L-CONVEX FUNCTIONS AND M-CONVEX FUNCTIONS

In the field of nonlinear programming (in continuous variables) convex analysis [22, 23] plays a pivotal role both in theory and in practice. An analogous theory for discrete optimization (nonlinear integer programming), called "discrete convex analysis" [18, 17], is developed for L-convex and M-convex functions by adapting the ideas in convex analysis and generalizing the results in matroid theory. The L- and M-convex functions are introduced in [18] and [13, 14], respectively.

Definitions of L- and M-convexity. Let V be a nonempty finite set and $\mathbf{Z}$ be the set of integers. For any function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ define $\mathrm{dom}\, g = \{p \in \mathbf{Z}^V \mid g(p) < +\infty\}$, called the effective domain of g.

A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ is called *L-convex* if

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (p, q \in \mathbf{Z}^V),$$

$$\exists r \in \mathbf{Z} : \; g(p + \mathbf{1}) = g(p) + r \quad (p \in \mathbf{Z}^V),$$

where $p \vee q = (\max(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V$, $p \wedge q = (\min(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V$, and $\mathbf{1}$ is the vector in $\mathbf{Z}^V$ with all components being equal to 1.

A set $D \subseteq \mathbf{Z}^V$ is said to be an *L-convex set* if its indicator function δ_D (defined by: $\delta_D(p) = 0$ if $p \in D$, and $= +\infty$ otherwise) is an L-convex function, i.e., if (i) $D \neq \emptyset$, (ii) $p, q \in D \Rightarrow p \vee q, p \wedge q \in D$, and (iii) $p \in D \Rightarrow p \pm \mathbf{1} \in D$.

A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M-convex* if it satisfies

(M-EXC) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$$

where, for any $u \in V$, χ_u is the characteristic vector of u (defined by: $\chi_u(v) = 1$ if $v = u$, and $= 0$ otherwise), and

$$\begin{aligned} \mathrm{supp}^+(z) &= \{v \in V \mid z(v) > 0\} \quad (z \in \mathbf{Z}^V), \\ \mathrm{supp}^-(z) &= \{v \in V \mid z(v) < 0\} \quad (z \in \mathbf{Z}^V). \end{aligned}$$

A set $B \subseteq \mathbf{Z}^V$ is said to be an *M-convex set* if its indicator function is an M-convex function, i.e., if B satisfies

(B-EXC) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

This means that an M-convex set is the same as the set of integer points of the base polyhedron of an integral submodular system (see [8] for submodular systems).

L-convexity and M-convexity are conjugate to each other under the *integral Fenchel-Legendre transformation* $f \mapsto f^\bullet$ defined by

$$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V),$$

where $\langle p, x \rangle = \sum_{v \in V} p(v) x(v)$. That is, for L-convex function g and M-convex function f, it holds [18] that $g^\bullet$ is M-convex, $f^\bullet$ is L-convex, $g^{\bullet\bullet} = g$, and $f^{\bullet\bullet} = f$.

Example 1: Minimum cost flow problem. L-convexity and M-convexity are inherent in the integer minimum-cost flow problem, as pointed out in [14, 18]. Let $G = (V, A)$ be a graph with vertex set V and arc set A, and let $T \subseteq V$ be given. For $\xi : A \to \mathbf{Z}$ its boundary $\partial \xi : V \to \mathbf{Z}$ is defined by

$$\begin{aligned} \partial \xi(v) = & \sum \{\xi(a) \mid a \in \delta^+ v\} \\ & - \sum \{\xi(a) \mid a \in \delta^- v\} \quad (v \in V), \end{aligned}$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of out-going and in-coming arcs incident to v, respectively. For $\tilde{p} : V \to \mathbf{Z}$ its coboundary $\delta \tilde{p} : A \to \mathbf{Z}$ is defined by

$$\delta \tilde{p}(a) = \tilde{p}(\partial^+ a) - \tilde{p}(\partial^- a) \quad (a \in A),$$

where $\partial^+ a$ and $\partial^- a$ mean the initial and terminal vertices of a, respectively. Denote the class

L-convex→ L-convex function
L-convex set
M-convex→ M-convex function
M-convex set
integral Fenchel-Legendre transformation

of one-dimensional discrete convex functions by

$$\mathcal{C}_1 = \{\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\} \mid \mathrm{dom}\,\varphi \neq \emptyset, \\ \varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)\ (t \in \mathbf{Z})\}.$$

For $\varphi_a \in \mathcal{C}_1$ $(a \in A)$, representing the arc-cost in terms of flow, the total cost function $f : \mathbf{Z}^T \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x) = \inf_{\xi}\{\sum_{a\in A} \varphi_a(\xi(a)) \mid \\ \partial\xi(v) = -x(v)\ (v \in T), \\ \partial\xi(v) = 0\ (v \in V \setminus T)\} \quad (x \in \mathbf{Z}^T)$$

is M-convex, provided that $f > -\infty$ (i.e., f does not take the value of $-\infty$). For $\psi_a \in \mathcal{C}_1$ $(a \in A)$, representing the arc-cost in terms of tension, the total cost function $g : \mathbf{Z}^T \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$g(p) = \inf_{\tilde{p}}\{\sum_{a\in A} \psi_a(\eta(a)) \mid \eta = -\delta\tilde{p}, \\ \tilde{p}(v) = p(v)\ (v \in T)\} \quad (p \in \mathbf{Z}^T)$$

is L-convex, provided that $g > -\infty$. The two cost functions $f(x)$ and $g(p)$ are conjugate to each other in the sense that, if $\psi_a = {\varphi_a}^\bullet$ $(a \in A)$, then $g = f^\bullet$.

Example 2: Polynomial matrix. Let $A(s)$ be an $m \times n$ matrix of rank m with each entry being a polynomial in a variable s, and let $\mathcal{B} \subseteq 2^V$ be the family of bases of $A(s)$ with respect to linear independence of the column vectors; namely, $J \subseteq V$ belongs to $\mathcal{B}$ if and only if $|J| = m$ and the column vectors with indices in J are linearly independent. Then $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} -\deg_s \det A[J] & (x = \chi_J, J \in \mathcal{B}) \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex, where $\chi_J \in \{0,1\}^V$ is the characteristic vector of J (defined by: $\chi_J(v) = 1$ if $v \in J$, and $= 0$ otherwise), $A[J]$ denotes the $m \times m$ submatrix with column indices in $J \in \mathcal{B}$, and $\deg_s(\cdot)$ means the degree as a polynomial in s. The Grassmann-Plücker identity implies the exchange property of f. This example was the motivation of valuated matroids in [2, 3], which in turn can be identified with the negative of M-convex functions f with $\mathrm{dom}\, f \subseteq \{0,1\}^V$.

For $p = (p(v) \mid v \in V) \in \mathbf{Z}^V$ denote by $D(p)$ the diagonal matrix of order $n = |V|$ with diagonal elements $s^{p(v)}$ $(v \in V)$. Then the function $g : \mathbf{Z}^V \to \mathbf{Z}$ defined by

$$g(p) = \max\{\deg_s \det(A \cdot D(p))[J] \mid J \in \mathcal{B}\}$$

is L-convex [17], where $(A \cdot D(p))[J]$ means the $m \times m$ submatrix of $A \cdot D(p)$ with column indices in J. We have $g = f^\bullet$.

L-convex sets. An L-convex set $D \subseteq \mathbf{Z}^V$ has "no holes" in the sense that $D = \overline{D} \cap \mathbf{Z}^V$, where $\overline{D}$ denotes the convex hull of D in $\mathbf{R}^V$. Hence it is natural to consider the polyhedral description of $\overline{D}$, "L-convex polyhedron" (see [18, 17]). For any function $\gamma : V \times V \to \mathbf{Z} \cup \{+\infty\}$ with $\gamma(v,v) = 0$ $(v \in V)$, define

$$\mathbf{D}(\gamma) = \{p \in \mathbf{R}^V \mid \\ p(v) - p(u) \leq \gamma(u,v)\ (\forall u, v \in V)\}.$$

If $\mathbf{D}(\gamma) \neq \emptyset$, $\mathbf{D}(\gamma)$ is an integral polyhedron and $D = \mathbf{D}(\gamma) \cap \mathbf{Z}^V$ is an L-convex set. If γ satisfies triangle inequality:

$$\gamma(u,v) + \gamma(v,w) \geq \gamma(u,w) \quad (u,v,w \in V),$$

then $\mathbf{D}(\gamma) \neq \emptyset$ and

$$\gamma(u,v) = \sup\{p(v) - p(u) \mid p \in \mathbf{D}(\gamma)\} \\ (u, v \in V).$$

Conversely, for any nonempty $D \subseteq \mathbf{Z}^V$,

$$\gamma(u,v) = \sup\{p(v) - p(u) \mid p \in D\} \quad (u,v \in V)$$

satisfies triangle inequality as well as $\gamma(v,v) = 0$ $(v \in V)$, and if D is L-convex, then $\overline{D} = \mathbf{D}(\gamma)$. Thus there is a one-to-one correspondence between L-convex set D and function γ satisfying triangle inequality. In particular, $D \subseteq \mathbf{Z}^V$ is L-convex if and only if $D = \mathbf{D}(\gamma) \cap \mathbf{Z}^V$ for some γ satisfying triangle inequality. For L-convex sets $D_1, D_2 \subseteq \mathbf{Z}^V$, it holds that $D_1 + D_2 = \overline{D_1 + D_2} \cap \mathbf{Z}^V$ and $\overline{D_1} \cap \overline{D_2} = \overline{D_1 \cap D_2}$.

It is also true that a function γ satisfying triangle inequality corresponds one-to-one to a positively homogeneous M-convex function f (i.e, $f(\lambda x) = \lambda f(x)$ for $x \in \mathbf{Z}^V$ and $0 \leq \lambda \in \mathbf{Z}$). The correspondence $f \mapsto \gamma$ is given by

$$\gamma(u,v) = f(\chi_v - \chi_u) \quad (u,v \in V),$$

whereas $\gamma \mapsto f$ by

$$f(x) = \inf_{\lambda}\{ \sum_{u,v\in V} \lambda_{uv}\gamma(u,v) \mid \sum_{u,v\in V} \lambda_{uv}(\chi_v - \chi_u) = x, \quad 0 \le \lambda_{uv} \in \mathbf{Z}\ (u,v \in V)\} \qquad (x \in \mathbf{Z}^V).$$

The correspondence between L-convex sets and positively homogeneous M-convex functions via functions with triangle inequality is a special case of the conjugacy relationship between L- and M-convex functions.

M-convex sets. An M-convex set $B \subseteq \mathbf{Z}^V$ has "no holes" in the sense that $B = \overline{B} \cap \mathbf{Z}^V$. Hence it is natural to consider the polyhedral description of $\overline{B}$, "M-convex polyhedron." A set function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ is said to be *submodular* if

$$\rho(X) + \rho(Y) \ge \rho(X \cup Y) + \rho(X \cap Y) \qquad (X, Y \subseteq V),$$

where the inequality is satisfied if $\rho(X)$ or $\rho(Y)$ is equal to $+\infty$. It is assumed throughout that $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$ for any set function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$. For a set function ρ, define

$$\mathbf{B}(\rho) = \{x \in \mathbf{R}^V \mid x(X) \le \rho(X)\ (\forall X \subset V), x(V) = \rho(V)\},$$

where $x(X) = \sum\{x(v) \mid v \in X\}$. If ρ is submodular, $\mathbf{B}(\rho)$ is a nonempty integral polyhedron, $B = \mathbf{B}(\rho) \cap \mathbf{Z}^V$ is an M-convex set, and

$$\rho(X) = \sup\{x(X) \mid x \in \mathbf{B}(\rho)\} \quad (X \subseteq V).$$

Conversely, for any nonempty $B \subseteq \mathbf{Z}^V$, define a set function ρ by

$$\rho(X) = \sup\{x(X) \mid x \in B\} \quad (X \subseteq V).$$

If B is M-convex, then ρ is submodular and $\overline{B} = \mathbf{B}(\rho)$. Thus there is a one-to-one correspondence between M-convex set B and submodular set function ρ. In particular, $B \subseteq \mathbf{Z}^V$ is M-convex if and only if $B = \mathbf{B}(\rho) \cap \mathbf{Z}^V$ for some submodular ρ. The correspondence $B \leftrightarrow \rho$ is a restatement of a well-known fact [4, 8].

submodular→ submodular function
Lovász extension

For M-convex sets $B_1, B_2 \subseteq \mathbf{Z}^V$, it holds that $B_1 + B_2 = \overline{B_1 + B_2} \cap \mathbf{Z}^V$ and $\overline{B_1} \cap \overline{B_2} = \overline{B_1 \cap B_2}$.

It is also true that a submodular set function ρ corresponds one-to-one to a positively homogeneous L-convex function g. The correspondence $g \mapsto \rho$ is given by the restriction

$$\rho(X) = g(\chi_X) \quad (X \subseteq V)$$

(χ_X is the characteristic vector of X), whereas $\rho \mapsto g$ by the Lovász extension (explained below). The correspondence between M-convex sets and positively homogeneous L-convex functions via submodular set functions is a special case of the conjugacy relationship between M- and L-convex functions.

For a set function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$, the *Lovász extension* [12] of ρ is a function $\hat{\rho} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\hat{\rho}(p) = \sum_{j=1}^{n}(p_j - p_{j+1})\rho(V_j) \quad (p \in \mathbf{R}^V),$$

where, for each $p \in \mathbf{R}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that $p(v_1) \ge p(v_2) \ge \cdots \ge p(v_n)$; $p_j = p(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} = 0$. The right-hand side of the above expression is equal to $+\infty$ if and only if $p_j - p_{j+1} > 0$ and $\rho(V_j) = +\infty$ for some j with $1 \le j \le n-1$. The Lovász extension $\hat{\rho}$ is indeed an extension of ρ, since $\hat{\rho}(\chi_X) = \rho(X)$ for $X \subseteq V$.

The relationship between submodularity and convexity is revealed by the statement [12] that a set function ρ is submodular if and only if its Lovász extension $\hat{\rho}$ is convex.

The restriction to $\mathbf{Z}^V$ of the Lovász extension of a submodular set function is a positively homogeneous L-convex function, and any positively homogeneous L-convex function can be obtained in this way [18].

Properties of L-convex functions. For any $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ and $x \in \mathbf{R}^V$, define $g[-x] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$g[-x](p) = g(p) - \langle p, x \rangle \quad (p \in \mathbf{Z}^V).$$

The set of the minimizers of $g[-x]$ is denoted as $\mathrm{argmin}\,(g[-x])$.

Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be L-convex. Then $\mathrm{dom}\, g$ is an L-convex set. For each $p \in \mathrm{dom}\, g$,

$$\rho_p(X) = g(p + \chi_X) - g(p) \quad (X \subseteq V)$$

is a submodular set function with $\rho_p(\emptyset) = 0$ and $\rho_p(V) < +\infty$.

An L-convex function g can be extended to a convex function $\overline{g} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ through the Lovász extension of the submodular set functions ρ_p for $p \in \mathrm{dom}\, g$. Namely, for $p \in \mathrm{dom}\, g$ and $q \in [0, 1]^V$, it holds [18] that

$$\overline{g}(p + q) = g(p) + \sum_{j=1}^{n} (q_j - q_{j+1})(g(p + \chi_{V_j}) - g(p)),$$

where, for each q, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that $q(v_1) \geq q(v_2) \geq \cdots \geq q(v_n)$; $q_j = q(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $q_{n+1} = 0$. The expression of $\overline{g}$ shows that an L-convex function is an integrally convex function in the sense of [5].

An L-convex function g enjoys *discrete midpoint convexity*:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

for $p, q \in \mathbf{Z}^V$, where $\lceil p \rceil$ (or $\lfloor p \rfloor$) for any $p \in \mathbf{R}^V$ denotes the vector obtained by rounding up (or down) the components of p to the nearest integers.

The minimum of an L-convex function g is characterized by the local minimality in the sense that, for $p \in \mathrm{dom}\, g$, $g(p) \leq g(q)$ for all $q \in \mathbf{Z}^V$ if and only if $g(p+\mathbf{1}) = g(p) \leq g(p+\chi_X)$ for all $X \subseteq V$.

The minimizers of an L-convex function, if nonempty, forms an L-convex set. For any $x \in \mathbf{R}^V$, $\mathrm{argmin}\,(g[-x])$, if nonempty, is an L-convex set. Conversely, this property characterizes L-convex functions, under an auxiliary assumption that the function can be extented to a convex function over $\mathbf{R}^V$ (cf. [20]).

A number of operations can be defined for L-convex functions [18, 17]. For $x \in \mathbf{Z}^V$, $g[-x]$ is an L-convex function. For $a \in \mathbf{Z}^V$ and $\beta \in \mathbf{Z}$, $g(a + \beta p)$ is L-convex in p. For $U \subseteq V$, the projection of g to U:

$$g^U(p') = \inf\{g(p', p'') \mid p'' \in \mathbf{Z}^{V \setminus U}\} \quad (p' \in \mathbf{Z}^U)$$

is L-convex in p', provided that $g^U > -\infty$. For $\psi_v \in \mathcal{C}_1$ $(v \in V)$,

$$\tilde{g}(p) = \inf_{q \in \mathbf{Z}^V} \left[g(q) + \sum_{v \in V} \psi_v(p(v) - q(v)) \right]$$

is L-convex in $p \in \mathbf{Z}^V$, provided that $\tilde{g} > -\infty$. The sum of two (or more) L-convex functions is L-convex, provided that its effective domain is nonempty.

discrete midpoint convexity

Properties of M-convex functions. Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be M-convex. Then $\mathrm{dom}\, f$ is an M-convex set. For each $x \in \mathrm{dom}\, f$,

$$\gamma_x(u, v) = f(x - \chi_u + \chi_v) - f(x) \quad (u, v \in V)$$

satisfies [17] triangle inequality.

An M-convex function f can be extended to a convex function $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, and the value of $\overline{f}(x)$ for $x \in \mathbf{R}^V$ is determined by $\{f(y) \mid y \in \mathbf{Z}^V, \lfloor x \rfloor \leq y \leq \lceil x \rceil\}$. That is, an M-convex function is an integrally convex function in the sense of [5].

The minimum of an M-convex function f is characterized by the local minimality in the sense that for $x \in \mathrm{dom}\, f$, $f(x) \leq f(y)$ for all $y \in \mathbf{Z}^V$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ for all $u, v \in V$ [13, 14, 18].

The minimizers of an M-convex function, if nonempty, forms an M-convex set. Moreover, for any $p \in \mathbf{R}^V$, $\mathrm{argmin}\,(f[-p])$, if nonempty, is an M-convex set. Conversely, this property characterizes M-convex functions, under an auxiliary assumption that the effective domain is bounded or the function can be extented to a convex function over $\mathbf{R}^V$ (see [14, 18]).

The level set of an M-convex function is not necessarily an M-convex set, but enjoys a weaker exchange property. Namely, for any $p \in \mathbf{R}^V$ and $\alpha \in \mathbf{R}$, $S = \{x \in \mathbf{Z}^V \mid f[-p](x) \leq \alpha\}$ (the level set of $f[-p]$) satisfies: For $x, y \in S$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that either $x - \chi_u + \chi_v \in S$ or $y + \chi_u - \chi_v \in$

S. Conversely, this property characterizes M-convex functions [26].

A number of operations can be defined for M-convex functions [18, 17]. For $p \in \mathbf{Z}^V$, $f[-p]$ is an M-convex function. For $a \in \mathbf{Z}^V$, $f(a-x)$ and $f(a+x)$ are M-convex in x. For $U \subseteq V$, the restriction of f to U:

$$f_U(x') = f(x', \mathbf{0}_{V\setminus U}) \quad (x' \in \mathbf{Z}^U)$$

(where $\mathbf{0}_{V\setminus U}$ is the zero vector in $\mathbf{Z}^{V\setminus U}$) is M-convex in x', provided that $\mathrm{dom}\, f_U \neq \emptyset$. For $\varphi_v \in \mathcal{C}_1$ $(v \in V)$,

$$\tilde{f}(x) = f(x) + \sum_{v\in V} \varphi_v(x(v)) \quad (x \in \mathbf{Z}^V)$$

is M-convex, provided that $\mathrm{dom}\, \tilde{f} \neq \emptyset$. In particular, a separable convex function $\tilde{f}(x) = \sum_{v\in V} \varphi_v(x(v))$ with $\mathrm{dom}\, \tilde{f}$ being an M-convex set is an M-convex function. For two M-convex functions f_1 and f_2, the integral convolution

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x = x_1 + x_2; x_1, x_2 \in \mathbf{Z}^V\} \quad (x \in \mathbf{Z}^V)$$

is either M-convex or else $(f_1 \Box f_2)(x) = \pm\infty$ for all $x \in \mathbf{Z}^V$.

Sum of two M-convex functions is not necessarily M-convex; such function with nonempty effective domain is called *M_2-convex*. Convolution of two L-convex functions is not necessarily L-convex; such function with nonempty effective domain is called *L_2-convex*. M_2- and L_2-convex functions are in one-to-one correspondence through the integral Fenchel-Legendre transformation.

$\mathrm{L}^\natural$- and $\mathrm{M}^\natural$-convexity. $\mathrm{L}^\natural$- and $\mathrm{M}^\natural$-convexity are variants of, and essentially equivalent to, L- and M-convexity, respectively. $\mathrm{L}^\natural$- and $\mathrm{M}^\natural$-convex functions are introduced in [9] and [21], respectively.

Let v_0 be a new element not in V and define $\tilde{V} = \{v_0\} \cup V$. A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ is called *$L^\natural$-convex* if it is expressed in terms of an L-convex function $\tilde{g} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ as $g(p) = \tilde{g}(0, p)$. Namely, an $\mathrm{L}^\natural$-convex function is a function obtained as the restriction of an L-convex function. Conversely, an $\mathrm{L}^\natural$-convex function determines the corresponding L-convex function up to the constant r in the definition of L-convex function.

An $\mathrm{L}^\natural$-convex function is essentially the same as a submodular integrally convex function of [5], and hence is characterized by discrete midpoint convexity [9]. An L-convex function, enjoying discrete midpoint convexity, is an $\mathrm{L}^\natural$-convex function.

Quadratic function

$$g(p) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} p_i p_j \quad (p \in \mathbf{Z}^n)$$

with $a_{ij} = a_{ji} \in \mathbf{Z}$ is $\mathrm{L}^\natural$-convex if and only if $a_{ij} \leq 0$ $(i \neq j)$ and $\sum_{j=1}^{n} a_{ij} \geq 0$ $(i = 1, \cdots, n)$. For $\{\psi_i \in \mathcal{C}_1 \mid i = 1, \cdots, n\}$, a separable convex function

$$g(p) = \sum_{i=1}^{n} \psi_i(p_i) \quad (p \in \mathbf{Z}^n)$$

is $\mathrm{L}^\natural$-convex.

The properties of L-convex functions mentioned above are carried over, mutatis mutandis, to $\mathrm{L}^\natural$-convex functions. In addition, the restriction of an $\mathrm{L}^\natural$-convex function g to $U \subseteq V$, denoted g_U, is $\mathrm{L}^\natural$-convex.

A subset of $\mathbf{Z}^V$ is called an *$L^\natural$-convex set* if its indicator function is an $\mathrm{L}^\natural$-convex function. A set $E \subseteq \mathbf{Z}^V$ is an $\mathrm{L}^\natural$-convex set if and only if

$$p, q \in E \quad \Longrightarrow \quad \left\lceil \frac{p+q}{2} \right\rceil, \left\lfloor \frac{p+q}{2} \right\rfloor \in E.$$

A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *$M^\natural$-convex* if it is expressed in terms of an M-convex function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ as

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 + \sum_{u\in V} x(u) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Namely, an $\mathrm{M}^\natural$-convex function is a function obtained as the projection of an M-convex function. Conversely, an $\mathrm{M}^\natural$-convex function determines the corresponding M-convex function up

M_2-convex→ M_2-convex function
L_2-convex→ L_2-convex function
$L^\natural$-convex→ $L^\natural$-convex function
$L^\natural$-convex set
$M^\natural$-convex→ $M^\natural$-convex function

to a translation of $\mathrm{dom}\, f$ in the direction of v_0. A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is M$^\natural$-convex if and only if (see [21]) it satisfies

(M$^\natural$-EXC) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$,

$$\begin{aligned} & f(x) + f(y) \\ & \geq \min[\, f(x - \chi_u) + f(y + \chi_u), \\ & \qquad \min_{v \in \mathrm{supp}^-(x-y)} \{ f(x - \chi_u + \chi_v) \\ & \qquad\qquad + f(y + \chi_u - \chi_v)\} \,]. \end{aligned}$$

Since (M-EXC) implies (M$^\natural$-EXC), an M-convex function is an M$^\natural$-convex function.

Quadratic function

$$f(x) = \sum_{i=1}^{n} a_i {x_i}^2 + b \sum_{i<j} x_i x_j \qquad (x \in \mathbf{Z}^n)$$

with $a_i \in \mathbf{Z}$ $(1 \leq i \leq n)$, $b \in \mathbf{Z}$ is M$^\natural$-convex if $0 \leq b \leq 2 \min_{1 \leq i \leq n} a_i$ (cf. [21]). For $\{\varphi_i \in \mathcal{C}_1 \mid i = 0, 1, \cdots, n\}$, a function of the form

$$f(x) = \varphi_0(\sum_{i=1}^{n} x_i) + \sum_{i=1}^{n} \varphi_i(x_i) \quad (x \in \mathbf{Z}^n)$$

is M$^\natural$-convex [21]; a separable convex function is a special case of this (with $\varphi_0 = 0$). More generally, for $\{\varphi_X \in \mathcal{C}_1 \mid X \in \mathcal{T}\}$ indexed by a laminar family $\mathcal{T} \subseteq 2^V$, the function

$$f(x) = \sum_{X \in \mathcal{T}} \varphi_X(x(X)) \quad (x \in \mathbf{Z}^V)$$

is M$^\natural$-convex [1], where $\mathcal{T}$ is called laminar if for any $X, Y \in \mathcal{T}$, at least one of $X \cap Y$, $X \setminus Y$, $Y \setminus X$ is empty.

The properties of M-convex functions mentioned above are carried over, mutatis mutandis, to M$^\natural$-convex functions. In addition, the projection of an M$^\natural$-convex function f to $U \subseteq V$, denoted f^U, is M$^\natural$-convex.

A subset of $\mathbf{Z}^V$ is called an *M$^\natural$-convex set* if its indicator function is an M$^\natural$-convex function. A set $Q \subseteq \mathbf{Z}^V$ is an M$^\natural$-convex set if and only if Q is the set of integer points of an integral generalized polymatroid (cf. [7] for generalized polymatroids).

As a consequence of the conjugacy between L- and M-convexity, L$^\natural$-convex functions and M$^\natural$-convex functions are conjugate to each other under the integral Fenchel-Legendre transformation.

Duality. Discrete duality theorems hold true for L-convex/concave and M-convex/concave functions. A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ is called L-concave (resp., L$^\natural$-, M-, or M$^\natural$-concave) if $-g$ is L-convex (resp., L$^\natural$-, M-, or M$^\natural$-convex); $\mathrm{dom}\, g$ means the effective domain of $-g$. The concave counterpart of the discrete Fenchel-Legendre transform is defined as

$$g^\circ(p) = \inf\{\langle p, x\rangle - g(x) \mid x \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V).$$

A discrete separation theorem for L-convex/concave functions, named *L-separation theorem* [18] (see also [10]), reads as follows. Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an L$^\natural$-convex function and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an L$^\natural$-concave function such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ or $\mathrm{dom}\, f^\bullet \cap \mathrm{dom}\, g^\circ \neq \emptyset$. If $f(p) \geq g(p)$ $(p \in \mathbf{Z}^V)$, there exist $\beta^* \in \mathbf{Z}$ and $x^* \in \mathbf{Z}^V$ such that

$$f(p) \geq \beta^* + \langle p, x^*\rangle \geq g(p) \quad (p \in \mathbf{Z}^V).$$

Since a submodular set function can be identified with a positively homogeneous L-convex function, the L-separation theorem implies Frank's discrete separation theorem for a pair of sub/supermodular functions [6], which reads as follows. Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ and $\mu : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be submodular and supermodular functions, respectively, with $\rho(\emptyset) = \mu(\emptyset) = 0$, $\rho(V) < +\infty$, $\mu(V) > -\infty$, where μ is called supermodular if $-\mu$ is submodular. If $\rho(X) \geq \mu(X)$ $(X \subseteq V)$, there exists $x^* \in \mathbf{Z}^V$ such that

$$\rho(X) \geq x^*(X) \geq \mu(X) \quad (X \subseteq V).$$

Another discrete separation theorem, *M-separation theorem* [14, 18] (see also [10]), holds true for M-convex/concave functions. Namely, let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an M$^\natural$-convex

M$^\natural$-convex set
L-separation theorem
M-separation theorem

function and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an M$^\natural$-concave function such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ or $\mathrm{dom}\, f^\bullet \cap \mathrm{dom}\, g^\circ \neq \emptyset$. If $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$, there exist $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^V$ such that

$$f(x) \geq \alpha^* + \langle p^*, x\rangle \geq g(x) \quad (x \in \mathbf{Z}^V).$$

The L- and M-separation theorems are conjugate to each other, while a self-conjugate statement can be made in the form of the *Fenchel-type duality* [14, 18], as follows. Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an L$^\natural$-convex function and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an L$^\natural$-concave function such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ or $\mathrm{dom}\, f^\bullet \cap \mathrm{dom}\, g^\circ \neq \emptyset$. Then it holds that

$$\begin{aligned}&\inf\{f(p) - g(p) \mid p \in \mathbf{Z}^V\}\\ &= \sup\{g^\circ(x) - f^\bullet(x) \mid x \in \mathbf{Z}^V\}.\end{aligned}$$

Moreover, if this common value is finite, the infimum is attained by some $p \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and the supremum is attained by some $x \in \mathrm{dom}\, f^\bullet \cap \mathrm{dom}\, g^\circ$.

Here is a simple example to illustrate the subtlety of discrete separation for discrete functions. Functions $f : \mathbf{Z}^2 \to \mathbf{Z}$ and $g : \mathbf{Z}^2 \to \mathbf{Z}$ defined by $f(x_1, x_2) = \max(0, x_1 + x_2)$ and $g(x_1, x_2) = \min(x_1, x_2)$ can be extended respectively to a convex function $\overline{f} : \mathbf{R}^2 \to \mathbf{R}$ and a concave function $\overline{g} : \mathbf{R}^2 \to \mathbf{R}$ according to the defining expressions. With $\overline{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$, we have $\overline{f}(x) \geq \langle \overline{p}, x\rangle \geq \overline{g}(x)$ for all $x \in \mathbf{R}^2$, and a fortiori, $f(x) \geq \langle \overline{p}, x\rangle \geq g(x)$ for all $x \in \mathbf{Z}^2$. However, there exists no integral vector $p \in \mathbf{Z}^2$ such that $f(x) \geq \langle p, x\rangle \geq g(x)$ for all $x \in \mathbf{Z}^2$. Note also that f is M$^\natural$-convex and g is L-concave.

Network duality. A conjugate pair of M- and L-convex functions can be transformed through a network ([14, 17]; see also [25]). Let $G = (V, A)$ be a directed graph with arc set A and vertex set V partitioned into three disjoint parts as $V = V^+ \cup V^0 \cup V^-$. For $\varphi_a \in \mathcal{C}_1$ $(a \in A)$ and M-convex $f : \mathbf{Z}^{V^+} \to \mathbf{Z} \cup \{+\infty\}$, define $\tilde{f} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{f}(y) = \inf_{\xi,x}\{f(x) + \sum_{a\in A} \varphi_a(\xi(a)) \mid\\ \partial\xi = (x, 0, -y) \in \mathbf{Z}^{V^+\cup V^0\cup V^-}, \xi \in \mathbf{Z}^A\}.\end{aligned}$$

For $\psi_a \in \mathcal{C}_1$ $(a \in A)$ and L-convex $g : \mathbf{Z}^{V^+} \to \mathbf{Z} \cup \{+\infty\}$, define $\tilde{g} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{g}(q) = \inf_{\eta,p,r}\{g(p) + \sum_{a\in A} \psi_a(\eta(a)) \mid \eta = -\delta(p, r, q),\\ \eta \in \mathbf{Z}^A, (p, r, q) \in \mathbf{Z}^{V^+\cup V^0\cup V^-}\}.\end{aligned}$$

Then $\tilde{f}$ is M-convex, provided that $\tilde{f} > -\infty$, and $\tilde{g}$ is L-convex, provided that $\tilde{g} > -\infty$. If $g = f^\bullet$ and $\psi_a = {\varphi_a}^\bullet$ $(a \in A)$, then $\tilde{g} = \tilde{f}^\bullet$. A special case $(V^+ = V)$ of the last statement yields the network duality: $\inf\{\Phi(x, \xi) \mid \partial\xi = x, x \in \mathbf{Z}^V, \xi \in \mathbf{Z}^A\} = \sup\{\Psi(p, \eta) \mid \eta = -\delta p, p \in \mathbf{Z}^V, \eta \in \mathbf{Z}^A\}$, where $\Phi(x, \xi) = f(x) + \sum_{a\in A} \varphi_a(\xi(a))$, $\Psi(p, \eta) = -g(p) - \sum_{a\in A} \psi_a(\eta(a))$ and the finiteness of $\inf \Phi$ or $\sup \Psi$ is assumed. The network duality is equivalent to the Fenchel-type duality.

Subdifferentials. The subdifferential of $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ at $x \in \mathrm{dom}\, f$ is defined by $\{p \in \mathbf{R}^V \mid f(y) - f(x) \geq \langle p, y - x\rangle \ (\forall y \in \mathbf{Z}^V)\}$. The subdifferential of an L$_2$- or M$_2$-convex function forms an integral polyhedron. More specifically:

- The subdifferential of an L-convex function is an integral base polyhedron (an M-convex polyhedron).
- The subdifferential of an L$_2$-convex function is the intersection of two integral base polyhedra (M-convex polyhedra).
- The subdifferential of an M-convex function is an L-convex polyhedron.
- The subdifferential of an M$_2$-convex function is the Minkowski sum of two L-convex polyhedra.

Similar statements hold true with L and M replaced respectively by L$^\natural$ and M$^\natural$.

Algorithms. On the basis of the equivalence of L$^\natural$-convex functions and submodular integrally convex functions, the minimization of an L-convex function can be done by the algorithm of [5], which relies on the ellipsoid method. The minimization of an M-convex function can be done by purely combinatorial algorithms;

Fenchel-type duality→ Fenchel-type duality for M- and L-convex functions a greedy-type algorithm [2] for valuated matroids and a domain reduction-type polynomial-time algorithm [27] for M-convex functions. Algorithms for duality of M-convex functions (in other words, for M_2-convex functions) are also developed; polynomial algorithms [16, 24] for valuated matroids, and a finite primal algorithm [13] and a polynomial-time conjugate-scaling algorithm [11] for the submodular flow problem.

Applications. A discrete analogue of the conjugate duality framework [23] for nonlinear optimization is developed in [18]. An application of M-convex functions to engineering system analysis and matrix theory is in [15, 19]. M-convex functions find applications also in mathematical economics [1].

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Kazuo Murota
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
E-mail address: murota@kurims.kyoto-u.ac.jp

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