus X)\}$$

(2) If ρ_1 and ρ_2 are integer-valued, then

$$\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) = \overline{\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) \cap \mathbb{Z}^n}$$

and there exists $x^* \in \mathbb{Z}^n$ that attains the maximum

Frank's Discrete Separation

(Frank 82)

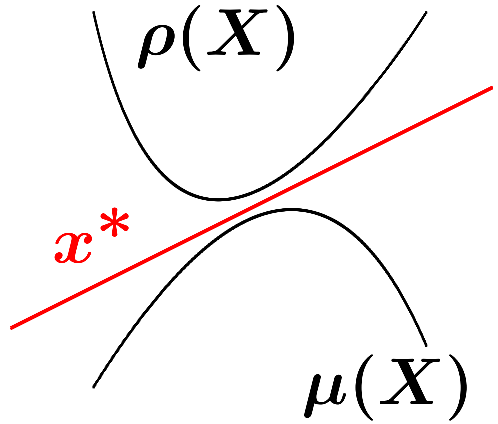
$\rho : 2^{V} \to \mathbb{R}$: submodular $\qquad (\rho(\emptyset) = 0)$

$\mu : 2^{V} \to \mathbb{R}$: supermodular $\qquad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \ \exists\ x^{*} \in \mathbb{R}^{V}$:

$$\rho(X) \geq x^{*}(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

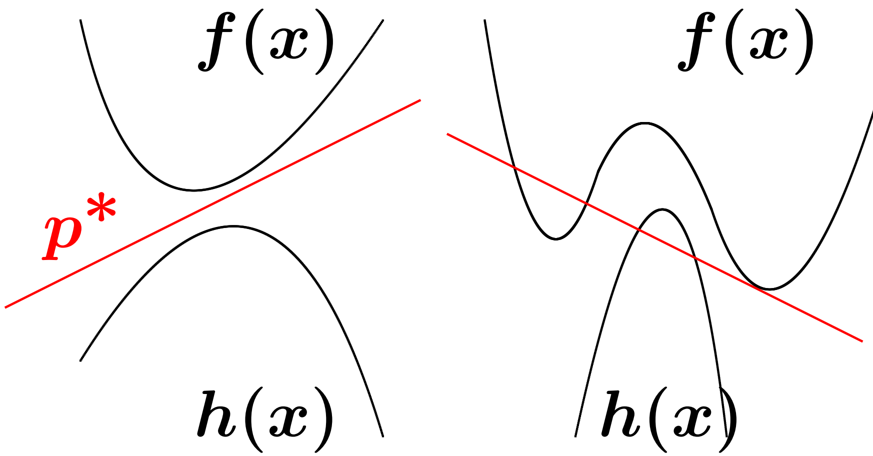
- ρ, μ: **integer-valued** $\Rightarrow x^{*} \in \mathbb{Z}^{V}$



Discrete Separation Theorem

$f : \mathbb{Z}^n \to \mathbb{R}$ **"convex"**

$h : \mathbb{Z}^n \to \mathbb{R}$ **"concave"**



- $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}^n) \Rightarrow \exists\ \alpha^* \in \mathbb{R}, \quad \exists\ p^* \in \mathbb{R}^n$:

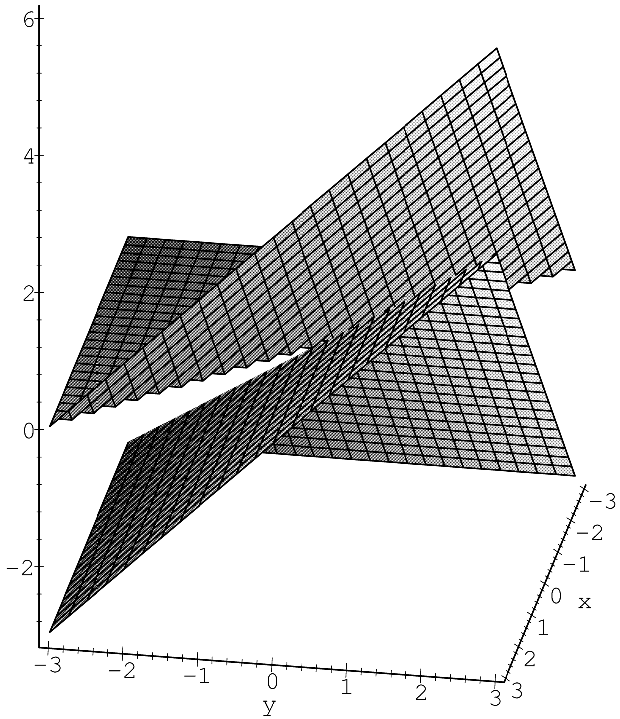
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbb{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \quad p^* \in \mathbb{Z}^n$

Difficulty of Discrete Separation (1)

$f(x,y) = \max(0, x+y)$ **convex**

$h(x,y) = \min(x,y)$ **concave**



nonintegral separation

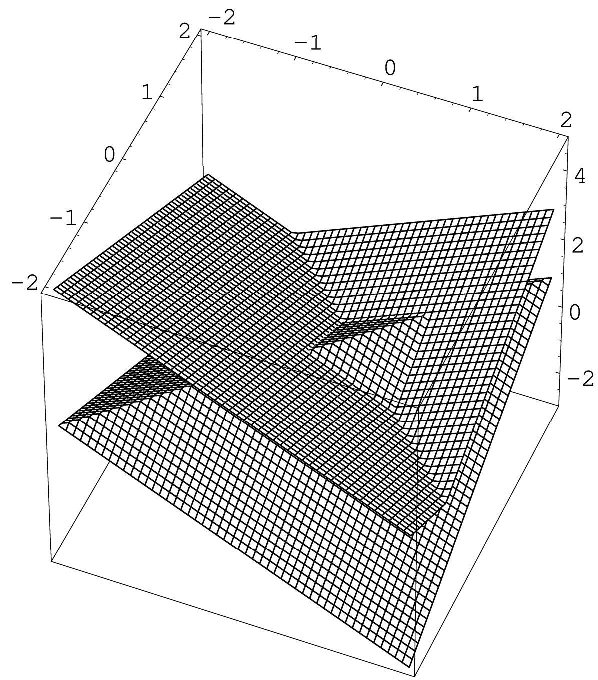
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x, y) = |x + y - 1|$ **convex**

$h(x, y) = 1 - |x - y|$ **concave**



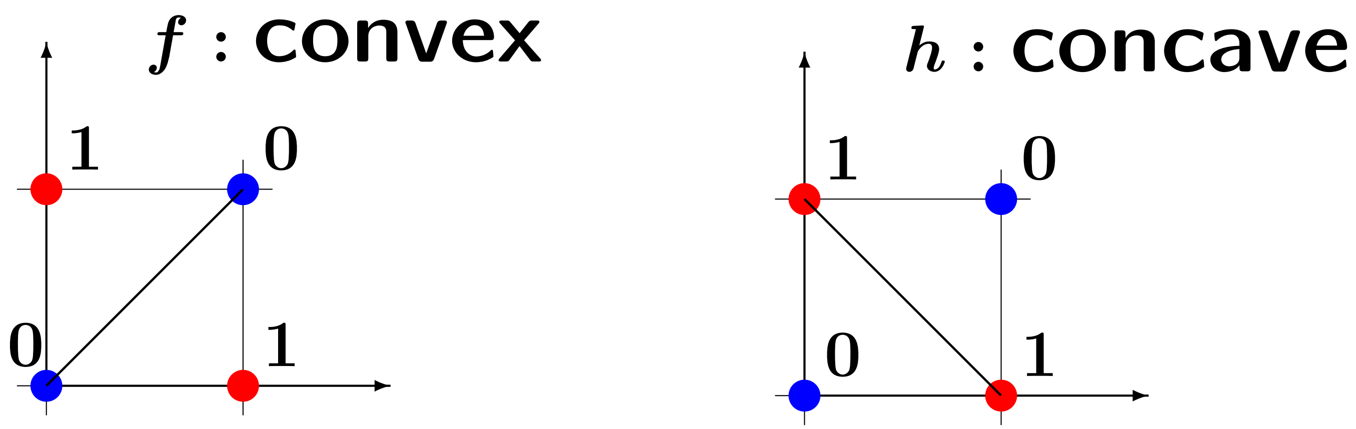
- $f(x, y) \geq h(x, y)$ $(\forall (x, y) \in \mathbb{Z}^2)$ **true**
- **No** $\alpha^* \in \mathbb{R}$, $p^* \in \mathbb{R}^2$: $\quad f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$

$\because$ $f = 0 < h = 1$ **at** $(x, y) = (1/2, 1/2)$

Difficulty of Discrete Separation (3)

Set function $\Longleftrightarrow$ Function on $\{0,1\}^n$

Every set function $\{0,1\}^n \to \mathbb{R}$ can be extended to convex/concave function



- $f(x,y) = h(x,y)$ $(\forall(x,y) \in \{0,1\}^2)$
- **No separation possible**

Discrete Separation Theorems

(Murota 96/98)

M-separation Thm (for M$^\natural$-convex)

$\Rightarrow$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn = M$^\natural$-convex fn)

L-separation Thm (for L$^\natural$-convex)

$\Rightarrow$ Discrete separ. for submod. set fn (Frank 82)

(submod. set fn = L$^\natural$-convex fn on 0–1 vectors)

Min-Max Duality

f: M$^\natural$-convex, h: M$^\natural$-concave $(\mathbb{Z}^n \to \overline{\mathbb{Z}})$

Legendre–Fenchel transform

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\}$$

$$h^\circ(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbb{Z}^n\}$$

Fenchel-type duality thm (Murota 96, 98)

$$\inf_{x\in\mathbb{Z}^n}\{f(x) - h(x)\} = \sup_{p\in\mathbb{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

self-conjugate ($f^\bullet$: L$^\natural$-convex, h°: L$^\natural$-concave)

$\Longrightarrow$ Edmonds' intersection thm
Fujishige's Fenchel duality thm

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation	
$f(x) \geq \boxed{\text{Lin}} \geq h(x)$	Fenchel duality (Fujishige 84)
	matroid intersect. (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality	$\Rightarrow$ discrete separ. for submod (Frank 82)
$\inf\{f - h\}$	
$= \sup\{h^\circ - f^\bullet\}$	$\Rightarrow$ valuated matroid intersect. (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation	weighted matroid intersect.
$f^\bullet(p) \geq \boxed{\text{Lin}} \geq h^\circ(p)$	(Edmonds 79, Iri-Tomizawa 76, Frank 81)

Separation and Min-Max Theorems

	separation	min-max
submodular (set fn)	Y (Frank)	Y (Edmonds, Fujishige)
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	Y

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	Y	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	N	N	N	N	Y	Y*	Y	Y	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	—	Y	N	Y*	Y	—	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	N	N	N	N	Y	Y*	Y	Y	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	—	—	Y	Y*	Y*	?	?	?

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca ling	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	Y	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	N	N	N	N	Y	Y*	Y	Y	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
3. Conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for

- separable-convex functions
- L$\natural$-convex functions
- M$\natural$-convex functions

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