⋆ Combinatorial Optimization (Oberwolfach, November 9–15, 2014)

Discrete DC Programming by Discrete Convex Analysis

—Use of Conjugacy—

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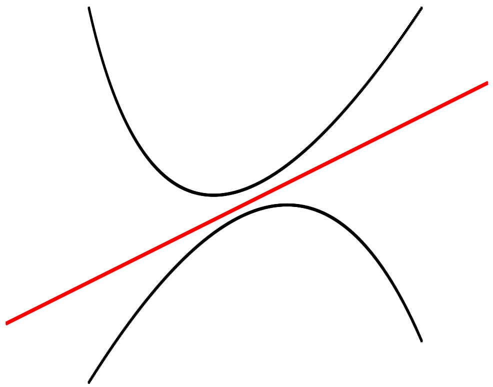

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Discrete DC Program

DC = Difference of Convex

$$\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} \qquad g, h\text{: "convex"}$$

Convexity: $\mathrm{M}^\natural$–$\mathrm{M}^\natural$, $\mathrm{M}^\natural$–$\mathrm{L}^\natural$, $\mathrm{L}^\natural$–$\mathrm{M}^\natural$, $\mathrm{L}^\natural$–$\mathrm{L}^\natural$

(Examples)

Submod. max. under matroid constraint: $\mathrm{M}^\natural$–$\mathrm{L}^\natural$

Subm-superm proc. (Narasimhan-Bilmes): $\mathrm{L}^\natural$–$\mathrm{L}^\natural$

Discrete DC Functions

Discrete DC = DDC = Difference of Discrete Convex

L–L : (almost) all functions

L–M : ⊆ submodular

M–L : ⊆ supermodular

M–M : ? (open)

DC Algorithm (Pham Dinh Tao, 1985(ca.))

$$\min\{g(x) - h(x)\} \quad \Longrightarrow \quad \min\{g(x) - \langle p, x\rangle\}$$

subgradient $p \in \partial h(x)$

Algorithm 1 DC algorithm

Let $x^{(1)}$ be an initial solution

for $k = 1, 2, \ldots$ **do**

  (Dual phase) Pick $p^{(k)} \in \partial h(x^{(k)})$

  (Primal phase) Pick $x^{(k+1)} \in \mathrm{argmin}\,(g - p^{(k)})$

  if $(g - p^{(k)})(x^{(k)}) = (g - p^{(k)})(x^{(k+1)})$ **then**

    Return $x^{(k)}$

  end if

end for

- **concave-convex proc (Yuille–Rangarajan 03)**
- **submod-supermod proc (Narasimhan–Bilmes 05)**

cf. supermod-submod proc, mod-mod proc

(Iyer–Bilmes 12; Iyer–Jegelka–Bilmes 13)

Subdifferentiability & Biconjugacy

$$\partial f(x) = \{p \mid f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y)\}$$

$$f^{\bullet}(p) = \sup_{x} \ \{p \cdot x - f(x)\}$$

$$p \in \arg\min_{q}\{f^{\bullet}(q) - \langle q, x \rangle\} \iff p \in \partial f(x)$$

$$\Updownarrow$$

$$x \in \arg\min_{x}\{f(y) - \langle p, y \rangle\} \iff x \in \partial f^{\bullet}(p)$$

biconjugacy: $f^{\bullet\bullet} = f$

Integral Subgradients & Biconjugacy

$$f : \mathrm{Z}^n \to \overline{\mathrm{Z}} \qquad \partial_{\mathrm{Z}} f(x) \neq \emptyset \text{ ?} \qquad f^{\bullet\bullet} = f \text{ ?}$$

Example: $D = \{(0,0,0), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\}$

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)/2, & x \in D, \\ +\infty, & \text{o.w.} \end{cases}$$

D is "convex": $\mathrm{conv}(D) \cap \mathrm{Z}^n = D$

$\partial f_{\mathrm{R}}(0) = \{(1/2, 1/2, 1/2)\}$, $\quad \partial_{\mathrm{Z}} f(0) = \emptyset$

$$f^{\bullet\bullet}(0) = -\inf_{p \in \mathbb{Z}^3} \max\{0, |p_1+p_2-1|, |p_2+p_3-1|, |p_3+p_1-1|\}$$

$$f^{\bullet\bullet}(0) = -1 \neq 0 = f(0)$$

Conjugacy in Matroids

Biconjugacy

Independent sets $\mathcal{I}$		Rank function ρ
Exchange axiom	$\Longleftrightarrow$	Submodularity
(vertex)	$\Longleftrightarrow$	(face)

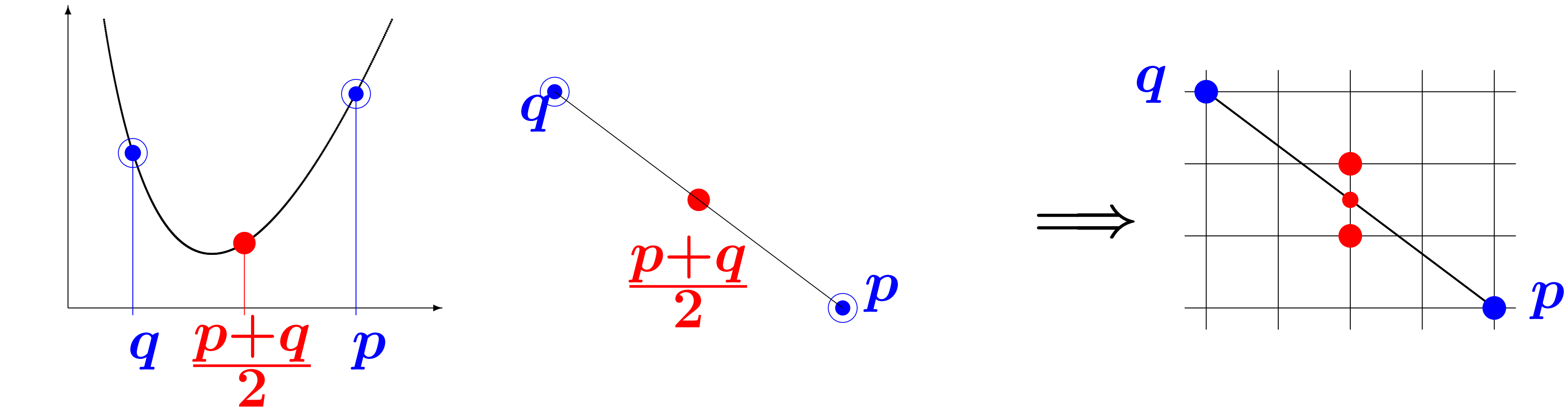
Subgradients

$\partial\rho(\emptyset) =$ Independence polyhedron

L-convex and M-convex Functions

L♮-convexity from Mid-pt-convexity

(Mu. 1998, Fujishige–Mu. 2000)



Mid-point convex ($g : \mathrm{R}^n \to \mathrm{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathrm{Z}^n \to \mathrm{R}$)

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

L♮-convex function (L = Lattice)

L$^\natural$-convexity from Submodularity

—Original definition of L$^\natural$-convexity—

Def: $g : \mathbf{Z}^n \to \mathbf{R}$ **is L$^\natural$-convex** $\iff$

$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ **is submodular in** (p_0, p)

$\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R}$, $\qquad \mathbf{1} = (1, 1, \ldots, 1, 1)$

L$^\natural$-convex Function: Examples

Quadratic: $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L$^\natural$-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

Separable convex: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathrm{R}}$

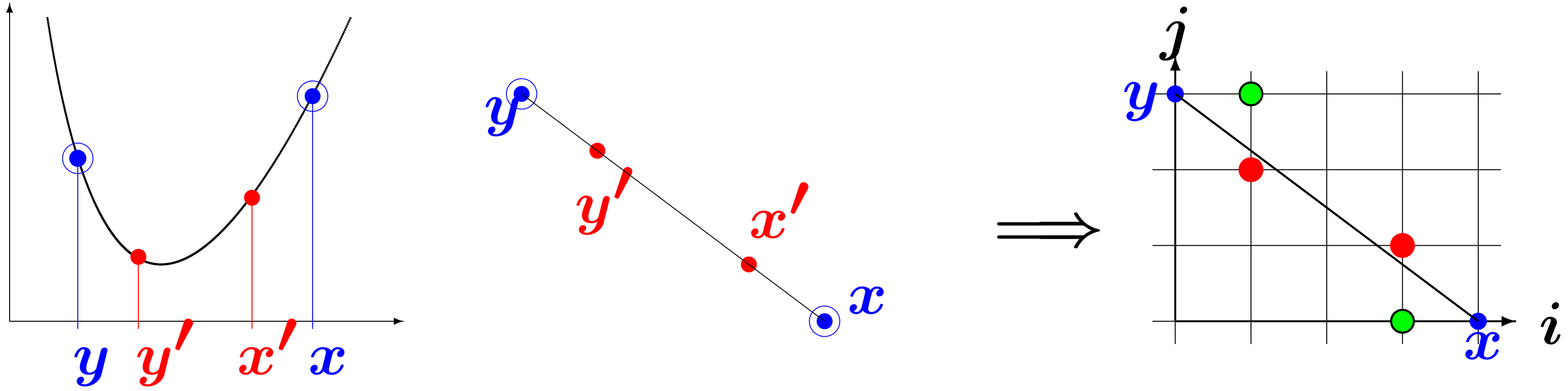
$\Leftrightarrow \rho(X) = g(\chi_X)$ for some L$^\natural$-convex g

Multimodular: $h : \mathrm{Z}^n \to \overline{\mathrm{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L$^\natural$-convex g

M♮-convexity from Equi-dist-convexity

(Mu. 1996, Mu. –Shioura 1999)



Equi-distance convex ($f : \mathbb{R}^n \to \mathbb{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange** ($f : \mathbb{Z}^n \to \mathbb{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \Big[f(x - e_i) + f(y + e_i), \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \Big]$$

M♮-convex function (M = Matroid)

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum_i \sum_j a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow a_{ij} \geq 0, \qquad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \not\in \{i, j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

Matroid rank: $f(X) = -$ rank of X

Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum_i \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_A \varphi_A(x(A))$ (φ_A: convex)

$\{A, B, ...\}$: laminar $\Leftrightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

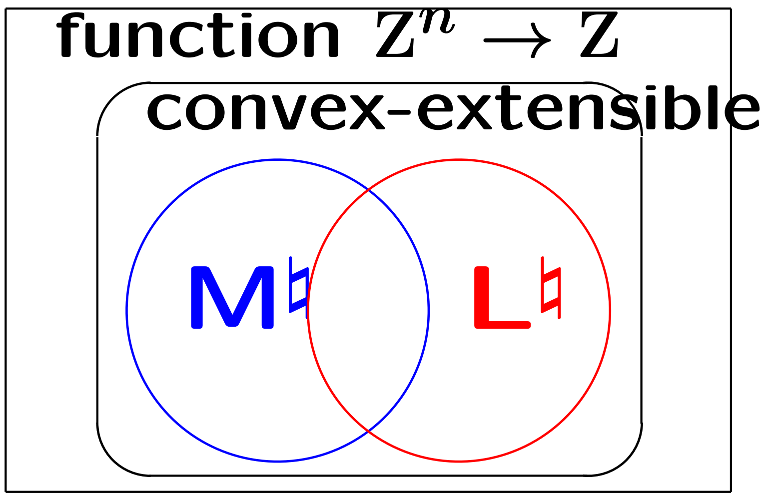
M-L Conjugacy Theorem

Integer-valued discrete fn $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$

Legendre transform: $f^{\bullet}(p) = \sup_{x \in \mathbf{Z}^n} [\langle p, x \rangle - f(x)]$

M$^\natural$-convex and L$^\natural$-convex are conjugate

$f \mapsto f^{\bullet} = g \mapsto g^{\bullet} = f$ **(Mu. 1998)**



biconjugacy

$f^{\bullet\bullet} = f$

Discrete Toland-Singer Duality

$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$

$h : \mathbf{Z}^n \to \mathbf{Z}$: M$^\natural$-convex or L$^\natural$-convex (*g*: any fn)

Toland-Singer Duality Thm (Maehara-Mu. 2013)

$$\inf_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n} \{h^{\bullet}(p) - g^{\bullet}(p)\}$$

(Proof) Integral biconjugacy: $h^{\bullet\bullet} = h$.

$$\begin{aligned}
&\inf_x \{g(x) - h(x)\} = \inf_x \{g(x) - h^{\bullet\bullet}(x)\} \\
&= \inf_x \{g(x) - \sup_p \{\langle p, x\rangle - h^{\bullet}(p)\}\} \\
&= \inf_x \inf_p \{g(x) - \langle p, x\rangle + h^{\bullet}(p)\} \\
&= \inf_p \{h^{\bullet}(p) - \sup_x \{\langle p, x\rangle - g(x)\}\} = \inf_p \{h^{\bullet}(p) - g^{\bullet}(p)\}.
\end{aligned}$$

Discrete DC Algorithm

Discrete DC Algorithm

$$\min_{x \in \mathrm{Z}^n} \{g(x) - h(x)\} \quad \Longrightarrow \quad \min_{x \in \mathrm{Z}^n} \{g(x) - \langle p, x \rangle\}$$

integral subgradient $p \in \partial h(x)$

Algorithm 2 Discrete DC algorithm

Let $x^{(1)}$ be an initial solution
for $k = 1, 2, \ldots$ **do**
  (Dual phase) Pick $p^{(k)} \in \partial h(x^{(k)}) \setminus \partial g(x^{(k)})$
  (Primal phase) Pick $x^{(k+1)} \in \mathrm{argmin}\,(g - p^{(k)})$
  if $(g - p^{(k)})(x^{(k)}) = (g - p^{(k)})(x^{(k+1)})$ **then**
    Return $x^{(k)}$
  end if
end for

- **monotone decreasing** $g(x) - h(x)$
- $\partial g(x) \supseteq \partial h(x)$ **guaranteed**
- **local optimality within** $U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$

Optimality Conditions $\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\}$

x: global opt $\iff$ $\partial_\epsilon g(x) \supseteq \partial_\epsilon h(x) \quad (\forall \epsilon \geq 0)$

$$\Downarrow$$

$$\boxed{\partial g(x) \supseteq \partial h(x)}$$

$$\Downarrow$$

x: local opt, i.e., x: minimum in

$$U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$$

$$\partial_\epsilon f(x) = \{p \mid f(y) - f(x) \geq \langle p, y - x \rangle - \epsilon \quad (\forall y)\}$$

Subgradient of M♮-/L♮-convex Func

$f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$,

integral subgradients:

$$\partial f(x) = \{p \in \mathbf{Z}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y)\}$$

f: M♮-convex $\Rightarrow$ $\partial f(x) \neq \emptyset$: L♮-convex set

$$-p_i \leq f(x - \chi_i) - f(x), \qquad p_j \leq f(x + \chi_j) - f(x),$$

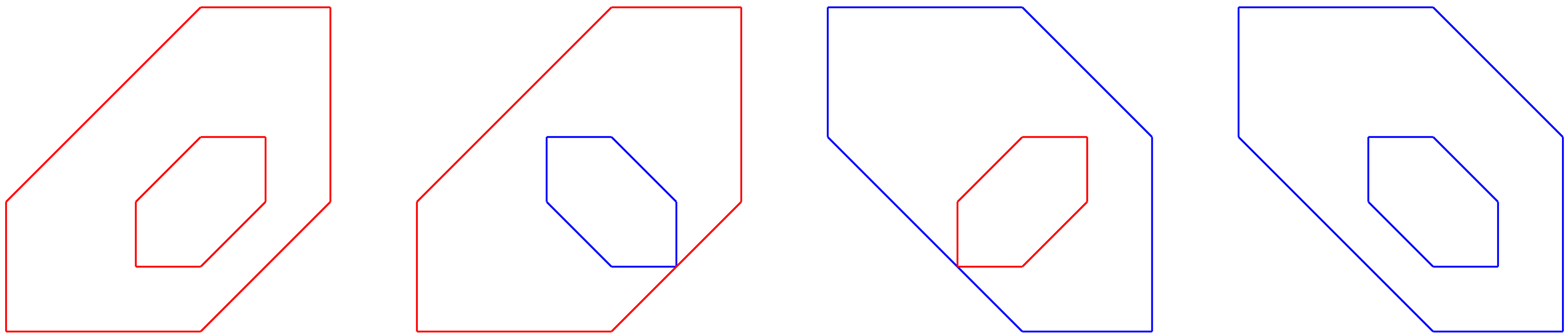
$$p_j - p_i \leq f(x - \chi_i + \chi_j) - f(x) \qquad (\forall i, j)$$

f: L♮-convex $\Rightarrow$ $\partial f(x) \neq \emptyset$: M♮-convex set

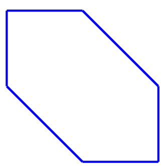
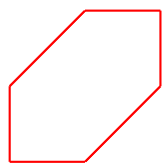
$$f(x) - f(x - \chi_A) \leq \langle p, \chi_A \rangle \leq f(x + \chi_A) - f(x) \quad (\forall A)$$

Testing for Local Opt: $\partial g(x) \supseteq \partial h(x)$

$\mathrm{M}^\natural$–$\mathrm{M}^\natural$	$\mathrm{M}^\natural$–$\mathrm{L}^\natural$	$\mathrm{L}^\natural$–$\mathrm{M}^\natural$	$\mathrm{L}^\natural$–$\mathrm{L}^\natural$



$\mathrm{L}^\natural \supseteq \mathrm{L}^\natural$	$\mathrm{L}^\natural \supseteq \mathrm{M}^\natural$	$\mathrm{M}^\natural \supseteq \mathrm{L}^\natural$	$\mathrm{M}^\natural \supseteq \mathrm{M}^\natural$
poly-time	poly-time	poly-time	co-NP-compl. (McCormick 96)

 g-polymatroid  dual of shortest-path

Larger Conjugate Classes of Functions

(i) sum of two M$^\natural$-convex functions M$^\natural$+M$^\natural$

(ii) convolution of two L$^\natural$-convex fns L$^\natural$ $\square$ L$^\natural$

Subdifferentiability & Biconjugacy

If $h : \mathbf{Z}^n \to \mathbf{Z}$: M$^\natural$+M$^\natural$ or L$^\natural$ $\square$ L$^\natural$ (g: any fn)

Toland-Singer Duality Thm

$$\inf_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n} \{h^\bullet(p) - g^\bullet(p)\}$$

⋆ Summary ("DC Programming")

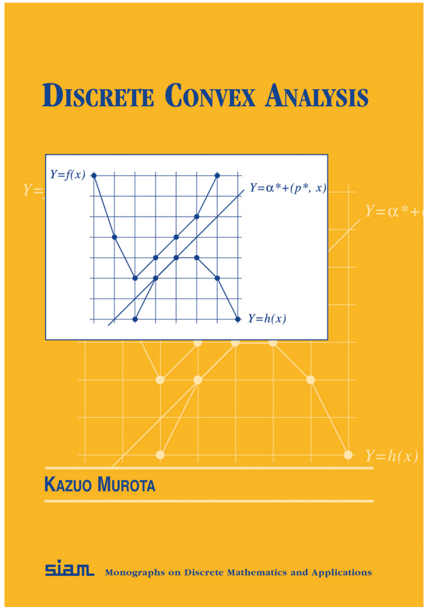
- **Theoretical framework**
  M$^\natural$–M$^\natural$, L$^\natural$–M$^\natural$ (subm), M$^\natural$–L$^\natural$ (superm), L$^\natural$–L$^\natural$, integral subgradient, biconjugacy, Toland-Singer
- **Local optimality condition**
  $\partial g(x) \supseteq \partial h(x)$, checking this condition
- **Algorithm**
  monotone, finite, local opt
- **Using discrete convex analysis for nonconvex prob**

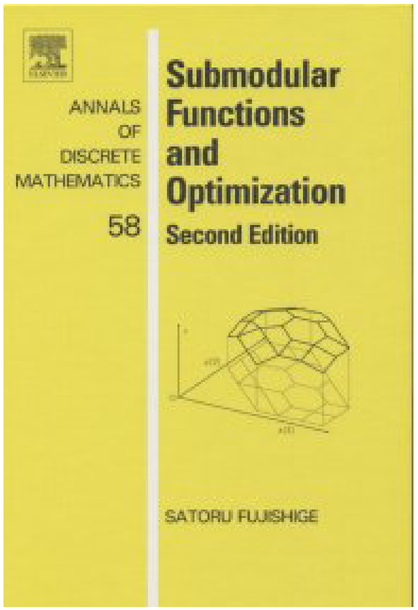
- DC Representability
- Hardness of minimization
- Approximation guarantee (for some 01 cases)
- Computational results (will come)

Books

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)





Survey/Slide/Video/Software

[Survey]
Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

[Slide] [Video]

http://www.misojiro.t.u-tokyo.ac.jp/ murota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)
http://www.misojiro.t.u-tokyo.ac.jp/DCP/

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