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Abstract

This short note gives an alternative proof of the M-convex intersection theorem, which is one of the central results in discrete convex analysis. This note is intended to provide a direct simpler proof accessible to nonexperts.

1 M-Convex Intersection Theorem

The M-convex intersection theorem [3, Theorem 8.17] reads as follows, where V is a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ are the sets of integers and reals, respectively; see §3 for the definitions of M$^\natural$-convex functions and notation arg min. This theorem is equivalent to the M-separation theorem, to the Fenchel-type min-max duality theorem, and to an optimality criterion of the M-convex submodular flow problem.

Theorem 1 (M-convex intersection theorem). *For M$^\natural$-convex functions f_1, f_2 and a point $x^* \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$ we have*

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \qquad (\forall x \in \mathbf{Z}^V) \tag{1}$$

if and only if there exists $p^* \in \mathbf{R}^V$ such that[1]

$$f_1[-p^*](x^*) \leq f_1[-p^*](x) \qquad (\forall x \in \mathbf{Z}^V), \tag{2}$$

$$f_2[+p^*](x^*) \leq f_2[+p^*](x) \qquad (\forall x \in \mathbf{Z}^V). \tag{3}$$

For such p^* we have

$$\arg\min(f_1 + f_2) = \arg\min f_1[-p^*] \cap \arg\min f_2[+p^*]. \tag{4}$$

Moreover, if f_1 and f_2 are integer-valued, we can choose integer-valued $p^* \in \mathbf{Z}^V$.

We shall give a constructive proof of Theorem 1 based on the successive shortest path algorithm. Different proofs available in [3] are:

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[1]Notation: $f_1[-p^*](x) = f_1(x) - \sum_{v \in V} p^*(v)x(v), \quad f_2[+p^*](x) = f_2(x) + \sum_{v \in V} p^*(v)x(v).$

1. original proof based on negative-cycle cancelling for the M-convex submodular flow problem (§9.5 and Note 9.21 of [3]), and
2. polyhedral proof for the discrete separation theorem based on the separation in convex analysis (Proof of Theorem 8.15 of [3]).

2 Essence of Theorem 1

The essence of Theorem 1 consists of two assertions:

1. optimality of x^* $\Rightarrow$ existence of p^*,
2. integrality of f_1, f_2 $\Rightarrow$ integrality of p^*.

To see this we make easier observations in this section.

Observation 1: Existence of p^* with (2) and (3) $\Rightarrow$ optimality (1) of x^*.

(Proof)

$$\begin{aligned} f_1(x^*) + f_2(x^*) &= f_1[-p^*](x^*) + f_2[+p^*](x^*) \\ &\leq f_1[-p^*](x) + f_2[+p^*](x) = f_1(x) + f_2(x). \end{aligned}$$

Observation 2: For any $p^* \in \mathbf{R}^V$ we have

$$\arg\min(f_1 + f_2) \supseteq \arg\min f_1[-p^*] \cap \arg\min f_2[+p^*]. \tag{5}$$

(Proof) This follows from the inequality shown in the proof of Observation 1.

Observation 3: If

$$f_1[-p^*](x^\circ) \leq f_1[-p^*](x) \qquad (\forall x \in \mathbf{Z}^V), \tag{6}$$
$$f_2[+p^*](x^\circ) \leq f_2[+p^*](x) \qquad (\forall x \in \mathbf{Z}^V) \tag{7}$$

for some x° and p^*, then

$$f_1[-p^*](x^*) \leq f_1[-p^*](x) \qquad (\forall x \in \mathbf{Z}^V), \tag{8}$$
$$f_2[+p^*](x^*) \leq f_2[+p^*](x) \qquad (\forall x \in \mathbf{Z}^V) \tag{9}$$

for every $x^* \in \arg\min(f_1 + f_2)$. Hence,

$$\arg\min(f_1 + f_2) \subseteq \arg\min f_1[-p^*] \cap \arg\min f_2[+p^*]. \tag{10}$$

(Proof) Put $x = x^*$ in (6) and (7) to obtain

$$f_1[-p^*](x^\circ) \leq f_1[-p^*](x^*), \tag{11}$$
$$f_2[+p^*](x^\circ) \leq f_2[+p^*](x^*). \tag{12}$$

Adding these yields

$$\begin{aligned} f_1(x^\circ) + f_2(x^\circ) &= f_1[-p^*](x^\circ) + f_2[+p^*](x^\circ) \\ &\leq f_1[-p^*](x^*) + f_2[+p^*](x^*) = f_1(x^*) + f_2(x^*), \end{aligned}$$

whereas $x^* \in \arg\min(f_1 + f_2)$. Hence we have equalities in (11) and (12).
Observation 4: It suffices to consider M-convex functions rather than M$^\natural$-convex functions.

(Proof) This follows from the equivalence between M$^\natural$-convexity and M-convexity; see [3, §6.1].

Thus the proof of Theorem 1 is reduced to showing the following.

Proposition 2. *For M-convex functions f_1, f_2 with $\arg\min(f_1 + f_2) \neq \emptyset$, there exist $x^\circ \in \arg\min(f_1 + f_2)$ and $p^* \in \mathbf{R}^V$ such that*

$$f_1[-p^*](x^\circ) \leq f_1[-p^*](x) \qquad (\forall x \in \mathbf{Z}^V), \tag{13}$$

$$f_2[+p^*](x^\circ) \leq f_2[+p^*](x) \qquad (\forall x \in \mathbf{Z}^V). \tag{14}$$

If f_1 and f_2 are integer-valued, we can choose integer-valued $p^* \in \mathbf{Z}^V$.

3 Notation and Basic Facts

We denote by $\mathbf{Z}^V$ the set of integral vectors indexed by V, and by $\mathbf{R}^V$ the set of real vectors indexed by V. For a vector $x = (x(v) : v \in V) \in \mathbf{Z}^V$, where $x(v)$ is the vth component of x, we define the positive support $\mathrm{supp}^+(x)$ and the negative support $\mathrm{supp}^-(x)$ by

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\}.$$

We use notation $x(S) = \sum_{v \in S} x(v)$ for a subset S of V. For each $S \subseteq V$, we denote by χ_S the characteristic vector of S defined by: $\chi_S(v) = 1$ if $v \in S$ and $\chi_S(v) = 0$ otherwise, and write χ_v for $\chi_{\{v\}}$ for all $v \in V$. For a vector $p = (p(v) : v \in V) \in \mathbf{R}^V$ and a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we define functions $\langle p, x \rangle$ and $f[p](x)$ in $x \in \mathbf{Z}^V$ by

$$\langle p, x \rangle = \sum_{v \in V} p(v)x(v), \quad f[p](x) = f(x) + \langle p, x \rangle.$$

We also denote the set of minimizers of f and the effective domain of f by

$$\begin{aligned} &\arg\min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V)\}, \\ &\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}. \end{aligned}$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-convex* if it satisfies

(M$^\natural$-EXC) for all $x, y \in \mathrm{dom}\, f$ and all $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where χ_0 is defined to be the zero vector in $\mathbf{Z}^V$.

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M-convex* if it satisfies

(M-EXC) for all $x, y \in \mathrm{dom}\, f$ and all $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

A nonempty set $B \subseteq \mathbf{Z}^V$ is called *M-convex* if it satisfies

(B-EXC) for all $x, y \in B$ and all $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v,\ y + \chi_u - \chi_v \in B$.

The minimizers of an M-convex function have a good characterization.

Lemma 3 ([3, Theorem 6.26]). *For an M-convex function f and $x \in \mathrm{dom}\, f$, $x \in \arg\min f$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ for all $u, v \in V$.*

Lemma 4 ([3, Proposition 6.29]). *For an M-convex function f, $\arg\min f$ is an M-convex set if not empty.*

An M-convex set has the following property. (See [1, Lemma 4.5] and [2, Lemma 2.3.22, Remark 3.3.24]. This is a special case of [3, Proposition 9.23].)

Lemma 5 ("no-short cut lemma"). *Let B be an M-convex set. For any $x \in B$ and any distinct $u_1,\ v_1,\ u_2,\ v_2, \cdots, u_r, v_r \in V$, if $x - \chi_{u_i} + \chi_{v_i} \in B$ for all $i = 1, \cdots, r$ and $x - \chi_{u_i} + \chi_{v_j} \notin B$ for all i, j with $i < j$, then $y = x - \sum_{i=1}^{r}(\chi_{u_i} - \chi_{v_i}) \in B$.*

4 Proof of Proposition 2 by SSP

We give a proof of Proposition 2 on the basis of the successive shortest path algorithm (SSP) [3, §10.3.4] as adapted to finding a minimizer of $f_1 + f_2$. We may assume that the effective domains of f_1 and f_2 are bounded.

Let x_1 and x_2 be arbitrary minimizers of f_1 and f_2, respectively. We construct a directed graph $G(f_1, f_2, x_1, x_2) = (V, A)$ and an arc length $\ell \in \mathbf{R}^A$ as follows. Arc set A is the union of two disjoint parts:

$$\begin{aligned} A_1 &= \{(u, v) \mid u, v \in V,\ u \neq v,\ x_1 - \chi_u + \chi_v \in \mathrm{dom}\, f_1\}, \\ A_2 &= \{(v, u) \mid u, v \in V,\ u \neq v,\ x_2 - \chi_u + \chi_v \in \mathrm{dom}\, f_2\}, \end{aligned} \tag{15}$$

and $\ell \in \mathbf{R}^A$ is defined by

$$\ell(a) = \begin{cases} f_1(x_1 - \chi_u + \chi_v) - f_1(x_1) & \text{if } a = (u,v) \in A_1, \\ f_2(x_2 - \chi_u + \chi_v) - f_2(x_2) & \text{if } a = (v,u) \in A_2. \end{cases} \tag{16}$$

The length function ℓ is nonnegative due to Lemma 3.

Put $S = \text{supp}^+(x_1 - x_2)$ and $T = \text{supp}^-(x_1 - x_2)$. A path exists from S to T by Lemma 6 below. Let P be a shortest path from S to T in G with a minimum number of arcs, and let $t \in T$ be the terminal vertex of P.

Let $d : V \to \mathbf{R} \cup \{+\infty\}$ denote the shortest distance from S to all vertices with respect to ℓ. Then we have

$$\ell(a) + d(u) - d(v) \geq 0$$

for all arcs $a = (u,v) \in A$. Define $p \in \mathbf{R}^V$ by $p(v) = \min\{d(v), d(t)\}$ for all $v \in V$. It follows from the nonnegativity of ℓ that

$$\ell(a) + p(u) - p(v) \geq 0$$

for all arcs $a = (u,v) \in A$. The above system of inequalities is equivalent to

$$\begin{aligned} f_1(x_1 - \chi_u + \chi_v) - f_1(x_1) + p(u) - p(v) \geq 0, \\ f_2(x_2 - \chi_u + \chi_v) - f_2(x_2) - p(u) + p(v) \geq 0 \end{aligned}$$

for all $u, v \in V$, which is further equivalent to

$$x_1 \in \arg\min f_1[-p], \quad x_2 \in \arg\min f_2[+p],$$

by Lemma 3. Note that for all arcs $a = (u,v) \in A$,

$$\ell_p(a) = \ell(a) + p(u) - p(v)$$

are the lengths of a in the graph $G(f_1[-p], f_2[+p], x_1, x_2)$ associated with $f_1[-p]$, $f_2[+p]$, x_1, and x_2.

Since $\ell_p(a) = 0$ for all $a \in P$, we have

$$\begin{aligned} x_1 - \chi_u + \chi_v &\in \arg\min f_1[-p] \qquad \text{for all } (u,v) \in P \cap A_1, \\ x_2 - \chi_u + \chi_v &\in \arg\min f_2[+p] \qquad \text{for all } (v,u) \in P \cap A_2. \end{aligned} \tag{17}$$

Since P has a minimum number of arcs, we also have

$$x_1 - \chi_u + \chi_w \notin \arg\min f_1[-p], \quad x_2 - \chi_w + \chi_u \notin \arg\min f_2[+p] \tag{18}$$

for all vertices u and w of P such that $(u,w) \notin P$ and u appears earlier than w in P.

Furthermore, arcs of A_1 and A_2 appear alternately in P. This can be proved as follows. Suppose that consecutive two arcs $(u,v), (v,w) \in P$ belong to, say, A_1. Then, by (M-EXC),

$$f_1(x_1 + \chi_u - \chi_v) + f_1(x_1 + \chi_v - \chi_w) \geq f_1(x_1) + f_1(x_1 + \chi_u - \chi_w),$$

which yields

$$\ell(u,v) + \ell(v,w) \geq \ell(u,w),$$

a contradiction to the minimality (with respect to the number of arcs) of P. Consequently, we have

$$\begin{array}{lll} a_1{=}(u_1,v_1), a_2{=}(u_2,v_2) \in P{\cap}A_1, a_1 \neq a_2 & \Longrightarrow & \{u_1,v_1\} \cap \{u_2,v_2\} = \emptyset, \\ a_1{=}(u_1,v_1), a_2{=}(u_2,v_2) \in P{\cap}A_2, a_1 \neq a_2 & \Longrightarrow & \{u_1,v_1\} \cap \{u_2,v_2\} = \emptyset. \end{array} \tag{19}$$

From Lemmas 4 and 5 together with (17), (18), and (19), we have

$$x_1' \equiv x_1 - \sum_{(u,v)\in P\cap A_1} (\chi_u - \chi_v) \quad \in \quad \arg\min f_1[-p], \tag{20}$$

$$x_2' \equiv x_2 - \sum_{(v,u)\in P\cap A_2} (\chi_u - \chi_v) \quad \in \quad \arg\min f_2[+p]. \tag{21}$$

Thus the modification of (f_1, f_2, x_1, x_2) to (f_1', f_2', x_1', x_2'), where $f_1' = f_1[-p]$ and $f_2' = f_2[+p]$, keeps the conditions

$$x_1' \in \arg\min f_1', \quad x_2' \in \arg\min f_2'.$$

We have

$$x_1' - x_2' = (x_1 - x_2) - (\chi_s - \chi_t)$$

with $s \in \mathrm{supp}^+(x_1 - x_2)$ and $t \in \mathrm{supp}^-(x_1 - x_2)$, since P is a path from $\mathrm{supp}^+(x_1 - x_2)$ to $\mathrm{supp}^-(x_1 - x_2)$ and arcs of A_1 and A_2 appear alternately in P. This implies that $\sum_{v\in V} |x_1(v) - x_2(v)|$ is decreased by two. Repeating the modification above we eventually arrive at $x_1 = x_2$, when we have

$$x_1 \in \arg\min f_1[-p] \cap \arg\min f_2[+p].$$

Finally note that, if the functions f_1 and f_2 are integer-valued, the length function ℓ is integer-valued, and hence p is also integer-valued.

The SSP algorithm is summarized below.

Algorithm SSP $(f_1, f_2$: M-convex)

Step 0. Find $x_1 \in \arg\min f_1$ and $x_2 \in \arg\min f_2$. Set $p := \mathbf{0}$.

Step 1. If $x_1 = x_2$ then stop.

Step 2. Construct G and compute ℓ for $f_1[-p]$, $f_2[+p]$, x_1 and x_2 by (15) and (16). Set $S := \mathrm{supp}^+(x_1 - x_2)$, $T := \mathrm{supp}^-(x_1 - x_2)$.

Step 3. Compute the shortest distances $d(v)$ from S to all $v \in V$ in G with respect to ℓ. Find a shortest path P from S to T with a minimum number of arcs, and let t be the terminal vertex of P.

Step 4. For all $v \in V$, set $p(v) := p(v) + \min\{d(v), d(t)\}$.

Update x_1 and x_2 by (20) and (21).

Go to Step 1.

Lemma 6. *If* $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \neq \emptyset$ *and* $x_1 \neq x_2$*, then there exists a path from* $S = \mathrm{supp}^+(x_1 - x_2)$ *to* $T = \mathrm{supp}^-(x_1 - x_2)$*.*

Proof: To prove by contradiction, suppose that there exists no path from S to T and let W be the set of the vertices reachable from S. Then $W \supseteq S$ and $W \cap T = \emptyset$.

Define set functions $\rho_i : 2^V \to \mathbf{Z} \cup \{+\infty\}$ as

$$\rho_i(X) = \sup\{z(X) \mid z \in \mathrm{dom}\, f_i\}$$

for $i = 1, 2$. For $z \in \mathrm{dom}\, f_i$ we obviously have[2]

$$z(X) \leq \rho_i(X) \quad (\forall X \subseteq V).$$

We also have $z(V) = \rho_i(V)$ since $y(V)$ is constant for all $y \in \mathrm{dom}\, f_i$. Hence, for all $z \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ we have

$$\rho_1(V) = z(V) = z(V \setminus X) + z(X) \leq \rho_1(V \setminus X) + \rho_2(X) \quad (\forall X \subseteq V). \tag{22}$$

Since $x_1 \in \mathrm{dom}\, f_1$ and there exists no arc of A_1 from W to $V \setminus W$, we have

$$x_1(V \setminus W) = \rho_1(V \setminus W)$$

by Lemma 3 applied to an M-convex function

$$f(z) = \begin{cases} -z(V \setminus W) & \text{if } z \in \mathrm{dom}\, f_1, \\ +\infty & \text{otherwise.} \end{cases}$$

Symmetrically, since $x_2 \in \mathrm{dom}\, f_2$ and there exists no arc of A_2 from W to $V \setminus W$, we have

$$x_2(W) = \rho_2(W).$$

Adding these yields

$$x_1(V) - [x_1(W) - x_2(W)] = \rho_1(V \setminus W) + \rho_2(W).$$

This contradicts (22), since $x_1(V) = \rho_1(V)$ and $[x_1(W) - x_2(W)] > 0$ by $W \supseteq S$ and $W \cap T = \emptyset$.

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[2] As is well known (see [3, §4.4]), the M-convexity of $\mathrm{dom}\, f_i$ implies that ρ_i is submodular and

$$\mathrm{dom}\, f_i = \{z \in \mathbf{Z}^V \mid z(X) \leq \rho_i(X) \ (\forall X \subset V), z(V) = \rho_i(V)\}.$$

However, we do not need this fact for the proof of Lemma 6.

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