On Exchange Axioms for Valuated Matroids and Valuated Delta-matroids*

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Abstract

Two further equivalent axioms are given for valuations of a matroid. Let $\mathbf{M} = (V, \mathcal{B})$ be a matroid on a finite set V with the family of bases $\mathcal{B}$. For $\omega : \mathcal{B} \to \mathbf{R}$ the following three conditions are equivalent:

(V1) $\forall B, B' \in \mathcal{B}$, $\forall u \in B - B', \exists v \in B' - B$:
$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u - v)$;

(V2) $\forall B, B' \in \mathcal{B}$ with $B \neq B'$, $\exists u \in B - B', \exists v \in B' - B$:
$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u - v)$;

(V3) $\forall B, B' \in \mathcal{B}$, $\forall u \in B - B', \exists v \in B' - B, \exists u' \in B - B'$:
$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u' - v)$.

A similar result is obtained for valuations of a delta-matroid.

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1 Results

We consider valuated matroids first and then valuated delta-matroids. Let V be a finite set and R be a totally ordered additive group (typically $R = \mathbf{R}$ (reals), $\mathbf{Q}$ (rationals), or $\mathbf{Z}$ (integers)).

1.1 Axioms for valuated matroids

Recently Dress-Wenzel [4], [6] introduced the concept of valuation of a matroid. Let $\mathbf{M} = (V, \mathcal{B})$ be a matroid [12], [14] defined on a finite set V in terms of the family of bases $\mathcal{B}$. A valuation of $\mathbf{M} = (V, \mathcal{B})$ is a function $\omega : \mathcal{B} \to R$ which enjoys the following exchange property:

(V1) For distinct $B, B' \in \mathcal{B}$ and $u \in B - B'$, there exists $v \in B' - B$ such that $B - u + v \in \mathcal{B}$, $B' + u - v \in \mathcal{B}$ and

$$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u - v).$$

The valuated matroids afford a nice combinatorial framework to which the optimization algorithms for matroids can be generalized. In fact, a version of greedy algorithm works for maximizing a matroid valuation (and conversely this property characterizes a matroid valuation) [4], and the weighted matroid intersetion algorithm can be extended for maximizing the sum of a pair of matroid valuations [7], [8], [9].

In this note we show that either of the following seemingly weaker exchange properties characterizes a matroid valuation:

(V2) For distinct $B, B' \in \mathcal{B}$, there exist $u \in B - B'$ and $v \in B' - B$ such that $B - u + v \in \mathcal{B}$, $B' + u - v \in \mathcal{B}$ and

$$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u - v).$$

(V3) For distinct $B, B' \in \mathcal{B}$ and $u \in B - B'$, there exist $v \in B' - B$ and $u' \in B - B'$ such that $B - u + v \in \mathcal{B}$, $B' + u' - v \in \mathcal{B}$ and

$$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u' - v).$$

Theorem 1.1 *For $\omega : \mathcal{B} \to R$, the three conditions, (V1), (V2), (V3), are equivalent.* ∎

For a trivial valuation $\omega \equiv 0$, (V1) reduces to the symmetric exchange axiom for matroids [14] and (V2) to Tomizawa's self-dual base axiom for matroids [11], whereas (V3) is somehow related to the augmentation axiom for independent sets. Other axioms of valuations in terms of independent sets are investigated in [10].

1.2 Axioms for valuated delta-matroids

A valuated delta-matroid, due to Dress-Wenzel [5] and Wenzel [13], is a function $\delta : 2^V \to R \cup \{-\infty\}$ such that

(D0) $\delta(I) \neq -\infty$ for some $I \subseteq V$,

(D1) For distinct $I, I' \subseteq V$ with $\delta(I) \neq -\infty \neq \delta(I')$ and for $u \in I \Delta I'$, there exists $v \in (I \Delta I') - u$ such that

$$\delta(I) + \delta(I') \leq \delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v).$$

Here Δ denotes the symmetric difference: $I \Delta I' = (I - I') \cup (I' - I)$, and $I \Delta u \Delta v = I \Delta \{u\} \Delta \{v\}$, etc. We put

$$\mathcal{F} = \{I \subseteq V \mid \delta(I) \neq -\infty\}.$$

The valuated delta-matroid is also natural in connection to optimization. The underlying family $\mathcal{F}$ is a well-behaved delta-matroid ("even delta-matroid") such that $|I \Delta I'|$ is even for $I, I' \in \mathcal{F}$ (see [1], [2], [3] for delta-matroids). Again a version of greedy algorithm works for maximizing a valuated delta-matroid (and conversely this property characterizes a matroid valuation) [5].

We show that either of the following seemingly weaker exchange properties characterizes a valuated delta-matroid:

(D2) For distinct $I, I' \subseteq V$ with $\delta(I) \neq -\infty \neq \delta(I')$, there exist distinct $u, v \in I \Delta I'$ such that

$$\delta(I) + \delta(I') \leq \delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v).$$

(D3) For distinct $I, I' \subseteq V$ with $\delta(I) \neq -\infty \neq \delta(I')$ and for $u \in I \Delta I'$, there exists $v \in (I \Delta I') - u$ and $u' \in (I \Delta I') - v$ such that

$$\delta(I) + \delta(I') \leq \delta(I \Delta u \Delta v) + \delta(I' \Delta u' \Delta v).$$

Theorem 1.2 *For $\delta : 2^V \to R \cup \{-\infty\}$ satisfying (D0), the three conditions, (D1), (D2), (D3), are equivalent.* ∎

2 Proofs

2.1 Proof for valuated matroids

Obviously, (V1) $\Rightarrow$ (V2) and (V1) $\Rightarrow$ (V3). To prove (V2) $\Rightarrow$ (V1) and (V3) $\Rightarrow$ (V1) we first show two lemmas. For $\omega : \mathcal{B} \to R$ and $p : V \to R$ define

$$\omega_p(B) = \omega(B) + \sum\{p(u) \mid u \in B\} \qquad (B \subseteq V), \tag{2.1}$$

where we put $\omega(B) = \omega_p(B) = -\infty$ if $B \notin \mathcal{B}$, and

$$\begin{aligned} \omega(B, u, v) &= \omega(B - u + v) - \omega(B) \qquad (B \in \mathcal{B}), \\ \omega_p(B, u, v) &= \omega_p(B - u + v) - \omega_p(B) \qquad (B \in \mathcal{B}), \end{aligned}$$

where $\omega(B, u, v) = \omega_p(B, u, v) = -\infty$ if $B - u + v \notin \mathcal{B}$. If $B, B' \in \mathcal{B}$ we have

$$\begin{aligned} &\omega(B - u + v) + \omega(B' + u - v) - \omega(B) - \omega(B') \\ &= \omega(B, u, v) + \omega(B', v, u) \\ &= \omega_p(B, u, v) + \omega_p(B', v, u) \qquad (u \in B - B', v \in B' - B). \end{aligned} \tag{2.2}$$

Lemma 2.1 *Assume (V3). If $B \in \mathcal{B}$ and $|B' - B| = |B - B'| = 2$, then there exist $u \in B - B'$ and $v \in B' - B$ such that*

$$\omega(B) + \omega(B') \leq \omega(B - u + v) + \omega(B' + u - v).$$

(Proof) Put $B - B' =: \{u_0, u_1\}$, $B' - B =: \{v_0, v_1\}$, and $\omega(B, u_i, v_j) =: \alpha_{ij}$ $(i, j = 0, 1)$. Since

$$\omega(B - u_i + v_j) + \omega(B' + u_i - v_j) = \alpha_{ij} + \alpha_{1-i,1-j} + 2\omega(B),$$

the claim is equivalent to saying that

$$\max(\alpha_{00} + \alpha_{11}, \alpha_{01} + \alpha_{10}) \geq \gamma \equiv \omega(B') - \omega(B).$$

Using (V3) with $u = u_0$ we obtain $\alpha_{0j} + \alpha_{i,1-j} \geq \gamma$ for some $i, j \in \{0, 1\}$. If $i = 1$, we are done. Otherwise, we have

$$\alpha_{00} + \alpha_{01} \geq \gamma. \tag{2.3}$$

Similarly, using (V3) with $u = u_1$ we obtain $\alpha_{1j} + \alpha_{i,1-j} \geq \gamma$ for some $i, j \in \{0, 1\}$. If $i = 0$, we are done. Otherwise, we have

$$\alpha_{10} + \alpha_{11} \geq \gamma. \tag{2.4}$$

Addition of (2.3) and (2.4) yields

$$(\alpha_{00} + \alpha_{11}) + (\alpha_{01} + \alpha_{10}) \geq 2\gamma,$$

which implies that $\alpha_{00} + \alpha_{11} \geq \gamma$ or $\alpha_{01} + \alpha_{10} \geq \gamma$. ∎

Lemma 2.2 *Let $B \in \mathcal{B}$, $B - B' = \{u_0, u_1\}$, $B' - B = \{v_0, v_1\}$ (with $u_0 \neq u_1$, $v_0 \neq v_1$) and $p : V \to R$. If (V2) or (V3) is satisfied, then*

$$\omega_p(B') - \omega_p(B) \leq \max(\pi_{00} + \pi_{11}, \pi_{01} + \pi_{10}),$$

where $\pi_{ij} = \omega_p(B, u_i, v_j)$ for $i, j = 0, 1$.

(Proof) The case (V2) is immediate from (2.2). The case (V3) follows from Lemma 2.1 and (2.2). ∎

Define

$$\begin{aligned}\mathcal{D} \;=\; \{(B, B') \mid B, B' \in \mathcal{B}, \exists u_* \in B - B', \forall v \in B' - B :\\ \omega(B) + \omega(B') > \omega(B - u_* + v) + \omega(B' + u_* - v)\},\end{aligned}$$

which denotes the set of pairs (B, B') for which the exchangeability in (V1) fails. In what follows we are to show $\mathcal{D} = \emptyset$ if (V2) or (V3) is satisfied.

Suppose to the contrary that $\mathcal{D} \neq \emptyset$, and take $(B, B') \in \mathcal{D}$ such that $|B' - B|$ is minimum and let $u_* \in B - B'$ be as in the definition of $\mathcal{D}$. Define $p : V \to R$ by

$$p(v) = \begin{cases} -\omega(B, u_*, v) & (v \in B' - B, B - u_* + v \in \mathcal{B}) \\ \omega(B', v, u_*) + \varepsilon & (v \in B' - B, B - u_* + v \notin \mathcal{B}, B' + u_* - v \in \mathcal{B}) \\ 0 & (\text{otherwise}) \end{cases}$$

with some $\varepsilon > 0$ and consider ω_p defined in (2.1).

Claim 1:

$$\omega_p(B, u_*, v) \;=\; 0 \qquad \text{if } v \in B' - B, B - u_* + v \in \mathcal{B}, \tag{2.5}$$

$$\omega_p(B', v, u_*) \;<\; 0 \qquad \text{for } v \in B' - B. \tag{2.6}$$

The inequality (2.6) can be shown as follows. If $B - u_* + v \in \mathcal{B}$, we have $\omega_p(B, u_*, v) = 0$ by (2.5) and

$$\omega_p(B, u_*, v) + \omega_p(B', v, u_*) = \omega(B, u_*, v) + \omega(B', v, u_*) < 0$$

by (2.2) and the definition of u_*. Otherwise we have $\omega_p(B', v, u_*) = -\varepsilon$ or $-\infty$ according to whether $B' + u_* - v \in \mathcal{B}$ or not.

Next we claim under the assumption of (V2) or (V3) that

Claim 2: There exist $u_0 \in B - B'$ and $v_0 \in B' - B$ such that $u_0 \neq u_*$ and $B' + u_0 - v_0 \in \mathcal{B}$.

In fact, (V2) implies that $\exists u_0 \in B - B'$, $\exists v_0 \in B' - B$ with

$$-\infty \neq \omega(B) + \omega(B') \leq \omega(B - u_0 + v_0) + \omega(B' + u_0 - v_0),$$

whereas (V3) implies that $\exists u_0 \in B - B'$, $\exists v_0 \in B' - B$ with

$$-\infty \neq \omega(B) + \omega(B') \leq \omega(B - u_* + v_0) + \omega(B' + u_0 - v_0).$$

In either case we have $u_0 \neq u_*$ by the definition of u_*, and $B' + u_0 - v_0 \in \mathcal{B}$ by $\omega(B' + u_0 - v_0) \neq -\infty$.

In addition to the conditions imposed in Claim 2 we can further assume

$$\omega_p(B', v_0, u_0) \geq \omega_p(B', v, u_0) \qquad (v \in B' - B) \tag{2.7}$$

by choosing v_0 appropriately. Put $B'' = B' + u_0 - v_0$.

Claim 3: $(B, B'') \in \mathcal{D}$.

To prove this it suffices to show

$$\omega_p(B, u_*, v) + \omega_p(B'', v, u_*) < 0 \qquad (v \in B'' - B).$$

We may restrict ourselves to v with $B - u_* + v \in \mathcal{B}$, since otherwise the first term $\omega_p(B, u_*, v)$ is equal to $-\infty$. For such v the first term is equal to zero by (2.5). For the second term it follows from Lemma 2.2, (2.6) and (2.7) that

$$\begin{aligned}
\omega_p(B'', v, u_*) &= \omega_p(B' + \{u_0, u_*\} - \{v_0, v\}) - \omega_p(B' + u_0 - v_0) \\
&\leq \max[\omega_p(B', v_0, u_0) + \omega_p(B', v, u_*), \omega_p(B', v, u_0) + \omega_p(B', v_0, u_*)] \\
&\qquad -\omega_p(B', v_0, u_0) \\
&< \max[\omega_p(B', v_0, u_0), \omega_p(B', v, u_0)] - \omega_p(B', v_0, u_0) \\
&= 0.
\end{aligned}$$

Since $|B'' - B| = |B' - B| - 1$, Claim 3 contradicts our choice of $(B, B') \in \mathcal{D}$. Therefore we conclude $\mathcal{D} = \emptyset$. ∎

2.2 Proof for valuated delta-matroids

Obviously, (D1) $\Rightarrow$ (D2) and (D1) $\Rightarrow$ (D3). To prove (D2) $\Rightarrow$ (D1) and (D3) $\Rightarrow$ (D1) we first show two lemmas. For $\delta : 2^V \to R \cup \{-\infty\}$ and $p : V \to R$ define

$$\delta_p(I) = \delta(I) + \sum\{p(u) \mid u \in I\} \qquad (I \subseteq V) \tag{2.8}$$

and

$$\begin{aligned} \delta(I,u,v) &= \delta(I \Delta u \Delta v) - \delta(I) \qquad (I \in \mathcal{F}, u \neq v), \\ \delta_p(I,u,v) &= \delta_p(I \Delta u \Delta v) - \delta_p(I) \qquad (I \in \mathcal{F}, u \neq v). \end{aligned}$$

Note that $\delta(I,u,v) = \delta(I,v,u)$ and $\delta_p(I,u,v) = \delta_p(I,v,u)$. If $\{u,v\} \subseteq I \Delta I'$, $I \in \mathcal{F}$ and $I' \in \mathcal{F}$ we have

$$\begin{aligned} &\delta(I \Delta u \Delta v) + \delta(I' \Delta u \Delta v) - \delta(I) - \delta(I') \\ &= \delta(I,u,v) + \delta(I',v,u) \\ &= \delta_p(I,u,v) + \delta_p(I',v,u). \end{aligned} \tag{2.9}$$

Lemma 2.3 *Let $I \in \mathcal{F}$, $I \Delta I' = \{v_1, v_2, v_3, v_4\}$ (with v_i being distinct) and $p : V \to R$. If (D2) or (D3) is satisfied, then*

$$\delta_p(I') - \delta_p(I) \leq \max(\pi_{12} + \pi_{34}, \pi_{13} + \pi_{24}, \pi_{14} + \pi_{23}), \tag{2.10}$$

where $\pi_{ij} = \delta_p(I, v_i, v_j)$ for $i, j \in \{1,2,3,4\}$.

(Proof) Put $\gamma = \delta(I') - \delta(I)$ and $\alpha_{ij} = \delta(I, v_i, v_j)$ for $\{i,j\} \subset \{1,2,3,4\}$ with $i \neq j$. First note that for $\{i,j\} \subset \{1,2,3,4\}$ with $i \neq j$, we have

$$I' \Delta v_i \Delta v_j = I \Delta v_l \Delta v_m$$

with $\{l,m\} = \{1,2,3,4\} - \{i,j\}$, and hence

$$\delta(I' \Delta v_i \Delta v_j) - \delta(I) = \alpha_{lm}. \tag{2.11}$$

Note also that by (2.9) the desired inequality (2.10) is equivalent to

$$\gamma \leq \max(\alpha_{12} + \alpha_{34}, \alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}). \tag{2.12}$$

Suppose (D2) is satisfied. We have

$$\delta(I) + \delta(I') \leq \delta(I \Delta v_i \Delta v_j) + \delta(I' \Delta v_i \Delta v_j)$$

for some $\{i,j\}$ with $i \neq j$. Using (2.11) we obtain (2.12).

Suppose now (D3) is satisfied. Using (D3) with $u = v_k$ we obtain

$$\delta(I) + \delta(I') \leq \delta(I \Delta v_k \Delta v_i) + \delta(I' \Delta v_j \Delta v_i) \tag{2.13}$$

for some $\{i,j\} = \{i_k, j_k\}$ with $i \neq k$ and $i \neq j$. Note that (2.13) can be written as

$$\alpha_{ki} + \alpha_{lm} \geq \gamma$$

with $\{l,m\} = \{1,2,3,4\} - \{i,j\}$. If $j_k = k$ for some $k \in \{1,2,3,4\}$, we are done. Therefore we assume that $j_k \neq k$ for any $k \in \{1,2,3,4\}$. Then the following lemma establishes (2.12). ∎

Lemma 2.4 *If for each $k \in \{1,2,3,4\}$ there exists $\{i,m\} \subset \{1,2,3,4\} - \{k\}$ with $i \neq m$ such that*

$$\alpha_{ki} + \alpha_{km} \geq \gamma, \tag{2.14}$$

then we have

$$\max(\alpha_{12} + \alpha_{34}, \alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}) \geq \gamma. \tag{2.15}$$

(Proof) Though the proof is elementary and similar to the one for Lemma 2.1, it is included here for completeness.

In (2.14) with $k = 1$ we may assume $i = 2$ and $m = 4$, which yields

$$\alpha_{12} + \alpha_{14} \geq \gamma. \tag{2.16}$$

From (2.14) with $k = 3$ we see that at least one of the following three inequalities holds:

$$\alpha_{13} + \alpha_{34} \geq \gamma, \tag{2.17}$$

$$\alpha_{23} + \alpha_{34} \geq \gamma, \tag{2.18}$$

$$\alpha_{13} + \alpha_{23} \geq \gamma. \tag{2.19}$$

In view of the symmetry between (2.17) and (2.19) under (2.16) we may concentrate upon the first two cases (2.17) and (2.18).

In the second case (2.18) we add (2.16) and (2.18) to get

$$(\alpha_{12} + \alpha_{34}) + (\alpha_{14} + \alpha_{23}) \geq 2\gamma, \tag{2.20}$$

which implies

$$\max(\alpha_{12} + \alpha_{34}, \alpha_{14} + \alpha_{23}) \geq \gamma,$$

establishing (2.15).

We consider the other case (2.17). We use (2.14) with $k = 2$ to see that at least one of the following three inequalities holds:

$$\alpha_{12} + \alpha_{24} \geq \gamma, \tag{2.21}$$
$$\alpha_{23} + \alpha_{24} \geq \gamma, \tag{2.22}$$
$$\alpha_{12} + \alpha_{23} \geq \gamma. \tag{2.23}$$

In the first case (2.21) we add (2.17) and (2.21) to obtain

$$(\alpha_{12} + \alpha_{34}) + (\alpha_{13} + \alpha_{24}) \geq 2\gamma,$$

which implies

$$\max(\alpha_{12} + \alpha_{34}, \alpha_{14} + \alpha_{23}) \geq \gamma,$$

establishing (2.15). In the second case (2.22) we add (2.16), (2.17) and (2.22) to obtain

$$(\alpha_{12} + \alpha_{34}) + (\alpha_{13} + \alpha_{24}) + (\alpha_{14} + \alpha_{23}) \geq 3\gamma, \tag{2.24}$$

which implies (2.15).

The remaining case (2.23), where (2.16) and (2.17) are valid, can be resolved as follows. From (2.14) with $k = 4$ we see that at least one of the following three inequalities holds:

$$\alpha_{14} + \alpha_{24} \geq \gamma, \tag{2.25}$$
$$\alpha_{14} + \alpha_{34} \geq \gamma, \tag{2.26}$$
$$\alpha_{24} + \alpha_{34} \geq \gamma. \tag{2.27}$$

In case of (2.25) we add (2.17), (2.23) and (2.25) to obtain (2.24). In case of (2.26) we add (2.23) and (2.26) to obtain (2.20). Finally, in case of (2.27) we add (2.16), (2.17), (2.23) and (2.27) to obtain

$$2(\alpha_{12} + \alpha_{34}) + (\alpha_{13} + \alpha_{24}) + (\alpha_{14} + \alpha_{23}) \geq 4\gamma,$$

which implies (2.15). ∎

Define

$$\begin{aligned} \mathcal{D} = \{(I, I') \mid I, I' \in \mathcal{F}, \exists u_* \in I\Delta I', \forall v \in (I\Delta I') - u_* : \\ \delta(I) + \delta(I') > \delta(I\Delta u_*\Delta v) + \delta(I'\Delta u_*\Delta v)\}, \end{aligned}$$

which denotes the set of pairs (I, I') for which the exchangeability in (D1) fails. In what follows we are to show $\mathcal{D} = \emptyset$ if (D2) or (D3) is satisfied.

Suppose to the contrary that $\mathcal{D} \neq \emptyset$, and take $(I, I') \in \mathcal{D}$ such that $|I \Delta I'|$ is minimum and let $u_* \in I \Delta I'$ be as in the definition of $\mathcal{D}$. Define $p : V \to R$ by

$$p(v) = \begin{cases} \delta(I, u_*, v) & (v \in (I - I') - u_*, I\Delta u_* \Delta v \in \mathcal{F}) \\ -\delta(I, u_*, v) & (v \in (I' - I) - u_*, I\Delta u_* \Delta v \in \mathcal{F}) \\ -\delta(I', v, u_*) - \varepsilon & (v \in (I - I') - u_*, I\Delta u_* \Delta v \notin \mathcal{F}, I'\Delta u_* \Delta v \in \mathcal{F}) \\ \delta(I', v, u_*) + \varepsilon & (v \in (I' - I) - u_*, I\Delta u_* \Delta v \notin \mathcal{F}, I'\Delta u_* \Delta v \in \mathcal{F}) \\ 0 & (\text{otherwise}) \end{cases}$$

with some $\varepsilon > 0$ and consider δ_p defined in (2.8).

Claim 1:

$$\delta_p(I, u_*, v) \;=\; 0 \qquad \text{if } v \in (I \Delta I') - u_*, I\Delta u_* \Delta v \in \mathcal{F}, \tag{2.28}$$

$$\delta_p(I', v, u_*) \;<\; 0 \qquad \text{for } v \in (I \Delta I') - u_*. \tag{2.29}$$

The inequality (2.29) can be shown as follows. If $I\Delta u_* \Delta v \in \mathcal{F}$, we have $\delta_p(I, u_*, v) = 0$ by (2.28) and

$$\delta_p(I, u_*, v) + \delta_p(I', v, u_*) = \delta(I, u_*, v) + \delta(I', v, u_*) < 0$$

by (2.9) and the definition of u_*. Otherwise we have $\delta_p(I', v, u_*) = -\varepsilon$ or $-\infty$ according to whether $I'\Delta u_* \Delta v \in \mathcal{F}$ or not.

Next we claim under the assumption of (D2) or (D3) that

Claim 2: There exists $\{u_0, v_0\} \subseteq (I \Delta I') - u_*$ such that $u_0 \neq v_0$ and $I'\Delta u_0 \Delta v_0 \in \mathcal{F}$.

In fact, (D2) implies that $\exists \{u_0, v_0\} \subseteq I \Delta I'$ with $u_0 \neq v_0$ and

$$-\infty \neq \delta(I) + \delta(I') \leq \delta(I\Delta u_0 \Delta v_0) + \delta(I'\Delta u_0 \Delta v_0),$$

whereas (D3) implies that $\exists \{u_0, v_0\} \subseteq I \Delta I'$ with $u_0 \neq v_0 \neq u_*$ and

$$-\infty \neq \delta(I) + \delta(I') \leq \delta(I\Delta u_* \Delta v_0) + \delta(I'\Delta u_0 \Delta v_0).$$

In either case we have $u_0 \neq u_* \neq v_0$ by the definition of u_*, and $I'\Delta u_0 \Delta v_0 \in \mathcal{F}$ by $\delta(I'\Delta u_0 \Delta v_0) \neq -\infty$.

In addition to the conditions imposed in Claim 2 we can further assume

$$\delta_p(I', v_0, u_0) \geq \delta_p(I', v, u) \qquad (\{u, v\} \subseteq (I \Delta I') - u_*, u \neq v) \tag{2.30}$$

by choosing u_0 and v_0 appropriately. Put $I'' = I'\Delta u_0 \Delta v_0$.

Claim 3: $(I, I'') \in \mathcal{D}$.

To prove this it suffices to show

$$\delta_p(I, u_*, v) + \delta_p(I'', v, u_*) < 0 \qquad (v \in (I \Delta I'') - u_*).$$

We may restrict ourselves to v with $I \Delta u_* \Delta v \in \mathcal{F}$, since otherwise the first term $\delta_p(I, u_*, v)$ is equal to $-\infty$. For such v the first term is equal to zero by (2.28). For the second term it follows from Lemma 2.3, (2.29) and (2.30) that

$$\begin{aligned}
\delta_p(I'', v, u_*) &= \delta_p(I' \Delta \{u_0, u_*, v_0, v\}) - \delta_p(I' \Delta u_0 \Delta v_0) \\
&\leq \max[\delta_p(I', v_0, u_0) + \delta_p(I', v, u_*), \delta_p(I', v, u_0) + \delta_p(I', v_0, u_*), \\
&\qquad \delta_p(I', v_0, v) + \delta_p(I', u_0, u_*)] \ - \delta_p(I', v_0, u_0) \\
&< \max[\delta_p(I', v_0, u_0), \delta_p(I', v, u_0), \delta_p(I', v_0, v)] - \delta_p(I', v_0, u_0) \\
&= 0.
\end{aligned}$$

Since $|I \Delta I''| = |I \Delta I'| - 2$, Claim 3 contradicts our choice of $(I, I') \in \mathcal{D}$. Therefore we conclude $\mathcal{D} = \emptyset$. ∎

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